More Haste, Less Speed? Signaling through Investment Timing*

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Abstract

We consider a cash-constrained firm learning on the value of an irreversible project at a privately-known speed. Under perfect information, the optimal date of investment may be non-monotonic in the learning speed: better learning increases the value of experimenting further, but also the speed of updating. Under asymmetric information, the firm uses its investment timing to signal confidence in the project and obtain cheaper credit from uninformed investors, which may generate timing distortions: investment is hurried when learning is sufficiently fast, and delayed otherwise. The severity of the cash constraint affects the magnitude of the distortion, but not its direction.

Keywords: Real options, investment timing, signaling, financing of innovation.

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1 Introduction

A sizable literature on investment has underlined the importance of real options: when investment is uncertain and irreversible, refraining from investing allows to keep the option not to invest open, which is valuable in case relevant information accrues in the meantime.[1] In such a context, the optimal timing of investment reflects a tradeoff between the benefit of learning and the cost of delay. An important related concern is that firms' timing and investment decisions may be distorted in the presence of information frictions. A recent literature has begun to explore how asymmetric information affects investment timing decisions, yielding various predictions: some papers establish that it may distort investment in the direction of hurried investment as compared to the optimal timing, some others predict delayed investment.[2] One way of interpreting this pattern is by relating the direction of the timing distortion to the nature of the information or corporate governance problem. For instance, Grenadier and Malenko (2011) predict hurried investment in situations where the decision maker wants the market to believe that the value of the project is larger than it truly is, and delayed investment in the opposite case where he prefers the market to underestimate the value.[3] In this paper, we take a complementary approach and consider a model where an entrepreneur wants to minimize his average cost of capital (hence always prefers the market to be optimistic about the value of the firm), but assume asymmetric information on the precision of the information which is learnt, that is, on the value of learning. This contrasts with the existing literature, which has instead focused on private information on the intrinsic value of the project. While there is a monotonic relationship between the optimal timing decision and the value of the project, which in turn implies that the distortion can only go in one direction, the precision of the signal has a non-monotonic effect on investment timing: observing a more a precise signal increases the marginal value of experimenting further, but at the same time increases the speed of updating, which increases incentives to reap the benefit from investing. This non-

[1]For a review, see Dixit and Pindyck (1994).

[2]Specifically, Morellec and Schürhoff (2011) and Bustamante (2012) obtain hurried investment, while Bouvard (2014) obtains delayed investment; in a paper in which four different corporate governance applications are considered, Grenadier and Malenko (2011) show that investment is hurried in two of their applications, and delayed in the other two.

[3]As illustrations, a manager prompts investment when he wishes to maximize a combination of the true value of firm and the stock price, and delays investment when he wants to divert cash from the firm's proceeds. In a similar vein, Bebchuk and Stole (1993) show that information frictions may lead to overinvestment or underinvestment according to the way the information asymmetry is modeled.

monotonicity results in the distortion possibly going in both directions (hurried or delayed investment). Our paper therefore offers a simple and tractable model where information and financial frictions may generate either hurried or delayed investment, and highlights the role of the learning environment as the main determinant of the direction of the timing distortion. This provides a novel interpretation of differences in timing or investment decisions across firms or markets: some firms may hurry investment while others delay investment because they have access to testing technologies with different precisions, and not necessarily because they are facing different corporate governance problems.

Specifically, we consider a continuous time model which combines the following ingredients: (a) an entrepreneur owns an irreversible project over which he learns as long as the project has not been launched (the entrepreneur holds a real option); (b) the precision of the signal he observes, hence the value of the option, is private information; (c) the entrepreneur is cash-constrained and needs outside funding to finance the project. Private information on the signal's precision implies that the entrepreneur has superior information about the value of the project compared to investors, hence a signaling problem: the date at which the entrepreneur launches the project conveys information on how confident he is, and the entrepreneur would have investors believe that he is as confident as possible in order to obtain cheaper credit.

We first analyze the benchmark case with complete information, and examine how the optimal investment timing varies with the precision of the signal. The impact of a higher precision is two-fold: on the one hand, a more precise signal increases the value of learning, hence incentives to experiment further; on the other hand, as learning is faster, the entrepreneur becomes optimistic on the project more quickly, raising his incentives to invest. When the prior net present value of the project is positive, the optimal investment date is single-peaked in the precision: it is optimal to invest early both when the signal is very precise, because learning is then very fast, and when it is very imprecise, as there is little to be learnt from waiting. This suggests that the information content of the investment timing is intrinsically ambiguous: if the entrepreneur wants to signal that he observes precise signals, it is a priori unclear whether he should invest early or late.

When the precision of the signal is private information, we first establish that the entrepreneur can always reach his complete information payoff whenever he holds sufficient cash: when a high share of the investment is internally financed, an entrepreneur with an imprecise signal is unwilling to distort his investment timing to pretend his signal is precise, as the cost of an inefficient investment policy dwarfs the benefit of cheaper credit. However, the investment timing has to be distorted whenever the entrepreneur holds insufficient cash, and the distortion is more severe the higher the cash shortage. This is because signaling concerns become more salient when a higher share of the project is financed with outside funds. Interestingly, though, while the magnitude of the distortion depends on the severity of the cash constraint, the direction of the distortion is orthogonal to the entrepreneur's net worth.

Indeed, the direction of the distortion only depends on the ordering of investment dates under complete information, which by definition is independent of the information problem. Specifically, whenever an entrepreneur with a precise signal invests later (resp. earlier) than one with an imprecise signal under complete information, the former must delay (resp. hurry) investment as compared to the optimal timing in order to increase the mimicking cost for the latter. The ranking of investment dates under complete information depends on two sets of parameters: (a) the intrinsic value of the project, (b) the value of learning. First, everything else equal, a higher expected value of the project makes early investment relatively more attractive to a slow-learning type whose option value is smaller, resulting in delayed investment by the fast-learning type. Second, everything else equal, fast-learning firms tend to prompt investment in an environment where learning is more accurate: indeed, the firm learning faster starts to lose its comparative learning advantage early when the slow-learning type also learns fast enough, as the latter quickly catches up on beliefs.

Empirically, the distortion in investment policy may have several interpretations. For instance, hurried investment as compared to the benchmark can be understood in a literal sense, but also implies that the total probability of investing is larger, since the firm is overall less likely to learn bad news (over-investment as compared to the benchmark), and that the probability of success conditional on investing is lower than in the benchmark (under-experimentation). Accordingly, the model allows predictions regarding timing decisions (e.g., the timing of release of a new product or a new version of the same product, IPO timing, strategic acceleration or deferment of patent examination procedures...), but also on the probability of success conditional on investment. In either case, a major determinant of the direction of the distortion created by information and financial frictions is the precision of the learning technology (hence the speed of learning): in industries or markets characterized by fast learning, one should observe hurried investment. Hurried investment may for instance materialize in "grandstanding": Gompers (1996) has shown that young venture capital firms tend to take ventures public early because of signaling or reputation concerns; it may also materialize in under-experimentation, that is, an inefficiently high failure rate conditional on investment. Conversely, we should expect delayed investment, or over-experimentation in industries where learning is slower. This is consistent with evidence on the pharmaceutical industry, an industry where development and testing is extremely long.[4] For instance, Guedj and Scharfstein (2004) focus on drug development and establish that cash-constrained firms tend to invest less than unconstrained ones, and have a lower failure rate. This is also consistent with Henkel and Jell (2010), who establish that long patent examination deferments are particularly frequent in the pharmaceutical and chemical industries. Finally, our model predicts that investment might trigger a stock price reaction, as the investment possibly signals relevant information on the entrepreneur's beliefs. The direction of the price reaction also depends on the learning environment. In the region where investment is hurried, investment triggers a positive stock price reaction because the fast-learning entrepreneur invests first; conversely, when investment is delayed, slow learners invest first and the stock price reaction upon investment should be negative.

Our paper relates to the literature on real options (Dixit and Pindyck, 1994), and on exponential/Poisson learning, notably Keller *et al.* (2005) and Décamps and Mariotti (2004). It is more particularly related to a class of models with no commitment where the investment timing is used as a signaling device (Grenadier and Malenko, 2011; Morellec and Schürhoff, 2011; Bustamante, 2012). Like these papers, ours also features a "real option signaling game". It is also related to a strand of the recent literature on experimentation dealing with the impact of asymmetric information. Agency problems may involve adverse selection (private learning), moral hazard (unobservable learning effort), or both. Several papers have considered the question of the design of the optimal contract (or the optimal mechanism) for experimenting agents, for instance Manso (2011), Gerardi and Maestri (2012), or Halac *et al.* (2013, 2014). In particular, a series of pa-

[4] In the market for drugs, DiMasi *et al.* (2003) estimate the time between the start of clinical testing and submission for approval to the FDA to 72.1 months. This takes into account neither the preclinical phase, which lasts on average between 1 and 6 years, nor the length of the FDA's approval process.

pers has focused on the question of the financing of experimentation (Bergemann and Hege, 1998, 2005; Hörner and Samuelson, 2013; Bouvard, 2014; Drugov and Macchiavello, 2014). However, most of the aforementioned papers model private information on a variable which monotonically affects the optimal timing decision (e.g, the prior belief, or the cost of investment). In turn, this monotonicity implies that the distortion always goes in one direction.[5] Instead, we model private information on the parameter which directly captures the precision of learning, which notably implies a non-monotonic relationship between the optimal investment date and the learning speed. The resulting signaling game does not exhibit the usual single-crossing property, which makes the analysis richer, and precisely generates the possibility of both hurried and delayed investment. In addition, the analysis provides an intuitive account of the determinants of the direction of the distortion, namely the precision of the learning process.

The paper is organized as follows: Section 2 is devoted to the presentation of the model. In Section 3, we characterize the optimal investment timing policy in the benchmark case of perfect information. In Section 4, we derive the unique equilibrium of the signaling game, and discuss the determinants of the direction of the distortion. Section 5 reviews the empirical implications of the model. In Section 6, we discuss the robustness of the results. Finally, Section 7 concludes. All the proofs are relegated to the Appendix.

2 The model

A firm owns a project with *ex ante* uncertain value. With probability p_0, the project is of high quality, and yields a revenue $R > 0$. With probability $1 - p_0$, the project is of low quality, and yields zero revenue. Investment involves an irreversible cost $I \in (0, R)$. The firm is risk neutral and discounts future revenues and costs at rate $r > 0$. We typically have in mind young startups with no established reputation which have not yet raised capital from financial markets, but the firm could also be a more established firm facing cash constraints. In what follows, we call the firm "the entrepreneur", but the entrepreneur could be backed by a venture capitalist. In case it is, we assume that the interests of the

[5] A notable exception is Halac *et al.* (2013), who consider a setup with private information on the probability of success of the project, in which learning comes from observing past successes or failures. They derive a non-monotonicity result similar to ours under complete information, but find that asymmetric information always results in under-experimentation (hurried investment), while we establish that over-experimentation (delayed investment) may also arise, and characterize conditions under which it does.

current shareholders (the venture capitalist and the entrepreneur) are perfectly aligned, so that they behave as a single entity. Therefore, our view is complementary to the literature on venture capital, which addresses issues raised by information asymmetries and misaligned interests between venture capital firms and the venture they back (e.g., IPO timing strategies, the design of venture capital contracts (Gompers, 1996; Kaplan and Strömberg, 2003; Gompers and Lerner, 2004)). Instead, we focus on information problems between the current shareholders/management and outsiders (the capital market).

2.1 Learning environment

The entrepreneur decides the date $t \geq 0$ at which he invests, if he ever does. The rationale behind waiting is that the entrepreneur learns about the quality of the project as long as investment has not taken place. We assume, following Décamps and Mariotti (2004), that he observes a signal modeled as a Poisson process with intensity $\lambda > 0$ if the project is of low quality, and with intensity 0 in case the project is of high quality. Therefore, "no news is good news": a jump perfectly identifies a low-quality project, whereas the entrepreneur gets increasingly optimistic about project quality as long as nothing is observed. One may think of the pre-investment or learning period as a phase during which the entrepreneur runs tests (for instance the phase I of the FDA's drug review process), and of λ as the accuracy of the testing technology.[6] The signal is observed for free, so that the benefit of learning is only traded against the cost of delaying investment.[7] Let $s(\lambda, t)$ denote the probability of receiving no signal before date t :

$$s(\lambda, t) = p_0 + (1 - p_0)e^{-\lambda t}. \tag{1}$$

Using Bayes' rule, the entrepreneur's beliefs at date t conditional on no signal being observed read

$$p^* (\lambda, t) = \frac{p_0}{s(\lambda, t)} = \frac{p_0}{p_0 + (1 - p_0)e^{-\lambda t}}. \tag{2}$$

We assume that the precision of the testing technology λ is private information to the entrepreneur. There are several reasons why the quality of the entrepreneur's learning

[6]The assumption that only bad news can be learnt may then be interpreted in the following way: the value R created by the project is well-known, but there may be essential impairments which "kill" the value of the project. For instance, the entrepreneur is perfectly aware of the performance of a drug, car, or software, but needs to run clinical tests, crash tests, or design an alpha/beta version in order to confidently reject the presence of side-effects, safety risks, or bugs.

[7]As discussed in Section 6, one could equivalently assume costly experimentation.

may be prone to private information. First, various testing procedures may be available, and knowing which is the most efficient requires a specific talent. This is typically the case in the software industry. Tassey (2002) notably reports that "software testing is still more of an art than a science, and testing methods and resources are allocated based on the expert judgment of senior staff". This suggests that there is no such thing as a standardized testing procedure, and that heterogeneity of testing skills is prevalent. In addition, outsiders are likely to observe neither the accuracy of the testing procedure chosen, nor the resources allocated to testing. In pharmaceuticals, a great deal of trials are subcontracted to contract research organizations (CROs), and this layer of delegation may raise additional information concerns. Asymmetric information on λ may accordingly capture in a reduced-form way the moral hazard problem created by such delegation. In a similar spirit, when the entrepreneur is backed by a venture capitalist, some elements of the contract between the VC and the entrepreneur may be unobservable to outside investors. Finally, λ could capture the privately-known ability of the VC to advise the entrepreneur, or its opportunity cost of time or effort.

2.2 The financing problem

The entrepreneur initially holds assets with value $A < I$. These assets can be liquid assets (cash or quasi-cash), or illiquid assets which can be pledged as collateral. In principle, these assets could continuously capitalize at any rate $r_0 \leq r$, but we focus here on the case where $r_0 = 0$ for simplicity.[8] This allows to shut down the impact of time on the entrepreneur's cash position, and to focus on its impact on beliefs only.

The entrepreneur therefore needs to borrow cash at the date when he decides to invest in the project. Given that the payoff is either R or 0, the security which is issued (debt or equity) is irrelevant.[9] We assume for simplicity that the entrepreneur raises cash from risk neutral and competitive investors by issuing equity: the entrepreneur sells a fraction x of the firm in exchange for cash.

Importantly, we assume that investors can observe how long the entrepreneur has

[8] In Section 8.9 of the Appendix, we show how our results extend to the general case $r_0 \leq r$.

[9] It is easy to see that the assumption of binary payoffs is irrelevant as long as the entrepreneur's action is also binary. Since our focus here is on investment timing rather than the intensity of investment, we consider a stopping game with only two actions: invest or continue learning. It would be interesting to consider as well an intensive margin, and examine the sensitivity of investment to the cash and information constraints, but this is beyond the scope of this paper.

been learning before raising cash.[10] However, whether a signal revealing a bad project is observed by all parties or privately observed by the entrepreneur is irrelevant here, as the entrepreneur never solicits funding if he knows the project to be bad. Indeed, the cash he must invest out of pocket (or the collateral) would then be lost for sure.[11]

3 Complete information benchmark

Before we turn to the signaling problem raised by private information on λ, let us first examine how λ affects the entrepreneur's behavior in the benchmark case of perfect information. In this case, even if investors do not observe the entrepreneur's beliefs on the project, they can perfectly back out these beliefs using (2). Since investors are competitive, the entrepreneur obtains the full value of the project regardless of the date at which he solicits funding. Let $W^*(\lambda, t)$ denote the expected discounted payoff at date 0 of a entrepreneur learning at speed λ when he invests at date t (conditional on no bad news). We have:

$$\begin{aligned} W^*(\lambda, t) &= e^{-rt} s(\lambda, t) \left(p^*(\lambda, t) R - I\right) \\ &= e^{-rt} \left(p_0 R - I + (1 - s(\lambda, t)) I\right). \end{aligned}$$

This expression evidences the trade-off faced by the entrepreneur between discounting and learning: while waiting delays the realization of the payoff, it also allows to keep the option not to invest open.

The value of this option depends on $1 - s(\lambda, t)$, the probability that bad news accrues, in which case the entrepreneur does not invest and avoids sinking the outlay I. This probability increases in λ : better learning increases the value of the option.

The optimal investment date t^* reflects this tradeoff. It is such that:[12]

$$t^* = \underset{t \geq 0}{\operatorname{argmax}}\, W^*(\lambda, t) = \max\left(-\frac{1}{\lambda} \ln \frac{p_0 r (R - I)}{(1 - p_0)(\lambda + r) I}, 0\right). \tag{3}$$

[10]For instance, if the firm is backed by a venture capitalist, the market can observe the date of the first round of financing. Alternatively, the market can observe the date of the first clinical trials, the date at which the firm first filed a patent application, or the date of release of previous versions of a software. See also Section 6.4 to see how one could relax the assumption of perfect observability.

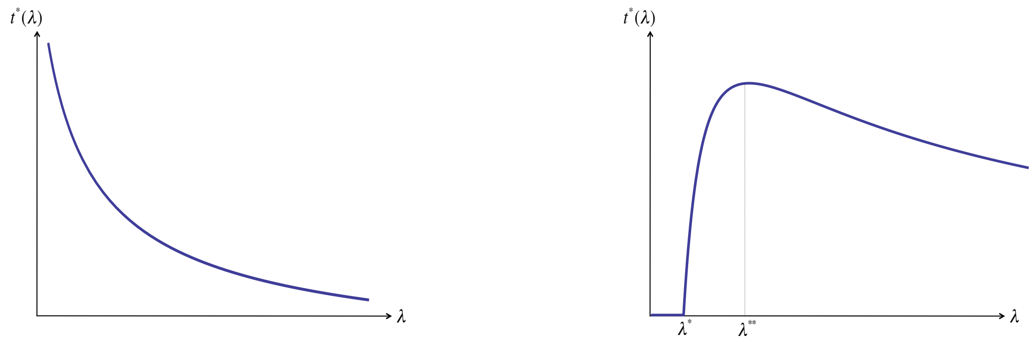
[11]This assumes though that the entrepreneur cannot take the money and run, i.e., raise cash and not invest in the project. Implicitly, we therefore assume that new shareholders can monitor the entrepreneur and impose investment at the date when capital is raised.

[12]Notice that it is fine to solve this stopping problem by maximizing the date-0 expected payoff since the entrepreneur perfectly forecasts at date 0 his conditional beliefs at all future dates.

It is immediate to see that t^* is nonincreasing in R, p_0 and r, and nondecreasing in I. However, t^* may be non-monotonic in λ :

Proposition 1 *The impact of λ on the optimal investment date t^* depends on the prior NPV of the project:*

- *If $p_0 R - I < 0$, $t^*(\lambda)$ is a decreasing function of λ;*
- *If $p_0 R - I \geq 0$, there exist λ^* and λ^{**}, with $0 \leq \lambda^* < \lambda^{**}$, such that $t^*(\lambda) = 0$ for $\lambda \leq \lambda^*$, $t^*(\lambda)$ is increasing on $[\lambda^*, \lambda^{**}]$ and decreasing on $[\lambda^{**}, +\infty)$.*



(a) $p_0 R - I < 0$.   (b) $p_0 R - I \geq 0$.

Figure 1: $t^*(\lambda)$ depending on the prior NPV.

The qualitative difference between the positive NPV case and the negative NPV case has to do with the fact that an entrepreneur who does not learn at all ($\lambda = 0$) never invests when $p_0 R - I < 0$ (i.e., $t^*(0) = \infty$), while he invests at date 0 as soon as the NPV becomes nonnegative. Generically, λ impacts the optimal investment date in two ways. An entrepreneur with a higher λ triggers investment when his beliefs reach a higher threshold, because the value of the option is higher.[13] Meanwhile, he also reaches a given threshold faster. When $p_0 R - I < 0$, the latter effect always dominates, whereas both effects alternatively dominate when $p_0 R - I \geq 0$. Notice that Halac *et al.* (2013, 2014) derive the same non-monotonicity result in models with costly information acquisition.[14] This non-monotonicity makes the signaling content of investment timing intrinsically ambiguous:

[13]Precisely, the entrepreneur invests when his beliefs first hit a threshold $\frac{(\lambda+r)I}{rR+\lambda I}$, which increases in λ.

[14]In their case, though, the case where t^* is decreasing in λ never obtains. Indeed, they focus on the case where costly experimentation is valuable, which is the "counterpart" of our positive NPV case. In the opposite case where experimentation is too costly, there is no learning, so the investment date does not reflect the learning speed. This non-monotonicity is also apparent in Bloch *et al.* (2011) in a duopoly model of entry timing.

for instance, early investment could stem from an entrepreneur with a high λ who has learnt fast, or from an entrepreneur with a low λ whose option has little value. In the next section, we explore in detail how this non-monotonicity shapes signaling incentives under asymmetric information.

4 Incomplete information

We now assume λ to be private information: $\lambda \in \left\{\underline{\lambda}, \overline{\lambda}\right\}$, with $0 \leq \underline{\lambda} < \overline{\lambda}$, and $\Pr(\lambda = \overline{\lambda}) = q_0$. To simplify notation, let us denote $\overline{t}^* = t^*(\overline{\lambda})$ and $\underline{t}^* = t^*(\underline{\lambda})$.

4.1 Payoffs

The cost of capital reflects investors' beliefs over the quality of the project. Investors know long the entrepreneur has been waiting, but do not know his true beliefs because of private information on λ. If q denotes the probability that the market assigns to the entrepreneur being of type $\overline{\lambda}$, the market perceives the probability of success to be

$$p\left(q,t\right) = \frac{p_0}{qs\left(\overline{\lambda},t\right) + (1-q)s\left(\underline{\lambda},t\right)}. \tag{4}$$

For simplicity, we consider here the case where the assets A are liquid, so that the entrepreneur borrows $I - A$.[15] Since investors are competitive, the entrepreneur needs to sell a share $x(q,t)$ of the firm such that

$$x(q,t)p\left(q,t\right)R = I - A. \tag{5}$$

The cost of outside capital then equals $\frac{x(q,t)R-(I-A)}{I-A} = \frac{1-p(q,t)}{p(q,t)}$. There are two implicit assumptions behind (5): first, the entrepreneur does not borrow more than needed. This can easily be shown to be optimal. Indeed, the entrepreneur is risk neutral, so the only motive for selling shares is the cash shortage. For a good entrepreneur, the cost of outside capital is no smaller than the cost of inside funds, as the market can never be more optimistic about the project than he is. Therefore, it is a weakly dominated strategy for him to sell more shares than needed, and the market should accordingly interpret extra

[15] If A was the value of illiquid assets pledged as collateral, the entrepreneur would have to borrow I, but financiers would then be able to seize A in case of failure of the project, so the two interpretations are completely equivalent.

issuance as coming from a bad entrepreneur.[16] Second, we assume that the entrepreneur does not underprice by selling shares below their expected value. Generically, equilibria involving money burning might exist, but we ignore them here for simplicity.

Let $W(\lambda, q, t)$ denote the expected discounted payoff at date 0 of type $\lambda \in \{\underline{\lambda}, \overline{\lambda}\}$ when he invests at date t, and is perceived as type $\overline{\lambda}$ with probability q. We have

$$W(\lambda, q, t) = e^{-rt} s(\lambda, t) \left[p^*(\lambda, t) (1 - x(q, t)) R - A \right]. \tag{6}$$

It is easy to see that $W(\overline{\lambda}, 1, t) = W^*(\overline{\lambda}, t)$ and $W(\underline{\lambda}, 0, t) = W^*(\underline{\lambda}, t)$, i.e., the entrepreneur's expected payoff under asymmetric information is the same as under perfect information as long as investors hold true beliefs on λ, a consequence of competition among investors. However, when the market holds wrong beliefs, the cost of inside and outside funds differ, and there is a benefit (cost) for the entrepreneur from being mistakenly perceived as a good (bad) type: $W(\lambda, q, t)$ is nondecreasing in q.

From (5) and (6), we derive:

$$W\left(\overline{\lambda}, q, t\right) - W(\underline{\lambda}, q, t) = Ae^{-rt} f(t), \tag{7}$$

where

$$f(t) \equiv s(\underline{\lambda}, t) - s(\overline{\lambda}, t) = (1 - p_0)(e^{-\underline{\lambda} t} - e^{-\overline{\lambda} t}). \tag{8}$$

For given investor beliefs, the only difference between two entrepreneurs with different signal precisions lies in the value of their options to wait: the good type (rightly) abstains from investing with probability $1 - s(\overline{\lambda}, t)$, in which case he saves the outlay (Ae^{-rt} in present value), while the bad type does so only with probability $1 - s(\underline{\lambda}, t)$. It is easy to see that $f(t)$ is nonnegative, as the good type invests in the project less often than the bad type, and that $f(0) = 0$, as both types differ only to the extent that some learning has taken place. Finally, f is single-peaked in t. This has to do with the fact that learning exhibits decreasing returns with such Poisson learning: the probability of learning bad news before t increases with t, but the marginal increase is decreasing with time. This implies that the comparative learning dynamics is characterized by two phases: a first phase in which the good type learns more at the margin (as time goes by, the good type becomes increasingly more optimistic than the bad type), and a second phase in which

[16]This holds because A is symmetric information. To make this statement, we implicitly use a refinement in the spirit of D1. See Section 4.2.1.

the bad type catches up on beliefs: in the limit, there is perfect learning for any positive λ, so the difference between types becomes negligible.

Finally, it is easy to show that $e^{-rt}f(t)$ is also single-peaked in t and reaches its maximum at

$$t_0 \equiv \frac{\ln(\overline{\lambda}+r)-\ln(\underline{\lambda}+r)}{\overline{\lambda}-\underline{\lambda}} > 0.$$

4.2 Equilibrium analysis

Before we turn to the equilibrium analysis, let us first notice that asymmetric information has no bite when $\overline{t}^* = 0$. Indeed, if $\overline{t}^* = 0$, then a fortiori $\underline{t}^* = 0$, using (3), so both types can reach their complete information payoff by investing at date 0, in which case investors' beliefs are irrelevant ($p(q,0) = p_0$ for all q). In what follows, we therefore restrict attention to $\overline{t}^* > 0$.

4.2.1 Equilibrium definition and concept

We look for perfect Bayesian equilibria satisfying D1. Whenever D1 is not enough to guarantee uniqueness, we select the Pareto-dominant equilibrium, or least-cost equilibrium.[17] A pure-strategy equilibrium features investment dates $\underline{t}$ and $\overline{t}$ (conditional on no news) for types $\underline{\lambda}$ and $\overline{\lambda}$, and a belief function $q(t)$, which assigns a probability that investment at date t comes from type $\overline{\lambda}$.[18]

4.2.2 Separating equilibria

We first look for separating equilibria where $\underline{t} \neq \overline{t}$. First of all, it is standard that, in any separating equilibrium, we must have $\underline{t} = \underline{t}^*$.[19] Therefore, the following incentive constraint has to hold:[20]

$$W(\underline{\lambda},0,\underline{t}^*) \geq W(\underline{\lambda},1,\overline{t}). \tag{9}$$

[17] D1 imposes to attribute a deviation to some date t to the type with the stronger incentive to deviate to t (see Section 8.2 for details on how D1 applies to the signaling game we consider). We will see that, whenever there is equilibrium multiplicity, the only equilibria are separating, meaning that they can be Pareto-ranked.

[18] Mixed strategies in such a continuous time game are not obvious to define, but we will later show that a mixed strategy equilibrium must involve one type randomizing between exactly two pure strategies, and the other type playing a pure strategy. See Lemma 3 in the Appendix.

[19] If $\underline{t} \neq \underline{t}^*$, type $\underline{\lambda}$ could always increase his payoff by playing $\underline{t}^*$: even it is does not improve the market's beliefs, it yields a higher expected payoff.

[20] This constraint is necessary for a separating equilibrium, but generically not sufficient. We show in the Appendix that, provided that the equilibrium is a least-cost separating equilibrium and survives the D1 refinement, it is also sufficient.

This constraint states that the slow-learning type prefers to play his optimal investment strategy rather than mimic the fast-learning type. Using (7), this is equivalent to

$$W(\underline{\lambda}, 0, \underline{t}^*) \geq W^* \left(\overline{\lambda}, \overline{t}\right) - Ae^{-r\overline{t}} f(\overline{t}). \tag{10}$$

Before exploring separating equilibria, let us remark that D1 allows us to restrict the set of possible equilibrium dates $\overline{t}$, as Lemma 1 shows.

Lemma 1 *In any separating equilibrium, the good type invests at a date comprised between $\overline{t}^*$ and t_0 : $\overline{t} \in [\min\left(t_0, \overline{t}^*\right), \max\left(t_0, \overline{t}^*\right)]$.*

Lemma 1 states that the equilibrium investment date must reflect the compromise between the preferred investment strategy of the good type (invest at $\overline{t}^*$) and the call for incentives, which requires distorting investment in the direction of t_0 to increase the difference in option values. Indeed, this difference in option values, $Ae^{-rt} f(t)$, is maximum at $t = t_0$. Therefore, a deviation to an off-equilibrium date between t_0 and $\overline{t}$ ("in the direction of t_0") is relatively more beneficial to the good type (no matter the out-of-equilibrium associated to this deviation), so that D1 imposes to attribute such a deviation to the good type. Given this restriction, it is clear that Lemma 1 has to hold. Otherwise, by deviating in the direction of t_0, the good type could secure a strictly higher expected payoff, while still being perceived as good.

The next Proposition provides necessary and sufficient conditions for the existence of a separating equilibrium.

Proposition 2 *There exists a separating equilibrium if and only if*

$$A \geq A_1 \equiv \max\left(0, \frac{W^*\left(\overline{\lambda}, t_0\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{e^{-rt_0} f(t_0)}\right).$$

In addition, $(\underline{t}^*, \overline{t}^*)$ is an equilibrium if and only if

$$A \geq A_0 \equiv \frac{W^*\left(\overline{\lambda}, \overline{t}^*\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{e^{-r\overline{t}^*} f(\overline{t}^*)} \in (A_1, I).$$

In a separating equilibrium, the sorting of types is achieved by making sure that the difference between each type's beliefs on the project is large enough that it is worthwhile for the good type to invest his cash A out of pocket, but too costly for the bad type. This is why a separating equilibrium is only possible when the entrepreneur has enough cash. Whenever the cash constraint is sufficiently soft ($A \geq A_0$), the entrepreneur can reach his complete information payoff. Indeed, a high share of the investment is then externally financed, and the benefit of cheap credit does not compensate the loss due to the inefficiency of the investment policy, so that mimicking is not a concern. When $A \in (A_1, A_0)$, the timing must be distorted, but the entrepreneur still has enough cash to make separation possible. Notice that $A_0 > 0$, so efficiency is never attainable when A is too small. However, one may have $A_1 = 0$, in which case there exists a separating equilibrium even if the entrepreneur has no cash. This is because the preferences of each type over investment timing is a sorting force *per se*, as it is intrinsically costly for an entrepreneur of a given type to mimic the other type, even when the project is fully externally financed. Accordingly, the preferences may be sufficiently different for a separating equilibrium to exist even when the entrepreneur has no cash.

4.2.3 Pooling and semi-pooling equilibria

We now characterize pooling and semi-pooling equilibria:

Proposition 3 *If $A \geq A_1$, any equilibrium is fully separating.*

In addition, $\exists\, A_2 \leq A_1$ such that:

- *If $A_2 \leq A < A_1$, the unique equilibrium is such that $\bar{t} = t_0$, and $\underline{\lambda}$ randomizes between t_0 and $\underline{t}^*$,*
- *If $A < A_2$, the unique equilibrium is pooling: $\underline{t} = \bar{t} = t_0$.*

In the Appendix, we derive the cutoff value A_2 :

$$A_2 = \max\left(0, I - \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{q_0 e^{-rt_0} f(t_0)}\right).$$

Notice that the pooling equilibrium never exists if $A_2 = 0$, i.e., when $q_0 \leq \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{I e^{-rt_0} f(t_0)}$. This corresponds to instances where the prior probability q_0 is small enough, in which case the bad type is unwilling to distort his investment strategy, as the gain of being perceived as the average type is too small.

Two findings emerge from Proposition 3. First, the equilibrium cannot involve any pooling when a separating equilibrium exists. Second, when there is pooling in equilibrium, it must be at date t_0. This springs from Lemma 1, which imposes that the distortion be “capped” at t_0 : investment at t_0 corresponds to maximal incentives, as it maximizes the difference between each type’s beliefs. We derive the following Corollary:

Corollary 1 *There is a unique least-cost equilibrium satisfying D1.*

The only possible range of multiplicity is $A \geq A_1$. In this case, there may be a continuum of separating equilibria, but pooling is then impossible. Therefore, all these equilibria can be Pareto-ranked, and there is consequently a unique least-cost separating equilibrium.

4.3 The cash constraint and the magnitude of the distortion

The equilibrium strategies reflect the severity of the incentive problem, which is measured by A. When the cash constraint is soft, the complete information payoffs are attained. Otherwise, the good type needs to distort his investment date in the direction of t_0 in order to prevent mimicking by the bad type. When the information problem becomes too severe (A is too small), the only solution for the good type is to invest at t_0 to maximize the difference in option values. When this is not sufficient to achieve separation, the good type must be (fully or partially) pooled with the bad type. Proposition 4 establishes formally that the distortion incurred by the good type increases with the cash shortage.

Proposition 4 *The magnitude of the distortion* $|\bar{t}^* - \bar{t}|$ *is nonincreasing in* A.

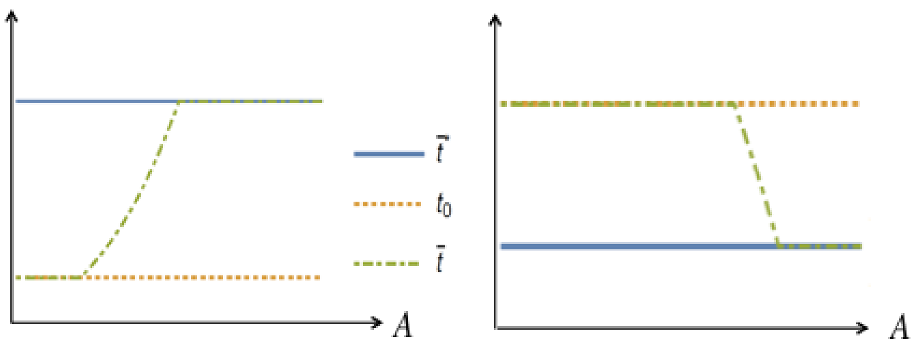


Figure 2: Cash constraint and distortion

From $W(\underline{\lambda}, 1, t) = W^*(\overline{\lambda}, t) - Ae^{-rt}f(t)$, one sees that changes in t have a larger impact on the incentive constraint of the bad type whenever A increases, which reduces the distortion. A captures the stake of the entrepreneur in his project, i.e., the share of the investment which is financed internally. As the cash shortage problem improves (that is, as A increases), the benefit from fooling investors decreases, and the cost from distorting the timing away from the preferred timing policy increases. By improving the sorting of types, a higher A therefore attenuates the information problem, and decreases the welfare loss.

4.4 The direction of the distortion

We have characterized so far how the severity of the cash constraint measured by A shapes equilibrium behavior, and possibly results in timing distortions. In addition, we have underlined in Lemma 1 that incentive constraints call for a deviation in the direction of t_0. Let us now examine how the direction of the distortion depends on the primitives of the model.

One first remark is that $\overline{t}^*$ and t_0 are independent of A, which implies that the direction of the distortion does not depend on the cash constraint. A therefore only affects the magnitude of the distortion, but not its direction.

Second, because $\overline{t}^*$ is decreasing in p_0 as long as it is positive, and t_0 is independent of p_0, we derive that there exists a critical value $\hat{p} \in (0,1)$ such that $\overline{t}^* \leq t_0 \Leftrightarrow p_0 \geq \hat{p}$. Similarly, there exists $\hat{R} > I$ such that $\overline{t}^* \leq t_0 \Leftrightarrow R \geq \hat{R}$. Accordingly, an empirical prediction of the model is that projects with a low expected value (a low p_0 or R) are hurried when the cash constraint is too severe, while projects with better prospects are delayed.

Third, it is easy to see that $\overline{t}^* \leq t_0 \Leftrightarrow \underline{t}^* \leq \overline{t}^*$.

Therefore, the direction of the distortion is simply given by the ordering of the optimal investment dates. If the good type invests earlier (resp. later) than the bad type under complete information, then he should hurry (resp. delay) investment as compared to his optimal timing to prevent mimicking from the bad type. This is helpful to derive whether investment should be hurried or delayed as a function of $\underline{\lambda}$ and $\overline{\lambda}$. Indeed, a direct consequence of Proposition 1 is that, whenever $p_0R - I < 0$, one always has $\overline{t}^* < \underline{t}^*$, so that investment can only be hurried. When $p_0R - I \geq 0$, there are two cases:

- $\overline{\lambda} \leq \lambda^{**}$, in which case $\underline{t}^* < \overline{t}^*$ and investment can only be delayed,

- $\overline{\lambda} > \lambda^{**}$, in which case there exists a cutoff value $\underline{\lambda}_0$ such that $t^*(\underline{\lambda}_0) = t^*(\overline{\lambda})$. Investment is delayed if $\underline{\lambda} \leq \underline{\lambda}_0$, and hurried otherwise.

Relatedly, an empirical prediction is that investment should be hurried in markets where learning is fast (both $\underline{\lambda}$ and $\overline{\lambda}$ are large enough), and delayed in markets where learning is slow ($\underline{\lambda}$ is small enough). To understand this, recall that f reflects the comparative learning dynamics: in a first phase, the beliefs of both types diverge apart, with the good type learning faster than the bad type at the margin; in a second phase, the bad type catches up on beliefs. When both types learn sufficiently fast, the phase during which the good type learns faster stops early. In this case, signaling fast learning imposes to invest early to make sure that the bad type is sufficiently less confident than the good type about the project. Conversely, when the slow-learning type learns little enough, the first phase stops later, and the good type should exploit as much as possible his comparative learning advantage by waiting longer.

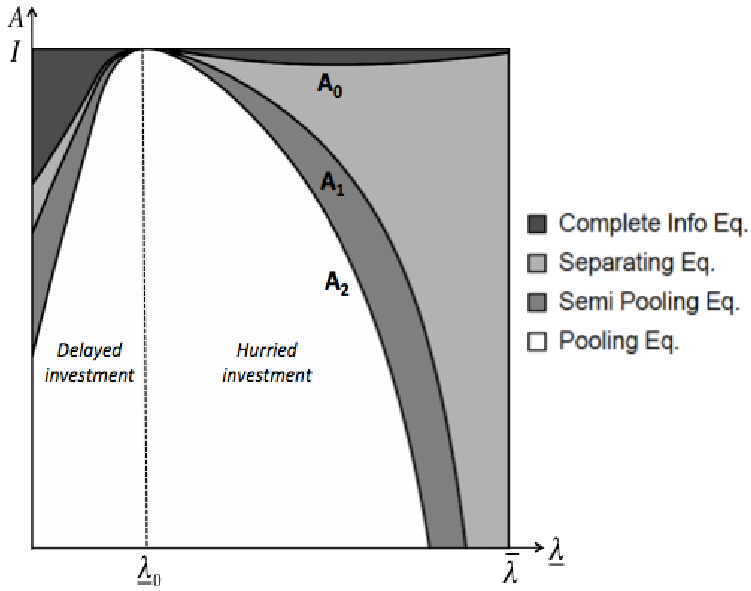


Figure 3: Equilibrium characterization

To conclude this section, we summarize in Figure 3 the different equilibrium regions in the space $(\underline{\lambda}, A)$ in the positive NPV case. The figure displays that higher values of A make it more likely that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium, and, if not, that there exists a separating equilibrium. It also shows that separation is obtained by delaying investment when $\underline{\lambda}$ is sufficiently small, and by hurrying investment otherwise, regardless of the value of A.

4.5 Belief reversal

Since the entrepreneur endogenously decides the date at which he invests, it is a priori not clear whether an entrepreneur with a precise signal is more or less confident upon investing than an entrepreneur with an imprecise signal. Indeed, the latter could compensate the lower accuracy of the learning technology by experimenting longer. Under complete information, it is not difficult to see that this can never be the case, and that, even when a fast-learning entrepreneur optimally invests sooner, he still is more confident at the date when he invests. Under asymmetric information, this is a fortiori true when investment is delayed. However, when investment is hurried, we show that the timing distortion may be large enough to generate a reversal of beliefs, in that the bad type becomes more optimistic on the project upon investing than the good type.

Proposition 5 *Suppose* $p_0 R - I > 0$*. For all* $\overline{\lambda} > \lambda^{**}$*, there exists a cutoff* $\lambda_1 \in (\underline{\lambda}_0, \overline{\lambda})$ *such that*

$$\underline{\lambda} > \lambda_1 \Rightarrow \exists \underline{A} > 0, p^*(\underline{\lambda}, \underline{t}^*) > p^*(\overline{\lambda}, \overline{t}) \textit{ for } A < \underline{A}.$$

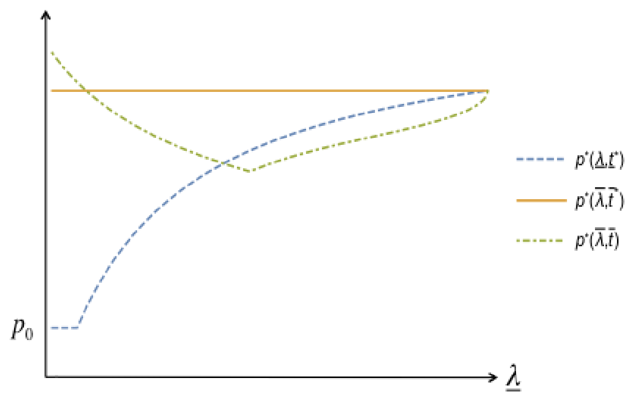


Figure 4: Belief reversal

4.6 The role of the expected value of the project

Before concluding this section, we examine the impact of p_0 and R on the equilibrium.

Proposition 6 *There exist* $\{\underline{\underline{p}}, \underline{p}, \overline{p}, \overline{\overline{p}}\}$ *with* $0 < \underline{\underline{p}} < \underline{p} < \hat{p} < \overline{p} < \overline{\overline{p}} < 1$ *and* $\{\underline{\underline{R}}, \underline{R}, \overline{R}, \overline{\overline{R}}\}$ *with* $I < \underline{\underline{R}} < \underline{R} < \hat{R} < \overline{R} < \overline{\overline{R}}$ *such that:*

- *If* $p_0 \notin \left(\underline{\underline{p}}, \overline{\overline{p}}\right) \Leftrightarrow R \notin \left(\underline{\underline{R}}, \overline{\overline{R}}\right)$*, the equilibrium involves investment at the optimal dates* $(\underline{t}^*, \overline{t}^*)$*;*

- *If* $\underline{\underline{p}} < p_0 < \underline{p} \Leftrightarrow \underline{\underline{R}} < R < \underline{R}$, *the equilibrium is separating and investment is hurried;*

- *If* $\underline{p} < p_0 < \hat{p} \Leftrightarrow \underline{R} < R < \hat{R}$, *the equilibrium is semi-pooling or pooling* ($\bar{t} = t_0$), *and investment is hurried;*

- *If* $\hat{p} < p_0 < \overline{p} \Leftrightarrow \hat{R} < R < \overline{R}$, *the equilibrium is semi-pooling or pooling* ($\bar{t} = t_0$), *and investment is delayed;*

- *If* $\overline{p} < p_0 < \overline{\overline{p}} \Leftrightarrow \overline{R} < R < \overline{\overline{R}}$, *the equilibrium is separating, and investment is delayed.*

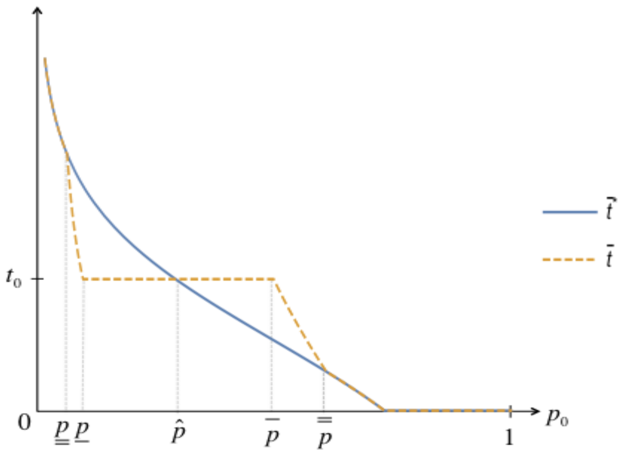


Figure 5: The equilibrium investment date $\bar{t}$ as a function of p_0

The investment timing is not distorted when the expected value of the project is either large or small enough. In the intermediate region where investment has to be distorted, investment should be hurried when the expected value of the project is small, and hurried otherwise. In addition, for any $A < I$, the equilibrium spans all the different regions (separating and pooling equilibrium, hurried and delayed investment) when p_0 varies from 0 to 1.

5 Empirical predictions

A first prediction of the model is that information and financial constraints affect investment: otherwise identical firms with different levels of cash have different investment policies. This is consistent with the finding that cash-rich firms have an investment policy which is less sensitive to their net worth than cash-constrained firms (see Hubbard

(1998) for a survey). In addition, our model predicts more specifically that inefficient investment may take both the form of hurried investment (under-experimentation) or delayed investment (over-experimentation) according to the expected value of the project and the shape of the learning curves in the market. First, projects with a high prior expected value tend to be delayed, while those with a low prior value tend to be hurried. This suggests that there should be too much and too risky investment in markets with poor prospects (in terms of probability of success, or profits conditional on success). Conversely, there should be too little and insufficiently risky investment in markets with better prospects. Second, in industries with slow learning, cash-constrained firms should invest later than unconstrained firms, which implies that they should succeed with a higher probability conditional on investing. This is consistent with Guedj and Scharfstein (2004), who show that small drug companies with less initial cash are less likely to move to subsequent phases of clinical tests, and are more likely to succeed in these phases, that is, cash-constrained firms experiment longer. This is also consistent with the fact that long deferments of patent examination are particularly frequent in the pharmaceutical and chemical industries (Henkel and Jell, 2010).[21] In industries characterized by fast learning, cash-constrained firms should invest earlier than unconstrained firms, and have accordingly a lower probability of success. In terms of the timing of patenting decisions, this implies that firms in industries where learning is relatively fast (e.g., software) should be more prone to soliciting accelerated patent examination procedures.[22] Overall, our model suggests that an empirical analysis on the impact of cash constraints on investment should group firms according both to how financially constrained they are, and to the speed of learning in the industry. Indeed, our results hint that an analysis where firms are grouped according to their net worth only would possibly underestimate the impact of the cash constraint, by pooling together firms which underinvest and firms which overinvest.

In addition, the model predicts that young firms which are more prone to asymmetric information about the precision of their learning technology behave differently from more

[21]Some patent offices allow patent applicants to solicit accelerated and/or deferred examination of their application, lowering or expanding *de facto* the duration of the experimentation period. Patents are often perceived as a way for cash-poor firms to secure financing by signaling their quality (Hall and Harhoff, 2012).

[22]Unfortunately, there has been little evidence on the characteristics of firms which file for accelerated procedures so far. Harhoff and Stoll (2014) exploit as a natural experiment the fact that the European Patent Office has switched from a regime where accelerated examination procedures were publicly disclosed to a regime where they are kept secret. But they do not focus on differences in the pool of applicants across industries.

established firms for which private information is less of an issue. For instance, young venture capital firms with no established reputation should distort the date at which they take ventures public because of signaling concerns. This is reminiscent of Gompers (1996), who evidences the importance of "grandstanding", i.e., the tendency of venture capitalists to take ventures public early, among young venture capital firms. However, while our model clearly predicts the possibility of grandstanding (hurried investment), it also suggests that such a phenomenon should be rather prevalent in industries where learning is fast, but less so in industries with slow learning, where we might instead observe the reverse pattern of deferred IPO timing.

Our model also yields predictions in terms of stock price reactions. Since investment possibly reveals information about the entrepreneur's confidence, the stock price should react to investment. However, our model predicts that the stock price reaction should be different in industries characterized by different learning speeds, and for firms with different financial constraints. In markets where investment is hurried, fast-learning firms invest earlier and investment triggers a positive stock price reaction; conversely, when investment is delayed, the stock price reaction upon investment should be negative, as slow learners invest first. However, such a stock price reaction occurs only in the case where the firm holds sufficient cash. Otherwise, the equilibrium is pooling, meaning that the investment date is uninformative, and triggers no price change.

Finally, we evidence two distinct "ranges of inaction" in which investment does not respond to a change in the expected profitability of the project (see Figure 5, where there are two parts on which $\bar{t}$ is flat). One range corresponds to the situation in which the option to wait is not exerted (if investment takes place at date 0, an increment in the NPV does not change the entrepreneur's behavior, both under complete and incomplete information); the other range corresponds to the zone where there is (full or partial) pooling: the entrepreneur is fully constrained by the incentive problem, and keeps investing at date t_0, even when the NPV marginally increases. Therefore, the non-responsiveness of investment to changes in the value of the investment may have two radically different causes: the option-like feature of investment and the cash constraint. In firms prone to cash constraints and asymmetric information, the non-responsiveness is more likely to originate from an incentive problem, while for firms holding projects which are easier to revert (for instance, when there is a liquid second-hand market for assets), the option has a smaller value, and the non-responsiveness could rather reflect the desire of the entrepreneur to reap the benefits from investment as soon as possible. It would be interesting to test these predictions empirically.

6 Discussion

6.1 Other corporate finance applications

In our specification, the entrepreneur's objective is to minimize the average cost of capital. Therefore, the problem we study is qualitatively similar to a problem of managerial myopia, in which the manager cares about both the true value of the firm and the stock price, or to a problem of optimal IPO timing, where the entrepreneur (or the venture capitalist) chooses the date at which the firm goes public, so as to maximize some average of the value of current equity and the future IPO price. Our model is therefore suited to study the impact of information frictions on timing decisions in a more general class of corporate governance problems. This being said, our results contrast with related results established in this literature. Grenadier and Malenko (2011) and Bustamante (2012) both analyze models of signaling through investment timing, the former with an application to managerial myopia (among other applications), the latter to IPO timing. They both find that asymmetric information generates hurried investment as compared to the complete information case, while we stress that both hurried or delayed investment may arise, depending on the shape of the learning curves in the market.

6.2 Capitalization/depreciation of A

When the asset A is capitalized at rate $r_0 \leq r$, we obtain $W(\overline{\lambda}, q, t) - W(\underline{\lambda}, q, t) = Ae^{-(r-r_0)t} f(t)$, a function which is still single-peaked in t, and reaches a maximum at $\tilde{t}_0(r_0)$, where $\tilde{t}_0(r_0)$ increases in r_0. Therefore, our main results qualitatively go through: the region where delayed investment obtains expands if $r_0 > 0$, and shrinks if $r_0 < 0$.[23] This highlights the complementary role played by the capitalization of the entrepreneur's cash: waiting longer is more of an effective signaling strategy when r_0 is higher, since the amount that is internally financed increases everything else equal, hence the cost incurred by the bad type when mimicking the good type. This extra effect may notably generate

[23] See Section 8.9 of the Appendix for a formal analysis.

timing reversals: for instance, when $r_0 > 0$, there are situations where the good type invests earlier than the bad type under complete information, but later under asymmetric information.

6.3 Costly experimentation

It is also easy to show that our results would hold with costly experimentation. For instance, we derive an essentially similar result when there is no discounting and learning involves a flow cost c.[24] In this case, since an entrepreneur who invests later has to pay higher experimentation costs, the cash he has left to finance the project shrinks with time.[25] This effect mirrors the effect of cash capitalization described in the previous paragraph: the cash of the entrepreneur depreciates, which makes the hurried investment equilibrium more likely everything else equal.

6.4 Non-observability of date 0

Since the investment date is the signal which the entrepreneur uses to display his confidence on the project, it is critical that investors are able to observe how long exactly the entrepreneur has been learning, i.e., "knows date 0". Suppose instead that the market does not know the exact waiting time, but that an entrepreneur who has been waiting for a length T could provide hard evidence that he has been waiting at least for any length $\tilde{T} \leq T$.[26] In other words, the entrepreneur could possibly understate, but not overstate his waiting time. In this case, our equilibrium with hurried investment would collapse: if investors believe an entrepreneur who pretends that he has been waiting for $\bar{t}$ to be a high type, then the entrepreneur should wait until $\bar{t}^*$, but pretend he has only been learning for a length $\bar{t}$. However, an equilibrium with delayed investment would be robust to this altered information structure, since such gaming is unprofitable in this direction. Therefore, the non-observability of date 0 would create some asymmetry between the (robust) delayed equilibrium and the (non robust) hurried equilibrium.

[24]The proof is omitted for concision, but is available upon request.

[25]This holds of course only as long as one assumes that the experimentation costs are monetary.

[26]For instance, the entrepreneur could exhibit past trial outcomes, which do not provide information on the quality of the project, but evidence that the entrepreneur was already running tests on the project at the time of the trial.

6.5 Good news model

Our assumption that "no news is good news" makes the analysis simpler, as there is a one-to-one relationship between dates and beliefs as long as the entrepreneur has not received bad news. Let us try to conjecture what would happen in a "good news" model, i.e., in a learning environment where the Poisson process has intensity λ in case the project is of high quality, and 0 otherwise. Under complete information, it is easy to see that the optimal rule consists of investing either at date 0 or only upon learning good news. We then obtain an equivalent of our non-monotonicity result: if the NPV is negative, investment occurs at the first date at which good news arises. This date is a random variable, with an expectation decreasing in λ. If the NPV is nonnegative, there is immediate investment if λ is small enough, while the expected investment date decreases in the range where the option is exerted. In the presence of a cash constraint, asymmetric information is irrelevant as long as jumps in the Poisson process are observable to investors. Indeed, the entrepreneur solicits funding only in instances where types with different learning speeds have the same beliefs on the project, that is, either at date 0, or upon observing a signal perfectly revealing a good project. However, if jumps in the Poisson process are unobservable to the market, it is impossible to back out beliefs from the investment date, since the first date at which there is a jump is stochastic. This is true even with no uncertainty on the entrepreneur's type, so that private learning gives rise to an interesting signaling problem even under perfect information on λ. We conjecture that the entrepreneur should not solicit funding before some critical date, in order to make sure that he is sufficiently pessimistic that he does not invest his own cash if he has not learnt good news yet. In case there is additional private information on λ, this also guarantees that the bad type still prefers to invest at date 0 (when this is his preferred strategy under complete information) than waiting to secure cheaper credit.

6.6 Commitment

By focussing on on a signaling game where the only way to signal confidence in the project is through the investment timing, we make important restrictions on commitment, communication, and instruments: first, we implicitly rule out commitment power, which would allow the entrepreneur and the financiers to agree *ex ante* on some contractual terms; second, we ignore the possibility for the entrepreneur to reveal information on his type using other messages than the timing decision; third, we rule out possible transfers prior to the investment date. In order to assess how sensitive our results are to these various restrictions, we might alternatively consider a screening problem, in which a monopolistic bank proposes a contract at date 0 to the entrepreneur. First of all, in order to make things comparable with our model (that is, to "test" the role of our assumption on commitment), one could impose that the entrepreneur always invests all his cash A, and that the only contractible variables are the payoff of the entrepreneur in case of success, and the investment date. This framework is the natural screening counterpart to our signaling model. In such a context, the function $e^{-rt}f(t)$, which measures the difference in option values, captures the rent that the good type should be given not to mimic the bad type. In order to lower this rent, the investment timing of the bad type should be distorted away from t_0.[27] In addition, notice that the result of no distortion at the top does not always hold, as the incentive constraints imply a "monotonicity condition" $e^{-r\bar{t}}f(\bar{t}) \geq e^{-r\underline{t}}f(\underline{t})$, which sometimes imposes that the good type's investment date also be distorted in the direction of t_0, as in our signaling model. The distortion therefore reflects the incentive problem captured by the same function f, suggesting that our results are no artifact of the no-commitment assumption. Notice, however, that allowing for a wider message space or set of instruments would relax the incentive problem.[28] For instance, if the entrepreneur was borrowing cash through a risky debt contract, the bank could use how much the entrepreneur internally finances to screen the entrepreneur's information, relying on the higher willingness to invest his cash out of pocket of a more confident entrepreneur.[29]

7 Conclusion

We consider a model in which a cash-constrained entrepreneur learns about the value of a project of his, but at a speed which is private information. The signaling problem arising from the conjunction of the information friction (private precision of the learning

[27] A standard difference between screening and signaling two-type models, is that the type who suffers a distortion is the bad type in a screening model (no distortion at the top), and the good type in a signaling game. By the same logic, the distortions go in opposite directions in each model.

[28] For instance, Halac *et al.* (2013) allow for transfers at all dates prior to investment, and show that there are timing distortions only under the conjunction of moral hazard and adverse selection. In our model with adverse selection only, we would obtain the first best with such a richer set of instruments.

[29] This contrats with the fact that incentives impose that both types invest all their cash A in the signaling game.

technology) and the financial friction (limited cash) results in the entrepreneur distorting his investment policy when the cash shortage is too severe. This distortion takes the form of hurried investment (under-experimentation) in markets with fast learning, and of delayed investment (over-experimentation) in markets where learning is slower.

The fact that both delayed and hurried investment may arise has to do with a noteworthy property of our signaling game: the entrepreneur's decision endogenously affects the amount of relevant asymmetric information: the relevant asymmetric information is not the entrepreneur's type λ *per se*, but his beliefs about the project, which depend on both his (privately observed) type λ and the (observable) timing decision t. This relevant asymmetric information, measured by the difference in each type's beliefs, increases and then decreases with time. This non-monotonicity explains why signaling may possibly involve hurried or delayed investment. Relatedly, a distinctive feature of our modeling of private information on learning speed is that it is impossible to rank types according to their preference over investment dates: depending on their relative learning speeds, a fast-learning type may be more willing or less willing than a slow-learning type to invest later, that is, the single-crossing property does not hold. While the analysis is made somewhat more complex, it is also richer. In addition, it highlights in an intuitive way differences in investment timing decisions across firms or industries, by relating these differences to the accuracy of learning in the market. Both from the methodological point of view and in terms of predictions, our paper therefore offers a substantial contribution to the literature on real options under asymmetric information.

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8 Appendix

Let us first introduce the function $\widetilde{f}$ defined as

$$\widetilde{f}(t) \equiv e^{-rt} f(t). \tag{11}$$

$\widetilde{f}$ is non-negative, single-peaked and attains its maximum at $t_0 = \frac{\ln(\overline{\lambda}+r)-\ln(\underline{\lambda}+r)}{\overline{\lambda}-\underline{\lambda}}$.

8.1 Proof of Proposition 1

Let us consider the derivative of $-\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}$ with respect to λ :

$$\frac{1}{\lambda^2}\ln\frac{p_0 r\,(R-I)}{(1-p_0)\,(\lambda+r)\,I}+\frac{1}{\lambda\,(\lambda+r)}.$$

Its sign is given by the sign of $a(\lambda)=\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}+\frac{\lambda}{\lambda+r}$.
Note that $a'(\lambda)=-\frac{\lambda}{(\lambda+r)^2}\leq 0$, and that $\lim\limits_{\lambda\to+\infty} a(\lambda)=-\infty$.
We distinguish three cases:

i) $p_0R-I<0 : -\frac{1}{\lambda}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}>0$ for all λ, and $a(0)=\ln\frac{p_0(R-I)}{(1-p_0)I}<0$.
Therefore, $t^*(\lambda)$ is positive and decreasing for all λ.

ii) $p_0R-I>0$: there exists $\lambda^*=\frac{r(p_0R-I)}{(1-p_0)I}>0$ such that $\lambda\leq\lambda^*\Leftrightarrow t^*(\lambda)=0$. Furthermore, $a(\lambda^*)=\frac{\lambda^*}{\lambda^*+r}>0$, so there also exists $\lambda^{**}>\lambda^*$ such that $a(\lambda^{**})=0$, so $t^*(\lambda)$ is increasing for $\lambda\in[\lambda^*,\lambda^{**}]$, and $t^*(\lambda)$ is decreasing for $\lambda\geq\lambda^{**}$.

iii) $p_0R-I=0$: this implies $t^*(0)=0$ and $t^*(\lambda)>0$ for all $\lambda>0$. One can show that $\lim\limits_{\lambda\to 0}\frac{1}{\lambda^2}\ln\frac{p_0 r(R-I)}{(1-p_0)(\lambda+r)I}+\frac{1}{\lambda(\lambda+r)}=+\infty$.
Therefore, this case is qualitatively similar to the case $p_0R-I>0 : t^*$ is increasing and then decreasing. □

8.2 D1 beliefs

In this section, we examine how D1 restricts beliefs $q(t)$. Suppose that the equilibrium prescribes that $\underline{\lambda}$ invests at $\underline{t}$ and $\overline{\lambda}$ invests at $\overline{t}$ with positive probability, and consider a date t_a at which no one invests with positive probability, so that $q(t_a)$ is not pinned down by Bayes' rule and can be arbitrary. D1 imposes to attribute a deviation to date t_a to the type with the stronger incentive to deviate from his equilibrium action to t_a.

Let $\Delta(t_a) \equiv W(\underline{\lambda}, q, t_a) - W(\underline{\lambda}, q(\underline{t}), \underline{t}) - [W(\overline{\lambda}, q, t_a) - W(\overline{\lambda}, q(\overline{t}), \overline{t})]$ denote the difference between the marginal incentive to deviate to date t_a for both types, when such a deviation generates beliefs q. Using (7), note that $\Delta(t_a) = W(\overline{\lambda}, q(\overline{t}), \overline{t}) - W(\underline{\lambda}, q(\underline{t}), \underline{t}) - A\widetilde{f}(t_a)$ is independent of q.

D1 imposes to consider $q(t_a) = 0$ if $\Delta(t_a) > 0$ and $q(t_a) = 1$ if $\Delta(t_a) < 0$. If $\Delta(t_a) = 0$, we assume that $q(t) = 1$ for simplicity, but this is innocuous. Overall, D1 imposes that, for any t_a off the equilibrium path, the following holds:

$$q(t_a) \in \{0, 1\} \text{ and } q(t_a) = 1 \Leftrightarrow \Delta(t_a) = W(\overline{\lambda}, q(\overline{t}), \overline{t}) - W(\underline{\lambda}, q(\underline{t}), \underline{t}) - A\widetilde{f}(t_a) \leq 0.$$

8.3 Proof of Lemma 1

A separating equilibrium $(\underline{t}^*, \overline{t})$ exists if and only if the following constraints hold:

$$\begin{cases} W(\overline{\lambda}, 1, \overline{t}) \geq W(\overline{\lambda}, 0, \underline{t}^*) & \text{(12a)} \\ W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, 1, \overline{t}) & \text{(12b)} \\ W(\overline{\lambda}, 1, \overline{t}) \geq W(\overline{\lambda}, q(t), t) \qquad \text{for all } t \notin \left\{\underline{t}^*, \overline{t}\right\} & \text{(12c)} \\ W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, q(t), t) \qquad \text{for all } t \notin \left\{\underline{t}^*, \overline{t}\right\} & \text{(12d)} \end{cases}$$

Consider the case $\overline{t}^* \geq t_0$.

Suppose that there is a separating equilibrium $(\underline{t}^*, \overline{t})$ such that $\overline{t}^* < \overline{t}$. Since $\overline{t}$ must satisfy (12b), we have

$$W(\underline{\lambda}, 0, \underline{t}^*) \geq W(\underline{\lambda}, 1, \overline{t}) = W(\overline{\lambda}, 1, \overline{t}) - A\widetilde{f}(\overline{t}).$$

Let us denote $\alpha(\overline{t}) \equiv W(\underline{\lambda}, 0, \underline{t}^*) - W(\overline{\lambda}, 1, \overline{t}) + A\widetilde{f}(\overline{t}) \geq 0$ the "slack" in the incentive constraint of the bad type.

One can write

$$\Delta(t) = W(\overline{\lambda}, 1, \overline{t}) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t) = A[\widetilde{f}(\overline{t}) - \widetilde{f}(t)] - \alpha(\overline{t}).$$

Since $\overline{t} > \overline{t}^* \geq t_0$, there exists $\epsilon > 0$ such that $\overline{t} - \epsilon \geq \overline{t}^* \geq t_0$. One has $\Delta(\overline{t} - \epsilon) < 0$, which implies $q(\overline{t} - \epsilon) = 1$. In addition, we must have $W(\overline{\lambda}, 1, \overline{t} - \epsilon) > W(\overline{\lambda}, 1, \overline{t})$ because $\overline{t} - \epsilon \geq \overline{t}^*$. Therefore, $\overline{t} - \epsilon$ is a profitable deviation for $\overline{\lambda}$, so $(\underline{t}^*, \overline{t})$ cannot be an equilibrium.

Suppose now that $\overline{t} < t_0 \leq \overline{t}^*$. By the same mechanic, one shows that type $\overline{\lambda}$ can strictly increase his payoff by deviating to $\overline{t} + \epsilon$ such that $\overline{t} + \epsilon \leq t_0 \leq \overline{t}^*$.

The proof is similar in the case $t_0 \geq \bar{t}^*$. $\square$

8.4 Proof of Proposition 2

Let us first look for conditions for investment at the optimal dates $(\underline{t}^*, \bar{t}^*)$ to be an equilibrium. It is clear that (12a) and (12c) are satisfied for $\bar{t} = \bar{t}^*$, as type $\bar{\lambda}$ gets his complete information payoff. One can also show that (12d) holds for $\bar{t} = \bar{t}^*$. Suppose it does not, i.e., there is some $t_a \neq \bar{t}^*$ such that $W(\underline{\lambda}, q(t_a), t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$.

Since D1 imposes to have $q(t) \in \{0, 1\}$, we must have $q(t_a) = 1$, which implies:

$$\Delta(t_a) = W(\bar{\lambda}, 1, \bar{t}^*) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t_a) \leq 0.$$

Therefore, we have

$$W(\underline{\lambda}, 1, t_a) = W(\bar{\lambda}, 1, t_a) - A\widetilde{f}(t_a) < W(\bar{\lambda}, 1, \bar{t}^*) - A\widetilde{f}(t_a) \leq W(\underline{\lambda}, 0, \underline{t}^*).$$

A contradiction.

Consequently, (12b) is a necessary and sufficient condition for $(\underline{t}^*, \bar{t}^*)$ to be an equilibrium. Let us now derive under which condition (12b) holds at $\bar{t} = \bar{t}^*$. This condition reads

$$W(\underline{\lambda}, 1, \bar{t}^*) = W(\bar{\lambda}, 1, \bar{t}^*) - A\widetilde{f}(\bar{t}^*) \leq W(\underline{\lambda}, 0, \underline{t}^*). \tag{13}$$

It is clear that if (13) holds for some $\tilde{A}$, then it must hold for all $A > \tilde{A}$. Note also that (13) does not hold for $A = 0$. When $A \to I$, $W(\underline{\lambda}, 1, \bar{t}^*) \to W(\underline{\lambda}, 0, \bar{t}^*) < W(\underline{\lambda}, 0, \underline{t}^*)$, so (13) holds. We conclude that there exists $A_0 \in (0, I)$ such that $(\underline{t}^*, \bar{t}^*)$ is an equilibrium iff $A \geq A_0$. In addition:

$$A_0 = \frac{W\left(\bar{\lambda}, 1, \bar{t}^*\right) - W\left(\underline{\lambda}, 0, \underline{t}^*\right)}{\widetilde{f}(\bar{t}^*)} = \frac{W\left(\bar{\lambda}, 1, \bar{t}^*\right) - W\left(\underline{\lambda}, 0, \underline{t}^*\right)}{e^{-r\bar{t}^*} f(\bar{t}^*)}.$$

Let us now suppose that

$$A < A_0 \Leftrightarrow W(\underline{\lambda}, 1, \bar{t}^*) > W(\underline{\lambda}, 0, \underline{t}^*).$$

Before going further, let us remark that the function $t \mapsto W(\bar{\lambda}, 1, t) - A\widetilde{f}(t)$ is increasing on $[t_0, \bar{t}^*]$ if $t_0 < \bar{t}^*$, and decreasing on $[\bar{t}^*, t_0]$ if $\bar{t}^* < t_0$. This implies that

$$\forall t \in [\min\left(t_0, \bar{t}^*\right), \max\left(t_0, \bar{t}^*\right)],\ W(\bar{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\bar{\lambda}, 1, t) - A\widetilde{f}(t). \tag{14}$$

We now establish the following lemma:

Lemma 2 *A separating equilibrium exists if and only if* $W(\underline{\lambda},1,t_0) \leq W(\underline{\lambda},0,\underline{t}^*)$.

Proof Let us first prove that $W(\underline{\lambda},1,t_0) \leq W(\underline{\lambda},0,\underline{t}^*)$ is a necessary condition for a separating equilibrium.

Suppose we have $W(\underline{\lambda},1,t_0) > W(\underline{\lambda},0,\underline{t}^*)$, with $W(\underline{\lambda},1,t_0) = W(\overline{\lambda},1,t_0) - A\widetilde{f}(t_0)$.

Using (14), we derive that

$$\forall t \in \left[\min\left(t_0,\overline{t}^*\right),\max\left(t_0,\overline{t}^*\right)\right],\ W(\overline{\lambda},1,t) - A\widetilde{f}(t) > W(\underline{\lambda},0,\underline{t}^*).$$

From Lemma 1, we derive that there is no separating equilibrium.

Let us now show that $W(\underline{\lambda},1,t_0) \leq W(\underline{\lambda},0,\underline{t}^*)$ is a sufficient condition for a separating equilibrium. Suppose $W(\underline{\lambda},1,t_0) \leq W(\underline{\lambda},0,\underline{t}^*) < W(\underline{\lambda},1,\overline{t}^*)$. This implies:

- If $t_0 < \overline{t}^* : \exists! \overline{t}^h \in [t_0,\overline{t}^*], W(\underline{\lambda},1,\overline{t}^h) = W(\underline{\lambda},0,\underline{t}^*)$,
- If $\overline{t}^* < t_0 : \exists! \overline{t}^d \in [\overline{t}^*,t_0], W(\underline{\lambda},1,\overline{t}^d) = W(\underline{\lambda},0,\underline{t}^*)$,

Therefore, for $i \in \{h,d\}$, one can write

$$W(\underline{\lambda},1,\overline{t}^i) = W(\overline{\lambda},1,\overline{t}^i) - A\widetilde{f}(\overline{t}^i) \text{ for } i \in \{h,d\}.$$

Let us now show that the good type has no profitable deviation starting from a candidate equilibrium $\overline{t} = \overline{t}^h$ (resp. $\overline{t}^d$), i.e., (12a) and (12c) are satisfied for $\overline{t} = \overline{t}^h$ (resp. $\overline{t}^d$).

Consider first the case $t_0 < \overline{t}^*$. Suppose the good type deviates from $\overline{t} = \overline{t}^h$ to some date t :

- If $t < \overline{t}^h$, we have $W(\overline{\lambda},q(t),t) \leq W(\overline{\lambda},1,t) < W(\overline{\lambda},1,\overline{t}^h)$. So such a deviation cannot be profitable for any out-of-equilibrium beliefs $q(t)$.
- If $\overline{t}^h < t$: one can write $\Delta(t) = W(\overline{\lambda},1,\overline{t}^h) - W(\underline{\lambda},0,\underline{t}^*) - A\widetilde{f}(t) = A[\widetilde{f}(\overline{t}^h) - \widetilde{f}(t)] > 0$ for $t_0 \leq \overline{t}^h < t$. Therefore, we must have $q(t) = 0$. The benefit from deviating to t becomes $W(\overline{\lambda},0,t) - W(\overline{\lambda},1,\overline{t}^h) = W(\underline{\lambda},0,t) + A\widetilde{f}(t) - [W(\underline{\lambda},1,\overline{t}^h) + A\widetilde{f}(\overline{t}^h)] = A[\widetilde{f}(t) - \widetilde{f}(\overline{t}^h)] + W(\underline{\lambda},0,t) - W(\underline{\lambda},0,\underline{t}^*) < 0$, so this deviation is not profitable.

In the case $\overline{t}^* < t_0$, the proof is similar.

We have shown that the good type never has an incentive to deviate to an off-path investment date. To establish that he does not want to invest at $\underline{t}^*$, let us remark that, if $\overline{t}^* < \underline{t}^*$ (resp. $\overline{t}^* > \underline{t}^*$), there exists a positive (resp. negative) ϵ such that $W(\overline{\lambda}, 0, \underline{t}^*) < W(\overline{\lambda}, q(\underline{t}^* - \epsilon), \underline{t}^* - \epsilon)$ for any $q(\underline{t}^* - \epsilon)$. So deviating to $\underline{t}^*$ is always strictly dominated by some off-path deviation which has been ruled out in the previous proof.

Therefore, the good type has no profitable deviation. The last thing we need to show is that the bad type cannot benefit from a deviation off path either. Suppose there exists t_a such that $W(\underline{\lambda}, q(t_a), t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$. Since D1 imposes to have $q(t) \in \{0, 1\}$ for all t in a separating equilibrium, we must have $q(t_a) = 1$, which implies

$$\Delta(t_a) = W(\overline{\lambda}, 1, \overline{t}^i) - W(\underline{\lambda}, 0, \underline{t}^*) - A\widetilde{f}(t_a) \leq 0 \Leftrightarrow \widetilde{f}(\overline{t}^i) \leq \widetilde{f}(t_a).$$

For $i = h$, $\widetilde{f}(\overline{t}^h) \leq \widetilde{f}(t_a) \Rightarrow W(\overline{\lambda}, 1, t_a) \leq W(\overline{\lambda}, 1, \overline{t}^h)$. Indeed, given $\overline{t}^h \geq t_0$, a necessary condition for $\widetilde{f}(\overline{t}^h) \leq \widetilde{f}(t_a)$ is $t_a \leq \overline{t}^h$.

Similarly, $\widetilde{f}(\overline{t}^d) \leq \widetilde{f}(t_a) \Rightarrow W(\overline{\lambda}, 1, t_a) \leq W(\overline{\lambda}, 1, \overline{t}^d)$ in the case $\overline{t}^* < t_0$.

Therefore, one has

$$\begin{aligned} W(\underline{\lambda}, 1, t_a) &= W(\overline{\lambda}, 1, t_a) - A\widetilde{f}(t_a) \\ &\leq W(\overline{\lambda}, 1, \overline{t}^i) - A\widetilde{f}(\overline{t}^i) \\ &= W(\underline{\lambda}, 1, \overline{t}^i) \\ &= W(\underline{\lambda}, 0, \underline{t}^*). \end{aligned}$$

This contradicts $W(\underline{\lambda}, 1, t_a) > W(\underline{\lambda}, 0, \underline{t}^*)$, so the bad type has no profitable deviation. □

Let us finally check under which conditions we have $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$.

One can rewrite the condition as

$$W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*). \tag{15}$$

Again, if $\tilde{A}$ satisfies (15), then so does $A > \tilde{A}$. Furthermore, (15) holds for $A = A_0$.

Indeed, (15) can be rewritten

$$W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\overline{\lambda}, 1, \overline{t}^*) - A_0\widetilde{f}(\overline{t}^*).$$

From $W(\overline{\lambda}, 1, t_0) - A\widetilde{f}(t_0) \leq W(\overline{\lambda}, 1, \overline{t}^*) - A\widetilde{f}(\overline{t}^*)$, it is then clear that (15) holds for $A = A_0$.

Therefore, $\exists A_1 \leq A_0$ such that a separating equilibrium exists if and only if $A \geq A_1$. We have:

$$A_1 = \max\left(0, \frac{W(\overline{\lambda}, 1, t_0) - W(\underline{\lambda}, 0, \underline{t}^*)}{\widetilde{f}(t_0)}\right) = \max\left(0, \frac{W(\overline{\lambda}, 1, t_0) - W(\underline{\lambda}, 0, \underline{t}^*)}{e^{-rt_0} f(t_0)}\right).$$

Finally, notice that there may be other separating equilibria involving $\overline{t} \in [t_0, \overline{t}^h)$ (resp. $\overline{t} \in (\overline{t}^d, t_0])$, but they would give a strictly lower profit to the good type than t^h (resp. t^d). □

8.5 Proof of Proposition 3

Let us first establish the following lemma:

Lemma 3 *In any non-separating equilibrium,* $\overline{\lambda}$ *invests at date* $\overline{t} = t_0$ *with probability* 1. *In addition,* $\underline{\lambda}$ *either invests at* t_0 *with probability* 1 *(pooling equilibrium), or randomizes between investing at* t_0 *and* $\underline{t}^*$ *(semi-pooling equilibrium).*

Proof Let $\mathbb{T}$ be the set of dates at which both types invest with positive probability. Given that D1 imposes to consider $q(t) \in \{0, 1\}$ for each t off path, we have that $q(t) \in (0, 1) \Leftrightarrow t \in \mathbb{T}$.

We first establish that $\mathbb{T}$ has at most two elements. Indeed, suppose $\mathbb{T}$ has at least three distinct elements (t_a, t_b, t_c). By definition of $\mathbb{T}$, one has

$$W(\lambda, q(t_a), t_a) = W(\lambda, q(t_b), t_b) = W(\lambda, q(t_c), t_c) \text{ for } \lambda \in \{\underline{\lambda}, \overline{\lambda}\} \tag{16}$$

Using (7), one derives that $\widetilde{f}(t_a) = \widetilde{f}(t_b) = \widetilde{f}(t_c)$, which is impossible, since f is continuous and single-peaked.

Suppose now that $\mathbb{T}$ has two distinct elements (t_a, t_b). One at least is different from t_0, say t_a. We then have $\Delta(t) = W(\overline{\lambda}, q(t_a), t_a) - W(\underline{\lambda}, q(t_a), t_a) - A\widetilde{f}(t) = A[\widetilde{f}(t_a) - \widetilde{f}(t)]$, using (7). If $t_a < t_0$, we have $\Delta(t_a + \epsilon) < 0$, so $q(t_a + \epsilon) = 1$. Since $q(t_a) < 1$, one can always find ϵ small enough to obtain $W(\lambda, 1, t_a + \epsilon) > W(\lambda, q(t_a), t_a)$ for all λ. So there is a profitable deviation. The same reasoning holds for $t_a > t_0$.

We conclude that $\mathbb{T}$ is a singleton. If the unique element of $\mathbb{T}$ is not t_0, there is always a profitable deviation, by the same reasoning as above. Therefore, $\mathbb{T} = \{t_0\}$.

Suppose now that $\overline{\lambda}$ invests with positive probability at some $t_a \neq t_0$. Since $\mathbb{T} = \{t_0\}$, $\underline{\lambda}$ must then invest with probability zero at date t_a.

We therefore have

$$W(\overline{\lambda}, 1, t_a) = W(\overline{\lambda}, q(t_0), t_0) = W(\underline{\lambda}, q(t_0), t_0) + A\widetilde{f}(t_0).$$

This implies that

$$W(\underline{\lambda}, 1, t_a) = W(\overline{\lambda}, 1, t_a) - A\widetilde{f}(t_a) = W(\underline{\lambda}, q(t_0), t_0) + A[\widetilde{f}(t_0) - \widetilde{f}(t_a)] > W(\underline{\lambda}, q(t_0), t_0).$$

So type $\underline{\lambda}$ then strictly prefers to invest at date t_a than at t_0. A contradiction. □

Let us now turn to the proof of the Proposition. Suppose first that a separating equilibrium exists, i.e., $A \geq A_1$. Using Lemma 2, this is equivalent to $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$. We then have:

$$W(\underline{\lambda}, q(t_0), t_0) < W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*).$$

Therefore, the bad type cannot invest at t_0 with positive probability, as this is strictly dominated by investing at $\underline{t}^*$. Therefore, there is neither semi-pooling nor pooling equilibria.

Before we derive equilibrium conditions for a pooling and a semi-pooling equilibrium, notice that, since the equilibrium payoffs are $W(\lambda, q(t_0), t_0)$ for each type λ, we have $\Delta(t) = A[\widetilde{f}(t_0) - \widetilde{f}(t)] > 0$, so any off-path deviation generates beliefs $q(t) = 0$.

Conditions for a pooling equilibrium The following conditions must be satisfied for a pooling equilibrium $\underline{t} = \overline{t} = t_0$ to exist:

$$W(\underline{\lambda}, q_0, t_0) \geq W(\underline{\lambda}, 0, \underline{t}^*) \tag{17}$$

$$W(\overline{\lambda}, q_0, t_0) \geq W(\overline{\lambda}, 0, t) \text{ for all } t \neq t_0. \tag{18}$$

A necessary condition for a pooling equilibrium is that a separating equilibrium does not exist, i.e., $W(\underline{\lambda}, 1, t_0) > W(\underline{\lambda}, 0, \underline{t}^*)$. Since $W(\underline{\lambda}, q, t_0)$ is increasing in q, and since $W(\underline{\lambda}, 0, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*)$, there exists a critical value of $\overline{q}$ such that (17) holds if and only if $q_0 \geq \overline{q}$. $\overline{q}$ satisfies

$$W(\underline{\lambda}, \overline{q}, t_0) = W(\underline{\lambda}, 0, \underline{t}^*).$$

Let $\hat{t} \in \arg\max_t W(\overline{\lambda}, 0, t)$:

$$\begin{aligned} W(\overline{\lambda}, \overline{q}, t_0) - W(\overline{\lambda}, 0, \hat{t}) &= W(\underline{\lambda}, \overline{q}, t_0) + A\widetilde{f}(t_0) - W(\underline{\lambda}, 0, \hat{t}) - A\widetilde{f}(\hat{t}) \\ &= W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, \hat{t}) + A\left[\widetilde{f}(t_0) - \widetilde{f}(\hat{t})\right] \\ &> 0. \end{aligned}$$

This implies that (17) $\Rightarrow$ (18). Consequently, there is a pooling equilibrium in which all types invest at $t = t_0$ if and only if $q_0 \geq \overline{q}$. To derive this equilibrium condition as a function of A, let us first notice that, at $A = A_1$, we have $\overline{q} = 1 > q_0$, by definition of A_1. So, the pooling equilibrium does not exist when A is sufficiently close to A_1. Furthermore, it is easy to see that (6) implies that

$$W(\underline{\lambda}, q, t) = W(\underline{\lambda}, 0, t) + q\widetilde{f}(t)(I - A),$$

which gives

$$\overline{q} = \frac{W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, t_0)}{\widetilde{f}(t_0)(I - A)}.$$

Therefore, $\overline{q}$ is increasing in A. When $A = 0$, we have $\overline{q} = \frac{W(\underline{\lambda},0,\underline{t}^*)-W(\underline{\lambda},0,t_0)}{\widetilde{f}(t_0)I}$. We derive that:

- if $q_0 \leq \frac{W(\underline{\lambda},0,\underline{t}^*)-W(\underline{\lambda},0,t_0)}{I\widetilde{f}(t_0)}$, no pooling equilibrium ever exists,
- if $q_0 > \frac{W(\underline{\lambda},0,\underline{t}^*)-W(\underline{\lambda},0,t_0)}{I\widetilde{f}(t_0)}$, a pooling equilibrium exists if and only if A is small enough.

Overall, a pooling equilibrium exists if and only if $A \leq A_2$, with

$$A_2 = \max\left(0, I - \frac{W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 0, t_0)}{q_0\widetilde{f}(t_0)}\right).$$

Conditions for a semi-pooling equilibrium The following conditions must be satisfied for a semi-pooling equilibrium to exist:

$$W(\underline{\lambda}, q(t_0), t_0) = W(\underline{\lambda}, 0, \underline{t}^*) \tag{19}$$

$$W(\overline{\lambda}, q(t_0), t_0) > W(\overline{\lambda}, 0, t) \text{ for all } t. \tag{20}$$

Using the same argument as above, it is easy to see that (19) $\Rightarrow$ (20).

Finally, it is obvious that $q(t_0) > q_0$, because $\underline{\lambda}$ does not invest at t_0 with probability 1, whereas $\overline{\lambda}$ does. So, if $q_0 \geq \overline{q} \Leftrightarrow A \leq A_2$, (19) cannot hold. Conversely, if $q_0 < \overline{q}$, type $\underline{\lambda}$ can always invest at date t_0 with a probability $x \in (0,1)$ such that $q(t_0) = \frac{q_0}{q_0+(1-q_0)x} = \overline{q}$, in which case (19) is satisfied. Hence the result. □

8.6 Proof of Proposition 4

- If $A \geq A_0$, $\overline{t} = \overline{t}^*$, which is independent of A,
- If $A < A_1$, $\overline{t} = t_0$, which is independent of A,
- If $A \in [A_1, A_0)$, in the least cost separating equilibrium, the good type invests at date $\overline{t}^i$ given by:

$$W(\underline{\lambda}, 1, \overline{t}^i) = W(\overline{\lambda}, 1, \overline{t}^i) - A\widetilde{f}(\overline{t}^i) = W(\underline{\lambda}, 0, \underline{t}^*). \tag{21}$$

Differentiating with respect to A yields

$$\frac{\partial \overline{t}^i}{\partial A} = \frac{\widetilde{f}(\overline{t}^i)}{W_3\left(\overline{\lambda}, 1, \overline{t}^i\right) - A\frac{\partial \widetilde{f}}{\partial t}(\overline{t}^i)}.$$

The numerator is positive. The denominator is negative for $i = d$, as $\overline{t}^d \in \left[\overline{t}^*, t_0\right]$, and positive for $i = h$, since $\overline{t}^h \in \left[t_0, \overline{t}^*\right]$. This implies $\frac{\partial \overline{t}^h}{\partial A} > 0$, and $\frac{\partial \overline{t}^d}{\partial A} < 0$, so $\frac{\partial |\overline{t}^* - \overline{t}|}{\partial A} < 0$. □

8.7 Proof of Proposition 5

$p^*(\overline{\lambda}, \overline{t}) - p^*(\underline{\lambda}, \underline{t}^*)$ has the same sign as $e^{-\underline{\lambda} t^*} - e^{\overline{\lambda} \overline{t}}$.

Notice first that, under complete information, this difference reads

$$e^{-\underline{\lambda t}^*} - e^{-\overline{\lambda t}^*} = \frac{rp_0(R-I)}{(1-p_0)I}\left(\frac{1}{\underline{\lambda}+r} - \frac{1}{\overline{\lambda}+r}\right) > 0,$$

so the good type is always more optimistic in this case.

If investment is delayed, the good type is more optimistic than in the complete information case upon investing, so he is a fortiori more confident than the bad type. Therefore, we focus on the case where investment is hurried. This implies that $p_0R - I > 0$, and that

$\overline{\lambda} > \lambda^{**}$, and $\underline{\lambda} > \underline{\lambda}_0$. In this case, $\bar{t}$ is increasing in A (see Proposition 4), which implies that $e^{-\underline{\lambda}\underline{t}^*} - e^{-\overline{\lambda}\bar{t}}$ is increasing in A as well. For $A \geq A_0$, investment is efficient, and there is no reversal. There are two situations then:

- If there is no belief reversal at $A = 0$, then there is never belief reversal.
- If there is belief reversal at $A = 0$, then there exists $\underline{A}$ such that belief reversal occurs if and only if $A < \underline{A}$.

We complete the proof by showing the following lemma.

Lemma 4 *Suppose that* $p_0 R - I > 0$, $\overline{\lambda} > \lambda^{**}$, $\underline{\lambda} > \underline{\lambda}_0$, *and* $A = 0$.
Then there exists $\lambda_1 \in (\lambda_0, \overline{\lambda})$ *such that*

$$p(\overline{\lambda}, \bar{t}) - p(\underline{\lambda}, \underline{t}^*) < 0 \Leftrightarrow \underline{\lambda} > \lambda_1$$

Proof When $A = 0$, the equilibrium is separating if and only if

$$W(\overline{\lambda}, 1, t_0) < W(\underline{\lambda}, 0, \underline{t}^*)$$

and pooling otherwise (or semi-pooling). The function $W(\underline{\lambda}, 0, \underline{t}^*) - W(\overline{\lambda}, 1, t_0)$ is increasing in $\underline{\lambda}$, because $t_0 < \bar{t}^*$ and t_0 is decreasing in $\underline{\lambda}$. When $\underline{\lambda} = \underline{\lambda}_0$, we have by definition $\underline{t}^* = t_0$, so $W(\underline{\lambda}, 0, \underline{t}^*) < W(\overline{\lambda}, 1, t_0)$. When $\underline{\lambda} \to \overline{\lambda}$, we have $W(\overline{\lambda}, 1, t_0) < \lim_{\underline{\lambda} \to \overline{\lambda}} W(\underline{\lambda}, 0, \underline{t}^*) = W(\overline{\lambda}, 1, \bar{t}^*)$. Therefore, when $A = 0$, the equilibrium is separating if and only if $\underline{\lambda} > \lambda_a$, where $\underline{\lambda}_0 < \lambda_a < \overline{\lambda}$.

If the equilibrium is separating, we have

$$W(\overline{\lambda}, 1, \bar{t}) = W(\underline{\lambda}, 0, \underline{t}^*),$$

$$\Leftrightarrow e^{-r\bar{t}} \left(R - \frac{I}{p(\overline{\lambda}, \bar{t})} \right) = e^{-r\underline{t}^*} \left(R - \frac{I}{p(\underline{\lambda}, \underline{t}^*)} \right).$$

It is easy to see that this implies

$$\frac{I}{p(\underline{\lambda}, \underline{t}^*)} - \frac{I}{p(\overline{\lambda}, \bar{t})} = \left(R - \frac{I}{p(\overline{\lambda}, \bar{t})} \right) \left(1 - e^{r(\underline{t}^* - \bar{t})} \right) < 0,$$

where the last inequality derives from $p_0 R - I > 0$ and $\bar{t} < \bar{t}^* < \underline{t}^*$.

Therefore, if the equilibrium is separating at $A = 0$, then it must involve a reversal.

If the equilibrium is pooling at $A = 0$, $\bar{t} = t_0$, which is decreasing in $\underline{\lambda}$. So the function $p(\overline{\lambda}, \bar{t}) - p(\underline{\lambda}, \underline{t}^*)$ is decreasing in $\underline{\lambda}$. This function is positive at $\underline{\lambda} = \underline{\lambda}_0$, because $\underline{t}^* = t_0$ at this point. At $\underline{\lambda} = \lambda_a$, it must be negative by continuity, using the proof above. Therefore, there exists a λ_1 such that the equilibrium involves belief reversal at $A = 0$ if and only if $\underline{\lambda} > \lambda_1$. □

8.8 Proof of Proposition 6

Before deriving how the equilibrium varies with p_0, let us examine how $\bar{t}^*, \underline{t}^*$ and t_0 compare when p_0 varies. Recall first that

$$t^*(\lambda) = \max\left(-\frac{1}{\lambda}\ln\frac{p_0 r\,(R-I)}{(1-p_0)\,(\lambda+r)\,I}, 0\right).$$

Therefore, $t^*(\lambda) > 0 \Leftrightarrow p_0 < \frac{(\lambda+r)I}{rR+\lambda I}$.

Let us consider how $\underline{t}^* - \bar{t}^*$ varies with p_0.

For $p_0 \geq \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$, $\underline{t}^* = 0$, so $\underline{t}^* - \bar{t}^* \leq 0$. For $p_0 < \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$, we have

$$\underline{t}^* - \bar{t}^* = (\frac{1}{\overline{\lambda}} - \frac{1}{\underline{\lambda}})\ln\frac{p_0}{1-p_0} + \frac{1}{\overline{\lambda}}\ln\frac{r(R-I)}{(\overline{\lambda}+r)I} - \frac{1}{\underline{\lambda}}\ln\frac{r(R-I)}{(\underline{\lambda}+r)I}.$$

This function is decreasing in p_0, tends to $+\infty$ when $p_0 \to 0$, and is negative at $p_0 = \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$, so there exists a $\hat{p} < \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I}$ such that $\bar{t}^* \leq \underline{t}^* \Leftrightarrow p_0 \leq \hat{p}$. In addition, $\bar{t}^* \geq t_0 \Leftrightarrow p_0 \leq \hat{p}$.

Let us define the following function:

$$g(p_0) = W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 1, \bar{t}^*).$$

We know that $(\underline{t}^*, \bar{t}^*)$ is an equilibrium whenever p_0 is such that $g(p_0) \geq 0$. Instead of working with p_0, it will prove easier to work with

$$x(p_0) \equiv \frac{p_0 r(R-I)}{(1-p_0)(\overline{\lambda}+r)I}. \tag{22}$$

Let $\tilde{g}$ be such that $g = \tilde{g} \circ x$. There are three different cases to be considered:

1. If $p_0 \leq \frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I} \Leftrightarrow x < \frac{\underline{\lambda}+r}{\overline{\lambda}+r}$, then $\underline{t}^* = -\frac{1}{\underline{\lambda}}\ln(\frac{\overline{\lambda}+r}{\underline{\lambda}+r}x) > 0$, and $\bar{t}^* = -\frac{1}{\overline{\lambda}}\ln x > 0$. In this

case, $\tilde{g}(x) = \frac{R-I}{r(R-I)+(\overline{\lambda}+r)xI}\phi(x)$, with

$$\phi(x) = \left(\frac{\overline{\lambda}+r}{\underline{\lambda}+r}\right)^{\frac{r}{\underline{\lambda}}+1} \underline{\lambda} I x^{\frac{r}{\underline{\lambda}}+1} - \left(\overline{\lambda} I + rA\right) x^{\frac{r}{\overline{\overline{\lambda}}}+1} + Arx^{\frac{r+\underline{\underline{\lambda}}}{\overline{\overline{\lambda}}}}. \tag{23}$$

2. If $\frac{(\underline{\lambda}+r)I}{rR+\underline{\lambda}I} < p_0 < \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I} \Leftrightarrow \frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r} < x < 1$, then $\underline{t}^* = 0$ and $\overline{t}^* = -\frac{1}{\overline{\lambda}}\ln x > 0$. In this case, we have $\tilde{g}(x) = \frac{R-I}{r(R-I)+(\overline{\lambda}+r)xI}\psi(x)$, with

$$\psi(x) = \left(\overline{\lambda}+r\right) xI - rI - \left(\overline{\lambda} I + rA\right) x^{\frac{r}{\overline{\overline{\lambda}}}+1} + Arx^{\frac{r+\underline{\underline{\lambda}}}{\overline{\overline{\lambda}}}}. \tag{24}$$

3. If $p_0 \geq \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I} \Leftrightarrow x \geq 1$, then $\overline{t}^* = \underline{t}^* = 0$. In this case, we have $\tilde{g}(x) = 0$.

Let $\hat{x} \equiv x(\hat{p}) = (\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r})^{\frac{\overline{\overline{\lambda}}}{\overline{\overline{\lambda}}-\underline{\underline{\lambda}}}} < \frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}$. It is easy to see that $g(\hat{p}) < 0$, which is equivalent to $\tilde{g}(\hat{x}) < 0$.

Let us start with the first case, and consider the function ϕ defined in (23).
On can rewrite $\phi(x) = \alpha x^a + \beta x^b + \gamma x^c$, with $a = \frac{r}{\underline{\lambda}} + 1 > b = \frac{r}{\overline{\overline{\lambda}}} + 1 > c = \frac{r+\underline{\underline{\lambda}}}{\overline{\overline{\lambda}}}$, and $\alpha > 0, \beta < 0, \gamma > 0$. It is easy to see that $x \mapsto x^{-c}\phi(x)$ is decreasing and then increasing as x varies from 0 to $+\infty$. This function is strictly positive when $x = 0$ and tends to $+\infty$ when x tends to $+\infty$.

In the second case, let us consider the function ψ defined in (24). Simple algebra allows us to show that ψ is increasing and then decreasing on $\mathbb{R}$. When $x \to 1$, $\tilde{g}$ tends to zero by continuity and $\psi'(1) < 0$.

Since we know that $\phi(\hat{x}) < 0$, there are only two possible cases:

- $\phi(\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}) = \psi(\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}) < 0$: in this case, there exists a unique $x_0 \in [0, \frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}]$ such that $\phi(x_0) = 0$ and a unique $x_1 \in [\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}, 1]$ such that $\psi(x_1) = 0$.

- $\phi(\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}) = \psi(\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}) > 0$: there exists a unique (x_0, x_1) with $0 < x_0 < x_1 < \frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}$ such that $\phi(x_0) = \phi(x_1) = 0$. In addition, we have $\psi(x) > 0$ for all $x \in (\frac{\underline{\underline{\lambda}}+r}{\overline{\overline{\lambda}}+r}, 1)$.

In any case, we have proven that $\tilde{g}(x) \geq 0$ if and only if $x \leq x_0$ or $x \geq x_1$. Consequently, there exist $(\underline{\underline{p}}, \overline{\overline{p}}) \in [0, \frac{(\overline{\lambda}+r)I}{rR+\overline{\lambda}I}]^2$ with $\underline{\underline{p}} < \hat{p} < \overline{\overline{p}}$ such that $(\underline{t}^*, \overline{t}^*)$ is an equilibrium if and only if $p_0 \notin (\underline{\underline{p}}, \overline{\overline{p}})$.

Let us now find conditions under which a separating equilibrium exists. From Lemma 2, we know that a separating equilibrium exists if and only if $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 0, \underline{t}^*) \Leftrightarrow$

$h(p_0) \geq 0$, where $h(p_0) \equiv W(\underline{\lambda}, 0, \underline{t}^*) - W(\underline{\lambda}, 1, t_0)$. It is straightforward to show that h is convex in p_0. Because $W(\underline{\lambda}, 1, t_0) \leq W(\underline{\lambda}, 1, \overline{t}^*)$, we have $h(p_0) \geq g(p_0)$ for all p_0 (with equality only at $p_0 = \hat{p}$). Therefore, $h(\underline{\underline{p}}) > g(\underline{\underline{p}}) = 0$ and $h(\overline{\overline{p}}) > g(\overline{\overline{p}}) = 0$ Finally, $h(\hat{p}) = g(\hat{p}) < 0$, so there exists a unique $(\underline{p}, \overline{p})$ such that $h(\underline{p}) = h(\overline{p}) = 0$. In addition, one has $\underline{\underline{p}} < \underline{p} < \hat{p} < \underline{p} < \overline{\overline{p}}$.

Therefore the equilibrium investment date $\overline{t}$ is

- $\overline{t}^*$ if $p_0 \leq \underline{\underline{p}}$ (optimal investment date),
- $\overline{t}^h < \overline{t}^*$ if $\underline{\underline{p}} < p_0 < \underline{p}$ (separating equilibrium, hurried investment),
- $t_0 \leq \overline{t}^*$ if $\underline{p} \leq p_0 \leq \hat{p}$ (pooling equilibrium, hurried investment),
- $t_0 > \overline{t}^*$ if $\hat{p} < p_0 < \overline{p}$ (pooling equilibrium, delayed investment),
- $\overline{t}^d > \overline{t}^*$ if $\overline{p} < p < \overline{\overline{p}}$ (separating equilibrium, delayed investment),
- $\overline{t}^*$ if $p_0 \geq \overline{\overline{p}}$ (optimal investment date). □

The proof to derive $\overline{t}$ as a function of R is exactly similar.

8.9 The general case with $r_0 \leq r$

The analysis can be extended to the case where the entrepreneur's initial assets A capitalize or depreciate at a rate $r_0 \leq r$. The first point to note is that if $r_0 > 0$, there exists a date $\tilde{t}$ after which the entrepreneur no longer needs outside financing, defined by $Ae^{r_0\tilde{t}} = I$. If $\overline{t}^* > \tilde{t}$, no distortion arises in equilibrium, so the only interesting case to consider is when $\overline{t}^* < \tilde{t}$.

(5) becomes

$$x(q,t)p\,(q,t)\,R = I - Ae^{r_0 t}.$$

The expected discounted payoff at date 0 of type $\lambda \in \{\underline{\lambda}, \overline{\lambda}\}$ when he invests at date t, and is perceived as type $\overline{\lambda}$ with probability q now depends on whether the investment date is larger than $\tilde{t}$ or not. If $t \geq \tilde{t}$, the entrepreneur does not need outside funds, so asymmetric information has no bite, and

$$W\,(\lambda, q, t) = W^*\,(\lambda, t) \;\text{ for all } q.$$

If $t < \widetilde{t}$,

$$W(\lambda, q, t) = e^{-rt} s(\lambda, t) \left(p^*(\lambda, t)(1 - x(q,t)) R - Ae^{r_0 t}\right).$$

$$\Rightarrow W\left(\overline{\lambda}, q, t\right) - W(\underline{\lambda}, q, t) = Ae^{-(r-r_0)t} f(t).$$

The function $t \mapsto e^{-(r-r_0)t} f(t)$ is single-peaked and reaches its maximum at

$$t_0 \equiv \frac{\ln(\overline{\lambda} + r - r_0) - \ln(\underline{\lambda} + r - r_0)}{\overline{\lambda} - \underline{\lambda}} > 0.$$

The results of Lemma 1, and of Propositions 2, 3, 4 naturally carry over when appropriately adapting $\tilde{f}$ to:

$$\widetilde{f}(t) = e^{-(r-r_0)t} f(t).$$

We notably derive:

$$\begin{aligned} A_0 &\equiv \frac{W^*\left(\overline{\lambda}, \overline{t}^*\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{e^{-(r-r_0)\overline{t}^*} f(\overline{t}^*)}, \\ A_1 &\equiv \max\left(0, \frac{W^*\left(\overline{\lambda}, t_0\right) - W^*\left(\underline{\lambda}, \underline{t}^*\right)}{e^{-(r-r_0)t_0} f(t_0)}\right), \\ A_2 &\equiv \max\left(0, Ie^{-r_0 t_0} - \frac{W^*(\underline{\lambda}, \underline{t}^*) - W^*(\underline{\lambda}, t_0)}{q_0 e^{-(r-r_0)t_0} f(t_0)}\right). \end{aligned}$$

Accordingly, the equilibrium is unique as well.

The main difference with the case where $r_0 = 0$ is that $\overline{t}^* \geq t_0$ is no longer equivalent to $\overline{t}^* \geq \underline{t}^*$. Therefore, the direction of the distortion is not the one predicted by the ranking of the complete information dates. However, one can show that, for any r_0, there exists a $\hat{p}(r_0)$ such that $\overline{t}^* > t_0$ if and only if $p_0 < \hat{p}(r_0)$. Therefore, our result that projects with a high expected value are delayed, and those with a low expected value are hurried carries over.

In addition, it is easy to see that $\overline{t}^* = \underline{t}^* \Rightarrow \overline{t}^* < t_0$ when $r_0 > 0$ and $\overline{t}^* = \underline{t}^* \Rightarrow \overline{t}^* > t_0$ when $r_0 < 0$. This notably implies that asymmetric information can cause timing reversals. For instance, when $r_0 > 0$, this happens when $\overline{t}^* < \underline{t}^* < \overline{t}$, that is, the good type invests later than the bad type, although he optimally invests earlier under complete information.[30]

[30] A formal proof that this may actually happen is available upon request.