

Liquidity and Trading Dynamics*

Veronica Guerrieri
University of Chicago

Guido Lorenzoni
MIT and NBER

June 12, 2007

Abstract

How do financial frictions affect the response of an economy to aggregate shocks? In this paper, we address this question, focusing on liquidity constraints and uninsurable idiosyncratic risk. We consider a search model where agents use liquid assets to smooth individual income shocks. We show that the response of this economy to aggregate shocks depends on the rate of return on liquid assets. In economies where liquid assets pay a low return, agents hold smaller liquid reserves and the response of the economy tends to be larger. In this case, agents expect to be liquidity constrained and, due to a self-insurance motive, their consumption decisions are more sensitive to changes in expected income. On the other hand, in economies where liquid assets pay a large return, agents hold larger reserves and their consumption decisions are more insulated from income uncertainty. Therefore, aggregate shocks tend to have larger effects if liquid assets pay a lower rate of return.

Keywords: Liquidity, Money, Search, Aggregate Volatility, Amplification.

JEL codes: D83, E41, E44.

*Email addresses: veronica.guerrieri@chicagogsb.edu; glorenzo@mit.edu. We thank for helpful comments Daron Acemoglu, Fernando Alvarez, Marios Angeletos, Boragan Aruoba, Gabriele Camera, Ricardo Cavalcanti, Chris Edmond, Ricardo Lagos, Robert Lucas, David Marshall, Robert Shimer, Nancy Stokey, Christopher Waller, Ivàn Werning, Randall Wright, and seminar participants at MIT, Cleveland Fed (Summer Workshop on Money, Banking, and Payments), University of Maryland, University of Notre Dame, Bank of Italy, AEA Meetings (Chicago), Chicago Fed, Stanford University, the Philadelphia Fed Monetary Macro Workshop, UCLA, San Francisco Fed, St. Louis Fed, and University of Chicago GSB.

1 Introduction

In times of economic distress, the demand for liquid assets typically increases. Facing the prospect of temporary shortfalls in revenues, agents tend to hold on to their reserves of cash, government bonds, and other safe assets as a form of self-insurance. An observation consistent with this type of behavior is that the precautionary motive in consumption decisions is countercyclical.¹ A further symptom of a countercyclical demand for liquidity can be identified in the path of several measures of liquidity premia.² In extreme episodes, such as financial market crises, this type of behavior takes the form of an outright “flight to liquidity.” What are the aggregate implications of a countercyclical demand for liquidity? Can it amplify the response of economic activity to exogenous shocks? In this paper, we explore these questions in a general equilibrium model with a single liquid asset and decentralized production and exchange. We find that the answer depends on the total supply of liquid assets. In economies where liquid assets are relatively abundant, a negative aggregate shock leads to a mechanical reduction in activity, but there is no additional effect due to the agents’ self-insurance motive. In economies where, instead, liquid assets are relatively scarce, the aggregate shock has a magnified effect on the economy, as agents reduce their consumption in an attempt to protect their reserves.

Our amplification mechanism is driven by a form of complementarity in trading decisions. An agent is less willing to spend his liquid assets when he expects other agents to spend less. This happens because, in that case, it is harder for him to rebuild his reserves by selling goods to other agents. The idea that “the difficulty of coordination of trade” may contribute to aggregate volatility goes back, at least, to Diamond (1982, 1984). The contribution of our paper is to show that the presence of this coordination problem depends in a crucial way on the supply of liquid assets. In particular, when they are relatively abundant, the coordination element vanishes.

We consider a model of decentralized trade in the tradition of search models of money. There is a large number of households, each with one consumer and one producer. Consumers and producers from different households meet and trade in spatially separated islands. In each island the gains from trade are determined by a local productivity shock. Agents make their trading decisions (consumption and production) without observing the shock realized in the island visited by their partners. Our focus is on whether trading decisions in a given island

¹See Parker and Preston (2005).

²See Krishnamurthy (2002) and Longstaff (2004) and papers reviewed in Cochrane (2005).

are affected by agents' expectations regarding the level of trading in other islands. This is the sense in which there is a "coordination problem" in our model.

Agents are anonymous and, thus, credit arrangements are not feasible. The government issues a fixed supply of interest-bearing notes, "money," which are used for transactions. These notes pay a constant interest rate which is financed by lump-sum taxation. As we will see, the equilibrium value of real money balances is an increasing function of the rate of return chosen by the government. Therefore, a regime with a high rate of return, is labeled a regime of "abundant liquidity," while one with a low rate of return is one of "scarce liquidity." To derive our main analytical results, we focus on two extreme cases. In the first one, the rate of return is equal to the inverse of the agents' discount factor. This is a "Friedman rule" regime and, in this case, real balances are large and the economy achieves the first-best allocation. In the second one, the rate of return and the value of real balances are so low that agents expect to be liquidity constrained for any realization of the idiosyncratic shocks. We refer to this case as a "fully constrained" regime.

Our first result is that under the Friedman rule the quantity traded in each island is independent of what happens in other islands. This result follows from an assumption of separability in preferences and from the fact that, in the Friedman rule regime, agents are fully insured against idiosyncratic shocks. This makes the marginal value of money constant, allowing the consumer and the producer from the same household to make their trading decisions independently. Away from the Friedman rule, instead, the decisions of the two agents are linked. The consumer needs to forecast the earnings of the producer to evaluate the household's marginal value of money, while the producer needs to forecast the consumer's outlays.

Next, we look at the aggregate implications of these "linkages." We consider an aggregate shock which shifts the distribution of island-specific productivities, reducing the probability of low realizations and increasing that of high realizations (a first-order stochastic increase). The aggregate shock is publicly observed. Under both regimes, there is a positive compositional effect: as more islands have high productivity, aggregate output increases. However, in the Friedman rule regime, there is no feedback from this aggregate increase in output to the level of trading in an island with a *given local shock*. In the fully constrained regime, instead, the linkage between the trading decisions in different islands generates an additional effect on trading and output. A good aggregate shock increases the probability of high earnings for the producer. This induces the consumer to increase spending and reduce the household's liquid

reserves. At the same time, the producer expects his partner to spend more, and this increases his incentive to produce. These two effects imply that a higher level of aggregate activity induces higher levels of activity in each individual island. This increase is further magnified once we take into account general equilibrium effects. Using a simple calibrated version of our economy, we show that this leads to a sizeable degree of amplification of aggregate shocks.

The crucial difference between the two financial regimes is the role of expectations in the trading decisions of individual agents. To further clarify this difference, we also consider the case where the aggregate distribution of island-specific shocks is not perfectly known at the beginning of each period, and agents observe a public signal regarding this distribution. In the Friedman rule regime, the public signal has no effect on aggregate activity, given that trading decisions are solely based on local shocks, which are observable. In the fully constrained regime, on the other hand, a favorable public signal tends to increase aggregate activity due to the effect of the signal on agents' expectations.

This paper is related to the vast literature on search models of money, going back to Diamond (1984) and Kiyotaki and Wright (1989). In particular, our model allows for divisible money and we use the Lagos and Wright (2005) approach to simplify the analysis. In Lagos and Wright (2005) agents alternate trading in a decentralized market to trading in a centralized competitive market. The combination of quasi-linear preferences and a periodic access to a centralized market ensures that the distribution of money holdings across agents is degenerate when they enter the decentralized market. Here we use these same two ingredients, with a modified periodic structure. In our model, agents have access to a centralized market every three periods. The extra period of decentralized trading is necessary to make the self-insurance motive matter for trading decisions in the decentralized market of the previous period. This is at the core of our amplification mechanism. A three-period structure is also used by Berentsen, Camera and Waller (2005) to study the short-run neutrality of money. They show that, away from the Friedman rule, random monetary injections can be non-neutral, since they have a differential effect on agents with heterogeneous money holdings. Although very different in the objective, their analysis also relies on the lack of consumption insurance. Finally, our model is related to Rocheteau and Wright (2005) for the use of competitive pricing *à la* Lucas and Prescott (1974) in a search model of money.

The paper is also related to the large literature exploring the relation between financial frictions and aggregate volatility, including Bernanke and Gertler (1989), Bencivenga and Smith

(1991), Acemoglu and Zilibotti (1997), Kiyotaki and Moore (1997). In particular, Kiyotaki and Moore (2001) also address this issue from the point of view of limited liquidity supply. Their paper emphasizes a different channel by which limited liquidity can affect the transmission of aggregate shocks, focusing on the effects on investment and capital accumulation.

The rest of the paper is organized as follows. In Section 2, we introduce our environment and solve for the first-best allocation of resources. In Section 3, we define and characterize the competitive equilibrium. Section 4 addresses the main question of the paper, that is, how the economy reacts to an aggregate shock. Section 5 present some numerical exercises on the amplification result and on the behavior of liquidity premia. Section 6 discusses the extension with imperfect information and public signals and versions of the model with alternative sources of liquidity supply. Section 7 concludes. The appendix contains all the proofs not in the text.

2 The Model

The economy is populated by a continuum of infinitely-lived households, composed of two agents, a consumer and a producer. Each household has an initial endowment M of perfectly divisible notes issued by the government, “money.” Time is discrete and each period agents produce and consume a single, perishable consumption good. The economy has a simple periodic structure: each time period t is divided into three sub-periods, $s = 1, 2, 3$. We will call them “periods,” whenever there is no risk of confusion.

In periods 1 and 2, the consumer and the producer from each household travel to spatially separated markets, or islands, where they interact with consumers and producers from other households. Each island has a competitive market, as in Lucas and Prescott (1974). Trading is characterized by anonymity, therefore the only type of trades that are feasible are spot exchanges of goods for money. There is a continuum of islands and each island receives the same mass of consumers and producers in both periods 1 and 2. The assignment of agents to islands is random and satisfies a law of large numbers, so that each island receives a representative sample of consumers and producers. The consumer and the producer from the same household do not communicate during periods 1 and 2. However, at the end of each period, they meet and share money holdings and information. In period 3, all consumers and producers trade in a single centralized market.

In period 1 of time t , the producer from household j , located in island k , has access to the

linear technology

$$y_{1,t}^j = \theta_t^k n_t^j$$

where y_t^j is output, n_t^j is labor effort, and θ_t^k is the island-specific productivity. The productivity θ_t^k is randomly drawn from a distribution with cumulative distribution function $F(\cdot)$ and support $\Theta = [0, \bar{\theta}]$. It is realized after producers and consumers have reached island k and is observed only by the agents in the island. A law of large numbers applies, so $F(\cdot)$ also represents the distribution of productivity shocks across islands. It will be useful to assume that $F(\cdot)$ has an atom at 0, i.e., $F(0) > 0$, and is continuous on $(0, \bar{\theta}]$. For the moment, we look at an economy with no aggregate shocks. In Section 4, we will introduce aggregate shocks by allowing for shifts in the distribution $F(\cdot)$.

In periods 2 and 3, the producer of household j has a fixed endowment of consumption goods, $y_{2,t}^j = e_2$ and $y_{3,t}^j = e_3$. We will assume that the value of e_3 is large, so as to ensure that equilibrium consumption in period 3 is strictly positive for all households.

The household's preferences are represented by the utility function

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t \left(u(c_{1,t}^j) - v(n_t^j) + U(c_{2,t}^j) + c_{3,t}^j \right) \right],$$

where $c_{s,t}^j$ is consumption in period (s, t) and $\beta \in (0, 1)$ is the discount factor. The functions u and U are increasing and strictly concave, and the function v , representing the disutility of effort, is increasing and convex. All of them have continuous first and second derivatives. A number of other assumptions will be useful in the analysis. Both u and U satisfy standard Inada conditions and v satisfies the condition $\lim_{n \rightarrow \bar{n}} v'(n) = \infty$. The function u is bounded below, with $u(0) = 0$, and there is a $\underline{\sigma} > 0$ such that $-u''(c) c / u'(c) \in [\underline{\sigma}, 1)$ for all $c \geq 0$. Finally, the function U satisfies $-U''(c) c / U'(c) \leq 1$ for all $c \geq 0$. We will discuss the role of these assumptions when they are needed in the analysis.

The fact that in period 3 consumers and producers trade in a centralized market and have linear utility is essential for tractability. This allows us to derive an equilibrium with a degenerate distribution of money balances at the beginning of period $(1, t)$, as in Lagos and Wright (2005).³

Finally, we need to specify the monetary regime. At the end of each period 3, the government levies a lump-sum tax T and pays a (gross) rate of return R on the net money balances

³See Shi (1997) for a different approach to obtain a degenerate distribution of money holdings.

held by each household. In order to focus on equilibria with stationary nominal prices, we focus on regimes with constant money supply M . We characterize a monetary regime using the two parameters R and M , and set T so as to satisfy the government budget constraint

$$M = R(M - T).$$

In this paper, we make no attempt to explain the government's choice of the monetary regime, but we simply explore the effect of different regimes on equilibrium behavior.

Notice that we allow for $R \leq 1$.⁴ The assumption of interest-paying money balances is a general way of introducing a government-supplied liquid asset. In the case $R > 1$ the asset resembles a nominal government bond, while in the case $R < 1$ it looks more like money subject to a positive inflation tax. In Section 6.2, we discuss a number of alternative ways of interpreting the liquid asset in our model.

2.1 First-best allocation

The first-best allocation provides a useful benchmark for the rest of the analysis. Consider a social planner who can choose the labor effort and the consumption of the households. Given that there is no capital, there is no real intertemporal link between periods. Therefore, we can look at a static planner problem which only includes periods $s = 1, 2, 3$.

Each household is characterized by a pair $(\theta, \tilde{\theta})$, where the first element represents the shock in the producer's island and the second represents the one in the consumer's island. An allocation is given by consumption functions $\{c_s(\theta, \tilde{\theta})\}_{s \in \{1, 2, 3\}}$ and an effort function $n(\theta, \tilde{\theta})$. The planner chooses an allocation that maximizes the ex-ante utility of the representative household

$$\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} \left(u(c_1(\theta, \tilde{\theta})) - v(n(\theta, \tilde{\theta})) + U(c_2(\theta, \tilde{\theta})) + c_3(\theta, \tilde{\theta}) \right) dF(\theta) dF(\tilde{\theta}),$$

subject to the economy's resource constraints. In period 1, there is one resource constraint for each island θ ⁵

$$\int_0^{\bar{\theta}} c_1(\tilde{\theta}, \theta) dF(\tilde{\theta}) \leq \int_0^{\bar{\theta}} y_1(\theta, \tilde{\theta}) dF(\tilde{\theta}),$$

where $y_1(\theta, \tilde{\theta}) = \theta n(\theta, \tilde{\theta})$. In period $s = 2, 3$, the resource constraint is

$$\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} c_s(\theta, \tilde{\theta}) dF(\theta) dF(\tilde{\theta}) \leq e_s.$$

⁴When $R < 1$, T is negative and corresponds to a lump-sum subsidy.

⁵From now on, "island θ " is short for "an island with productivity shock θ ."

The resource constraints for periods 1 and 2 reflect the assumption that each island receives a representative sample of consumers and producers.

The following proposition characterizes the optimal allocation.

Proposition 1 *The optimal allocation in period 1 is given by $n(\theta, \tilde{\theta}) = n^{FB}(\theta)$ and $c_1(\tilde{\theta}, \theta) = y_1(\theta, \tilde{\theta}) = y_1^{FB}(\theta)$ for all $(\theta, \tilde{\theta}) \in \Theta^2$, where $y_1^{FB}(\theta)$ and $n^{FB}(\theta)$ satisfy*

$$\theta u'(y_1^{FB}(\theta)) = v'(n^{FB}(\theta)), \quad (1)$$

and $y_1^{FB}(\theta) = \theta n^{FB}(\theta)$, for all $\theta \in \Theta$. Optimal consumption in period 2 is $c_2(\theta, \tilde{\theta}) = e_2$ for all $(\theta, \tilde{\theta}) \in \Theta^2$.

Due to the separability of the utility function, the optimal labor effort of a producer located in island θ depends only on the productivity θ and is not affected by the shock of any other island. Furthermore, it is easy to show that output is greater in islands with larger θ . These two results are summarized in the following lemma.

Lemma 1 *The first-best level of output in island θ , $y_1^{FB}(\theta)$, is independent of the economy-wide distribution of productivity shocks $F(\cdot)$. The function $y_1^{FB}(\theta)$ is increasing in θ .*

Moreover, notice that, at the optimum, c_2 is constant across households, that is, households are fully insured against the shocks θ and $\tilde{\theta}$. Finally, given linearity, the consumption levels in period 3 are not pinned down, as consumers are indifferent among any profile $c_3(\theta, \tilde{\theta})$, with $\mathbb{E}[c_3(\theta, \tilde{\theta})] = e_3$.

3 Equilibrium

We turn now to the definition and characterization of the competitive equilibrium. We concentrate on steady-state equilibria where prices, the allocation and the distribution of money are independent of time. Hence, from now on we drop the time index t .

We begin by characterizing optimal individual behavior for given prices. Money is the numeraire, $p_1(\theta)$ denotes the price of goods in period 1 in island θ , and p_2 and p_3 denote the prices in periods 2 and 3. Consider a household with an initial stock of money m at the beginning of period 1. The consumer travels to island $\tilde{\theta}$ and consumes $c_1(\tilde{\theta})$. Since money

holdings are non-negative, his budget constraint and liquidity constraint in period 1 are

$$\begin{aligned} m_1(\tilde{\theta}) + p_1(\tilde{\theta})c_1(\tilde{\theta}) &\leq m, \\ m_1(\tilde{\theta}) &\geq 0, \end{aligned}$$

where $m_1(\tilde{\theta})$ denotes the consumer's money holdings at the end of period 1. In the meantime, the producer, located in island θ , produces and sells $y_1(\theta) = \theta n(\theta)$. At the end of period 1, the consumer and the producer get back together, and pool their money holdings. Therefore, in period 2 the consumer's constraints are

$$\begin{aligned} m_2(\theta, \tilde{\theta}) + p_2 c_2(\theta, \tilde{\theta}) &\leq m_1(\tilde{\theta}) + p_1(\theta) \theta n(\theta), \\ m_2(\theta, \tilde{\theta}) &\geq 0, \end{aligned}$$

where consumption, $c_2(\theta, \tilde{\theta})$, and end-of-period money holdings, $m_2(\theta, \tilde{\theta})$, are now contingent on both shocks θ and $\tilde{\theta}$. Finally, in period 3, the consumer and the producer are located in the same island and the revenues $p_3 e_3$ are immediately available. Moreover, the household has to pay the nominal tax T . The constraints are now

$$\begin{aligned} m_3(\theta, \tilde{\theta}) + p_3 c_3(\theta, \tilde{\theta}) &\leq m_2(\theta, \tilde{\theta}) + p_2 e_2 + p_3 e_3 - T, \\ m_3(\theta, \tilde{\theta}) &\geq 0. \end{aligned}$$

Let $V(m)$ denote the expected utility of a household with nominal balances m at the beginning of period 1, facing prices $\{p_1(\theta)\}_{\theta \in \Theta}, p_2, p_3$ in all periods. The household's problem is then characterized by the Bellman equation

$$\begin{aligned} V(m) = \max_{\{c_s\}, \{m_s\}, n} &\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} [u(c_1(\tilde{\theta})) - v(n(\theta)) + U(c_2(\theta, \tilde{\theta})) + \\ &+ c_3(\theta, \tilde{\theta}) + \beta V(Rm_3(\theta, \tilde{\theta}))] dF(\theta) dF(\tilde{\theta}), \end{aligned} \quad (2)$$

subject to the budget constraints and liquidity constraints introduced above. The solution to this problem gives us optimal quantities as a function of the initial money balances m , which we denote by $c_1(\theta, m), c_2(\theta, \tilde{\theta}, m),$ etc.

We are now in a position to define a steady-state competitive equilibrium, where prices, quantities and the distribution of money balances are stationary.

Definition 1 *A steady-state competitive equilibrium is given by prices $\{\{p_1(\theta)\}_{\theta \in \Theta}, p_2, p_3\}$, a distribution of money holdings with c.d.f. $H(\cdot)$ and support \mathcal{M} , and an allocation $\{n(\theta, m), c_1(\theta, m), c_2(\theta, \tilde{\theta}, m), c_3(\theta, \tilde{\theta}, m), m_1(\theta, m), m_2(\theta, \tilde{\theta}, m), m_3(\theta, \tilde{\theta}, m)\}_{\theta \in \Theta, \tilde{\theta} \in \Theta, m \in \mathcal{M}}$ such that:*

(i) the allocation solves problem (2) for each $m \in \mathcal{M}$;

(ii) markets clear⁶

$$\int_{\mathcal{M}} c_1(\theta, m) dH(m) = \theta \int_{\mathcal{M}} n(\theta, m) dH(m) \text{ for all } \theta \in \Theta,$$

$$\int_{\mathcal{M}} \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} c_s(\theta, \tilde{\theta}, m) dF(\tilde{\theta}) dF(\theta) dH(m) = e_s \text{ for } s = 2, 3;$$

(iii) the distribution $H(m)$ satisfies

$$H(m) = \int_{\mathcal{M}} \int_0^{\bar{\theta}} \int_{\{\tilde{m}: Rm_3(\theta, \tilde{\theta}, \tilde{m}) \leq m\}} dF(\tilde{\theta}) dF(\theta) dH(\tilde{m}).$$

Condition (iii) ensures that the distribution $H(\cdot)$ is stationary. As we will see below, we can focus on equilibria where the distribution of money balances is degenerate at $m = M$. Therefore, from now on we drop the argument m from the equilibrium allocations.

In order to characterize the equilibrium behavior, it is useful to derive the household's first order conditions. From problem (2) we obtain the three Euler equations (with respective complementary slackness conditions)

$$u'(c_1(\tilde{\theta})) \geq \frac{p_1(\tilde{\theta})}{p_2} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) \quad (m_1(\tilde{\theta}) \geq 0) \text{ for all } \tilde{\theta} \in \Theta, \quad (3)$$

$$U'(c_2(\theta, \tilde{\theta})) \geq \frac{p_2}{p_3} \quad (m_2(\theta, \tilde{\theta}) \geq 0) \text{ for all } (\theta, \tilde{\theta}) \in \Theta^2, \quad (4)$$

$$1 \geq p_3 \beta R V'(Rm_3(\theta, \tilde{\theta})) \quad (m_3(\theta, \tilde{\theta}) \geq 0) \text{ for all } (\theta, \tilde{\theta}) \in \Theta^2, \quad (5)$$

the optimality condition for labor supply

$$v'(n(\theta)) = \theta \frac{p_1(\theta)}{p_2} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\tilde{\theta}) \quad \text{for all } \theta \in \Theta, \quad (6)$$

and the envelope condition

$$V'(m) = \int_0^{\bar{\theta}} \frac{u'(c_1(\tilde{\theta}))}{p_1(\tilde{\theta})} dF(\tilde{\theta}). \quad (7)$$

Our assumptions allow us to simplify the equilibrium characterization as follows. First, the assumption that $F(\cdot)$ has an atom at 0, together with the Inada condition for U , implies that $m_1(\tilde{\theta}) > 0$ for all $\tilde{\theta}$. All consumers keep some positive money reserves in period 1, to insure

⁶The market clearing conditions reflect the assumption that each island receive a representative sample of consumers and producers. Thanks to Walras' law we can omit the market clearing conditions for the money market.

against the risk that their producers make zero revenues. This guarantees that condition (3) always holds as an equality.⁷

Next, condition (5) shows why we obtain equilibria with a degenerate distribution of money balances, as in Lagos and Wright (2005). Given that the supply of money is constant at M , a steady state equilibrium with a degenerate distribution $H(\cdot)$ must satisfy

$$Rm_3(\theta, \tilde{\theta}) = M \text{ for all } (\theta, \tilde{\theta}) \in \Theta^2.$$

In equilibrium, all agents adjust their consumption in period 3, so as to reach the same level of m_3 , irrespective of their current shocks. This immediately implies that $m_3(\theta, \tilde{\theta}) > 0$, so that (5) holds as an equality for all pairs $(\theta, \tilde{\theta})$. Notice that the assumptions that utility is linear in period 3 and that e_3 is large are crucial to ensure that the left-hand side of (5) is constant, confirming that this behavior is optimal.⁸

This leaves us with condition (4). In general, this condition can be either binding or slack for different pairs $(\theta, \tilde{\theta})$, depending on the parameters of the model. However, we are able to give a full characterization of the equilibrium by looking at specific monetary regimes, namely, by making assumptions about the rate of return R . First, we look at equilibria where the liquidity constraint $m_2(\theta, \tilde{\theta}) \geq 0$ is never binding. We will show that this case arises when $R = 1/\beta$, that is, in a monetary regime that follows the Friedman rule. Second, we look at equilibria where the constraint $m_2(\theta, \tilde{\theta}) \geq 0$ is binding for all pairs $(\theta, \tilde{\theta})$. We will show that this case arises whenever the rate of return is sufficiently low, i.e., when $R \leq \hat{R}$, for a given cutoff $\hat{R} \in (0, 1/\beta)$.

These two polar cases provide two analytically tractable benchmarks, which illustrate well the mechanism at the core of our model. The quantitative analysis in Section 5 also considers the case of economies with $R \in (\hat{R}, 1/\beta)$, where the liquidity constraint in period 2 is binding for a subset of agents.

3.1 Unconstrained equilibrium

We begin by considering “unconstrained equilibria,” that is, equilibria where the liquidity constraint is never binding. In this case, condition (4) always holds as an equality. Combining

⁷In Section 5.2 and in Appendix B, we relax the assumption $F(0) > 0$.

⁸When $R < 1/\beta$, all steady-state equilibria are characterized by a degenerate distribution of money holdings. One can show that, in this case, the value function V is strictly concave in any steady-state equilibrium. This, together with (5) implies that m_3 is constant across households.

conditions (3)-(5) and (7) then gives

$$\frac{u'(c_1(\tilde{\theta}))}{p_1(\tilde{\theta})} = \beta R \int_0^{\tilde{\theta}} \frac{u'(c_1(\theta))}{p_1(\theta)} dF(\theta). \quad (8)$$

Taking expectations with respect to $\tilde{\theta}$ on both sides, shows that a necessary condition to obtain an unconstrained equilibrium is $\beta R = 1$. The following proposition shows that, indeed, $\beta R = 1$ is both necessary and sufficient for an unconstrained equilibrium. Moreover, under this monetary regime the equilibrium achieves an efficient allocation.⁹

Proposition 2 *An unconstrained equilibrium exists if and only if $R = 1/\beta$ and achieves a first-best allocation. In all unconstrained equilibria, the prices are*

$$p_1(\theta) = \kappa u'(y_1^{FB}(\theta)) \text{ for all } \theta \in \Theta, \quad (9)$$

$$p_2 = \kappa U'(e_2), \quad (10)$$

$$p_3 = \kappa, \quad (11)$$

for some $\kappa \in (0, \hat{\kappa}]$, where

$$\hat{\kappa} \equiv \frac{M}{u'(y_1^{FB}(\bar{\theta}))y_1^{FB}(\bar{\theta}) + U'(e_2)e_2}. \quad (12)$$

To capture the logic behind the efficiency result, consider a consumer and a producer located in island θ . Using condition (4) and market clearing, we can rewrite the consumer's Euler equation (3) as¹⁰

$$u'(y_1(\theta)) = \frac{p_1(\theta)}{p_3}, \quad (13)$$

and the producer's optimality condition (6) as

$$v'\left(\frac{y_1(\theta)}{\theta}\right) = \theta \frac{p_1(\theta)}{p_3}. \quad (14)$$

These two equations describe, respectively, the demand and the supply of consumption goods on island θ , as a function of the price $p_1(\theta)$. Jointly, they determine the equilibrium values of $p_1(\theta)$ and $y_1(\theta)$.

Conditions (13) and (14) highlight that, under the Friedman rule, a consumer and a producer in island θ do not need to forecast the income/spending of their partners when making

⁹See Rocheteau and Wright (2005) for a general discussion of the efficiency of the Friedman rule in a wide class of search models of money.

¹⁰Whenever we look at a consumer and producer located in the same island θ , the consumer's optimality conditions are given by (3)-(5), with the role of θ and $\tilde{\theta}$ inverted.

their trading decisions, given that their marginal value of money is constant and equal to $1/p_3$. This implies that trading in island θ is essentially independent of trading decisions in all other islands. We will see that this is no longer true when we move to a constrained equilibrium. Conditions (13) and (14) can be easily manipulated to obtain the planner's first order condition (1).

It is possible to prove that the unconstrained equilibrium allocation is unique, but the price levels are indeterminate, since κ in (9)-(11) can be anywhere in the interval $(0, \hat{\kappa}]$.¹¹ This implies that the real value of the money supply M is indeterminate. However, the upper bound $\hat{\kappa}$ determines a lower bound for the real value of M . In particular, given our focus on the liquidity constraint in period 2, we derive explicitly the upper bound for M/p_2 in the following corollary.

Corollary 1 *In an unconstrained equilibrium M/p_2 satisfies*

$$\frac{M}{p_2} \geq e_2 + \frac{u'(y_1^{FB}(\bar{\theta}))}{U'(e_2)} y_1(\bar{\theta}). \quad (15)$$

The fact that real value of money is large in period 2 ensures that all households can afford to consume $c_2(\theta, \tilde{\theta}) = e_2$, irrespective of the shocks θ and $\tilde{\theta}$, thus achieving full insurance.

3.2 Fully constrained equilibrium

We now turn to the case where the liquidity constraint is always binding in period 2, that is, $m_2(\theta, \tilde{\theta}) = 0$ for all pairs $(\theta, \tilde{\theta})$. The following proposition shows that this type of equilibrium arises when R is sufficiently low. We refer to it as a “fully constrained equilibrium.”

Proposition 3 *There is a cutoff $\hat{R} \in (0, 1/\beta)$ such that a fully constrained equilibrium exists if and only if $R \leq \hat{R}$.*

Let us illustrate the main steps of the equilibrium construction. Combining the consumer's budget constraints in periods 1 and 2 gives

$$c_2(\theta, \tilde{\theta}) = \frac{M}{p_2} - \frac{p_1(\tilde{\theta})}{p_2} c_1(\tilde{\theta}) + \frac{p_1(\theta)}{p_2} y_1(\theta), \quad (16)$$

where we use the fact that all agents begin period 1 with $m = M$ and end period 2 with $m_2(\theta, \tilde{\theta}) = 0$. Notice that now households are not fully insured and their consumption in period 2 depends both on the consumer's and on the producer's shock in period 1.

¹¹This indeterminacy can be eliminated using a “fiscal rule” under which the government commits to set the real value of the lump sum tax T/p_3 .

Integrating both sides of (16) over θ and $\tilde{\theta}$, and using market clearing in periods 1 and 2, we obtain a simple “quantity-theory” equation

$$p_2 e_2 = M. \quad (17)$$

It is easy to show that in any equilibrium the choice of M only affects nominal variables, but has no real effect. Therefore, we choose the convenient normalization $M = e_2$, which, in a fully constrained equilibrium, implies $p_2 = 1$.

As in the unconstrained case, let us focus on a consumer and a producer in island θ . The previous steps imply that we can rewrite the consumer’s Euler equation (3) as

$$u'(y_1(\theta)) = p_1(\theta) \int_0^{\bar{\theta}} U' \left(e_2 - p_1(\theta) y_1(\theta) + p_1(\tilde{\theta}) y_1(\tilde{\theta}) \right) dF(\tilde{\theta}), \quad (18)$$

and the producer’s optimality condition (6) as

$$v' \left(\frac{y_1(\theta)}{\theta} \right) = \theta p_1(\theta) \int_0^{\bar{\theta}} U' \left(e_2 - p_1(\tilde{\theta}) y_1(\tilde{\theta}) + p_1(\theta) y_1(\theta) \right) dF(\tilde{\theta}). \quad (19)$$

The demand and supply equations (18) and (19) correspond to (13) and (14) in the unconstrained case and show that the equilibrium values of $p_1(\theta)$ and $y_1(\theta)$ are now dependent on prices and quantities in all other islands. This highlights the “linkages” between trading decisions in island θ and trading decisions in other islands, which arise when liquidity constraints are binding in period 2. In this case, consumers and producers need to forecast the level of trading in other islands to evaluate their expected marginal value of money, and thus their willingness to trade at each price $p_1(\theta)$. These linkages will play a crucial role when we introduce aggregate shocks in the next section.

Equations (18) and (19) define two functional equations in $p_1(\cdot)$ and $y_1(\cdot)$. In the proof of Proposition 3, we show that this pair of functional equations has a unique solution. To do so, we define $x(\theta) \equiv p_1(\theta)y_1(\theta)$ and we analyze a fixed point problem in terms of the function $x(\cdot)$. The fixed point is found using a contraction mapping argument. Here we make use of the assumption that the elasticity of $u'(c)$ is bounded in $[\underline{\sigma}, 1)$. Proposition 3 can be proved under weaker conditions, using a different type of fixed point theorem. However, the contraction mapping approach laid out here will be essential in deriving our amplification result in Section 4. An interesting corollary of Proposition 3 is that the fully constrained equilibrium allocation is the same for all $R \leq \hat{R}$. The only equilibrium variable that changes with R is the price level p_3 .

The following lemma establishes two monotonicity results for real and nominal income in period 1, which will be useful in studying the effects of aggregate shocks.

Lemma 2 *In a fully constrained equilibrium, both $y_1(\theta)$ and $p_1(\theta)y_1(\theta)$ are monotone increasing in θ .*

Finally, comparing (15) and (17) shows that real money supply in period 2 is smaller in the fully constrained equilibrium than in any unconstrained equilibrium. The intuition behind this result is that a lower rate of return on money reduces agents' incentive to save. This tends to increase nominal prices and reduces the equilibrium value of real money balances. This result also explains why we label the monetary regime with $R \leq \hat{R}$ as one of "scarce liquidity," and the Friedman rule regime as one of "abundant liquidity." We will return to this issue in Section 6.2.

3.3 Liquidity premium

Let us extend our model by introducing an illiquid asset in zero net supply. This simple extension allows us to define a liquidity premium. The illiquid asset is traded at the beginning of period 1, before the idiosyncratic shocks θ are realized, and it represents a claim to 1 unit of money in period 3. It is illiquid in the sense that it cannot be carried by consumers to the islands they visit in periods 1 and 2. Denote by q its price in terms of money (the numeraire). Households choose illiquid asset holdings b at the beginning of period 1 and their budget constraints in periods 1 and 3 become

$$\begin{aligned} m_1(\tilde{\theta}) + p_1(\tilde{\theta})c_1(\tilde{\theta}) &\leq m - qb, \\ m_3(\theta, \tilde{\theta}) + p_3c_3(\theta, \tilde{\theta}) &\leq m_2(\theta, \tilde{\theta}) + p_2e_2 + p_3e_3 + b - T, \end{aligned}$$

while the budget constraint in period 2 is unaffected. The optimality condition for b can be written as

$$q \int_0^{\bar{\theta}} \frac{u'(c_1(\tilde{\theta}))}{p_1(\tilde{\theta})} dF(\tilde{\theta}) = \beta R \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} V'(Rm_3) dF(\theta) dF(\tilde{\theta}).$$

Combining it with the envelope condition (7), we obtain

$$q = \beta R. \tag{20}$$

Since the illiquid asset is in zero net supply, the equilibrium analysis is unchanged.

Notice that the gross nominal return on money between periods 1 and 3 is 1, while that on the illiquid asset is equal to $1/q$. Hence, $1/q - 1$ represents the liquidity premium in our economy. It follows from (20) that, under the Friedman rule, $q = 1$ and the liquidity premium is zero. In this case, the return on the illiquid asset is exactly the same as that of the liquid asset because agents are perfectly insured. Whenever $R < 1/\beta$, the liquidity premium is positive. In particular, in a fully constrained economy, the liquidity premium is above some cutoff $1/(\beta\hat{R}) - 1$. Agents are all liquidity constrained in period 2 and are willing to pay a lower price for the illiquid asset, because they are concerned about consumption volatility in period 2, when the illiquid asset cannot be used.

4 Aggregate Shocks

We now turn to the main question of the paper: how does output respond to aggregate shocks when liquidity is scarce or abundant? In this section, we compare analytically the effect of an aggregate shock in the unconstrained and fully constrained regimes derived above and we identify an amplification mechanism which is only present in the fully constrained case. In the next section, we will present numerical examples in order to evaluate the quantitative significance of the amplification effect and to study the intermediate cases with $R \in (\hat{R}, 1/\beta)$.

4.1 A decomposition

First, we introduce aggregate uncertainty in the model by introducing an aggregate shock ζ_t , which is realized and publicly revealed at the beginning of each period. We assume that ζ_t is i.i.d. with cumulative distribution function $G(\cdot)$ continuous on the support $[\underline{\zeta}, \bar{\zeta}]$. Conditional on ζ_t , the cross-sectional distribution of the productivity shocks θ_t^k is $F(\cdot|\zeta_t)$. Assume that $F(\theta|\zeta)$ is continuous and non-increasing in ζ , for each θ . This implies that a distribution with a higher ζ first-order stochastically dominates a distribution with lower ζ .

We assume i.i.d. aggregate shocks and we focus on simple stationary equilibria, defined along the lines of Definition 1, where the distribution of money balances at the beginning of period 1 is degenerate. Prices and allocations now depend not only on the current idiosyncratic shocks, but also on the current aggregate shock ζ , and are denoted by $p_1(\theta, \zeta)$, $c_1(\theta, \zeta)$, etc. In Appendix B we present the characterization of the equilibrium with aggregate shocks. In particular, both propositions 2 and 3 are easily extended to this case. The only noticeable

difference is that the definition of the cutoff \hat{R} needs to be modified so as to ensure that in a fully constrained equilibrium agents are constrained for all realizations of ζ .

Given that output in periods 2 and 3 is exogenous, we focus on aggregate output in period 1, which is equal to

$$Y_1(\zeta) \equiv \int_0^{\bar{\theta}} y_1(\theta, \zeta) dF(\theta|\zeta). \quad (21)$$

In particular, we look at the proportional response of output to a small aggregate shock, $d \ln Y_1 / d\zeta$. This measure can be decomposed as follows

$$\frac{d \ln Y_1}{d\zeta} = \frac{\int_0^{\bar{\theta}} y_1(\theta, \zeta) \frac{\partial f(\theta|\zeta)}{\partial \zeta} d\theta}{Y_1} + \frac{\int_0^{\bar{\theta}} \frac{\partial y_1(\theta, \zeta)}{\partial \zeta} dF(\theta|\zeta)}{Y_1}. \quad (22)$$

The first member on the right-hand side represents the mechanical effect of having a larger number of islands with high productivity. This effect is positive both in the unconstrained and in the fully constrained economy, given that output is increasing in θ in both regimes as shown by Lemmas 1 and 2. We call this the ‘‘own-productivity effect.’’ The second member captures the endogenous response of output for each given level of θ , and is at the core of our analysis. We call it the ‘‘expected income effect,’’ for reasons that will be apparent below.

4.2 The expected income effect

Consider first an unconstrained equilibrium. In this case, the economy achieves the first-best allocation and, as we know from Lemma 1, output in island θ is independent of the economy-wide distribution of productivity. Therefore, $\partial y_1(\theta, \zeta) / \partial \zeta = 0$ and the expected income effect is absent.

Next, consider the case of a fully constrained equilibrium. The following proposition shows that, in this case, output in each island is increasing in ζ , for any given realization of the local productivity shock θ . This implies that the expected income effect is positive.

Proposition 4 *Consider a fully constrained equilibrium of the economy with aggregate shocks. For each $\theta > 0$, the output $y_1(\theta, \zeta)$ is increasing in ζ .*

To understand the mechanism behind this effect, it is useful to consider the following partial equilibrium exercise. Let us focus on island θ and take as given $p_1(\tilde{\theta}, \zeta)$ and $y_1(\tilde{\theta}, \zeta)$ for all $\tilde{\theta} \neq \theta$. Rewriting the demand and supply equations (18) and (19) for the economy with

aggregate shocks gives

$$u'(y_1(\theta, \zeta)) = p_1(\theta, \zeta) \int_0^{\bar{\theta}} U'(c_2(\tilde{\theta}, \theta, \zeta)) dF(\tilde{\theta}|\zeta), \quad (23)$$

$$v'\left(\frac{y_1(\theta, \zeta)}{\theta}\right) = \theta p_1(\theta, \zeta) \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta), \quad (24)$$

where

$$c_2(\tilde{\theta}, \theta, \zeta) = e_2 - p_1(\theta, \zeta) y_1(\theta, \zeta) + p_1(\tilde{\theta}, \zeta) y_1(\tilde{\theta}, \zeta),$$

and the symmetric expression holds for $c_2(\tilde{\theta}, \theta, \zeta)$. Lemma 2 shows that $p_1(\theta, \zeta) y_1(\theta, \zeta)$ is an increasing function of θ . It follows that $U'(c_2(\tilde{\theta}, \theta, \zeta))$ is decreasing in $\tilde{\theta}$, while $U'(c_2(\theta, \tilde{\theta}, \zeta))$ is increasing in $\tilde{\theta}$. Hence, when ζ increases the integral on the right-hand side of (23) decreases, while the integral on the right-hand side of (24) increases.¹² On the demand side, the intuition is that when a liquidity constrained consumer expects higher income from his partner, his marginal value of money decreases. Then, he reduces his reserves and increases consumption for any given price $p_1(\theta)$. On the supply side, when a producer expects higher spending by his partner, he faces a negative income effect and, hence produces more for any given $p_1(\theta)$. The first effect shifts the demand curve to the right, the second shifts the supply curve to the right. The combination of the two implies that equilibrium output in island θ increases.

On top of this partial equilibrium mechanism, there is a general equilibrium feed-back due to the endogenous response of prices and quantities in the islands $\tilde{\theta} \neq \theta$. This magnifies the initial effect. As the nominal value of output in all other islands increases, there is a further increase in the marginal value of money for the consumers and a further decrease for the producers, leading to an additional increase in output.

Summing up, the mechanism identified in Proposition 4 tends to magnify the output response to aggregate shocks in a fully constrained economy. This amplification effect is driven by the agents' expectations regarding nominal income in other islands.

Notice that the proof of Proposition 4 is the only place where we use the assumption that the elasticity of $U'(c)$ is smaller or equal than 1. In particular, this condition is sufficient to establish that the labor supply in each island is positively sloped, which, in turns, is sufficient to obtain our result. When we turn to numerical examples, we will see that the result survives for elasticities larger than 1.

¹²Recall that an increase in ζ leads to a shift of the distribution of θ in the sense of first order stochastic dominance.

Going back to equation (22), we have established that the expected income effect is zero in the unconstrained case and positive in the constrained one. However, this is not sufficient to establish that output volatility is greater in the constrained economy, since we have not yet compared the relative magnitude of the own-productivity effects, which are positive in both cases. A special example where the comparison is unambiguous is the case of a binary shock for θ , where the own-productivity effect is identical in the two cases.

Let θ have a binary distribution on $\{0, \bar{\theta}\}$. Let $\zeta \in [0, 1]$ represent the probability of the high realization $\bar{\theta}$. In this case, the output response to a positive aggregate shock is always higher when liquidity is scarce.

Lemma 3 *When θ has a binary distribution on $\{0, \bar{\theta}\}$ the response of output to the aggregate shock ζ , $d \ln Y_1 / d\zeta$, is greater in a fully constrained equilibrium relative to an unconstrained equilibrium.*

Consider first an unconstrained equilibrium and let \bar{y}_1^U denote output in island $\bar{\theta}$. Output is zero in the island where $\theta = 0$. Moreover, from Lemma 1 we know that \bar{y}^U is independent of ζ . Therefore, aggregate output is equal to $Y_1(\zeta) = \zeta \bar{y}_1^U$, and

$$\frac{d \ln Y_1(\zeta)}{d\zeta} = \frac{1}{\zeta}. \quad (25)$$

Next, consider a fully constrained equilibrium and let $\bar{y}_1^C(\zeta)$ denote output in island $\bar{\theta}$. Aggregate output is now equal to $Y_1(\zeta) = \zeta \bar{y}_1^C(\zeta)$ and we have

$$\frac{d \ln Y_1(\zeta)}{d\zeta} = \frac{1}{\zeta} + \frac{d \ln \bar{y}_1^C(\zeta)}{d\zeta}. \quad (26)$$

The second element on the right-hand side of (26) is positive by Proposition 4. Therefore, comparing (25) and (26) immediately shows that the output response is larger in the fully constrained economy.

Apart from specific examples, it is generally difficult to compare the relative size of the own-productivity effect in the two regimes. In fact, it is possible to construct examples where this effect is larger in the unconstrained economy and where it is strong enough to dominate the expected income effect. Therefore, to evaluate the significance of the mechanism identified in Proposition 4, we turn to a basic quantitative exercise.

5 Numerical Examples

In this section, we present some numerical examples that show that, under reasonable parametrizations, the amplification effect identified above is sizeable and leads to higher volatility in economies with lower supply of liquid assets. To compute these examples, we generalize our theory to monetary regimes with $R \in (\hat{R}, 1/\beta)$, where the liquidity constraint is binding for a subset of realizations $(\theta, \tilde{\theta})$. The characterization of the equilibrium for this intermediate region is in Appendix B.

5.1 Parameters

We interpret each sequence of three sub-periods as a year, and set the discount factor β equal to 0.96. The instantaneous utility functions in periods 1 and 2 are $u(c) = c^{1-\sigma_1}/(1-\sigma_1)$ and $U(c) = c^{1-\sigma_2}/(1-\sigma_2)$. The disutility of labor effort is linear, $v(n) = n$. The island specific shock θ has a discrete uniform distribution, with 10 equally spaced realizations on the interval $[0, \bar{\theta}]$.

Given isoelastic preferences, it is possible to show that if $\bar{\theta}$, e_2 , and e_3 are scaled by appropriate factors, one obtains equilibria which are equivalent in terms of all the measures we will look at (output volatility, nominal incomes, etc). Therefore, we normalize $e_2 = 1$.

The coefficient of relative risk aversion σ_2 is set equal to 1. The parameters σ_1 and e_3 are chosen to obtain a realistic economy-wide demand for the liquid asset. In particular, we interpret the liquid asset in a narrow sense as money balances (M1, i.e., currency and demand deposits) and match the empirical relation between money velocity and the nominal interest rate. This approach makes our simple calibration comparable to those in Lagos and Wright (2005) and Craig and Rocheteau (2007), which, in turns, follow Lucas (2000). In the data, money velocity is defined as the ratio of M1 to nominal GDP and the nominal interest rate is measured by the short-term commercial paper rate. In the model, money velocity is measured as $M / \left(\int_0^{\bar{\theta}} p_1(\theta) y_1(\theta) dF(\theta) + p_2 e_2 + p_3 e_3 \right)$.¹³ To derive the nominal interest rate in the model, we use a result derived in Section 6.2, which shows that our model is equivalent to a model with non-interest-bearing money, where money growth and inflation are constant and equal to γ . In that setup, the real rate of return on money balances is equal to $R = 1/\gamma$, the inverse of the inflation rate, and the real rate of return on the illiquid asset (treasury

¹³For simplicity, we calibrate parameters using the model with no aggregate shock.

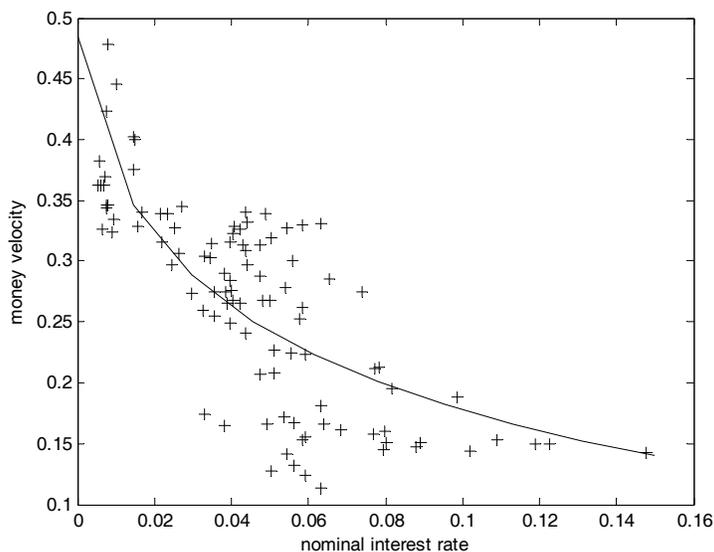


Figure 1: Money velocity and nominal interest rates

bills) is equal to $1/\beta$. Therefore, the nominal interest rate is equal to $1 + i = \gamma/\beta = (\beta R)^{-1}$. This shows that the Friedman rule regime corresponds, as it should, to $i = 0$, while a fully constrained regime corresponds to $i > \hat{i}$, where $\hat{i} = (\beta \hat{R})^{-1} - 1$.¹⁴

Finally, the value of $\bar{\theta}$ is chosen so that average nominal income is equal in periods 1 and 2, that is, $\int_0^{\bar{\theta}} p_1(\theta) y_1(\theta) dF(\theta) = p_2 e_2$, in a baseline scenario where the nominal interest rate is $i = 5\%$. The parameters we obtain are $\sigma_1 = 0.294$, $e_3 = 9.021$, and $\bar{\theta} = 1.904$. Figure 1 shows the relation between money velocity and the nominal interest rate in the data (scatter plot) and in the model (solid line).

5.2 Amplification

We can now illustrate the main result of the paper, by looking at the effect of aggregate shocks in economies with different levels of i (or R). We consider a positive aggregate shock that reduces the probability of $\theta = 0$ by 0.009 and increases proportionally the probability of all positive realizations of θ . If we interpret $\theta = 0$ as an unemployment state, this shock reduces its probability from 0.10 to 0.091. The size of the aggregate shock is chosen so that it increases Y_1

¹⁴For each value of i in the empirical time-series, we evaluate the equilibrium value of money velocity. We choose σ_1 and e_3 to minimize the quadratic distance between money velocity in the data and the model-generated series. We use US annual data for the sample period 1900-2000, as in Lucas (2000).

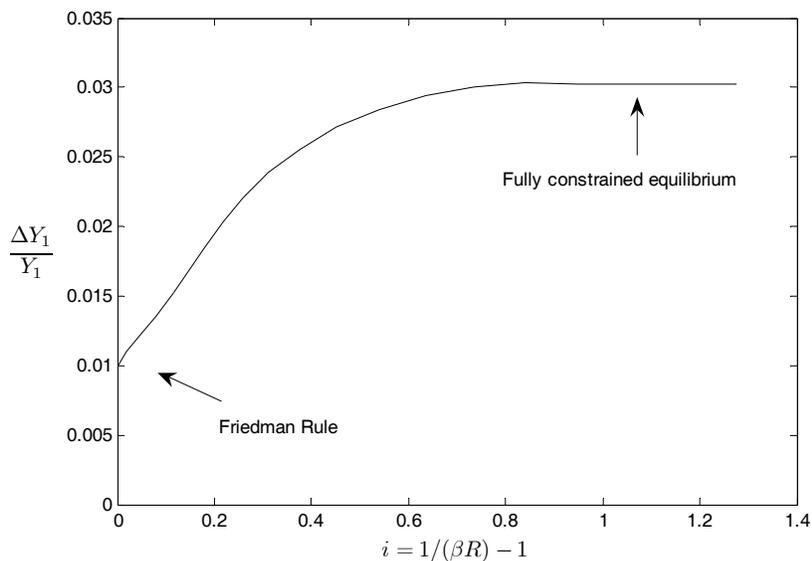


Figure 2: The aggregate output response in different regimes

by 1% under the Friedman rule. The equilibrium is computed assuming that the two aggregate shocks have equal probability.

Figure 2 shows the proportional response of Y_1 for different levels of the nominal interest rate i . Notice that the output response is about three times larger in the fully constrained economy, that is, when $i \geq \hat{i}$ ($R \leq \hat{R}$), relative to the Friedman rule regime, where $i = 0$ ($R = 1/\beta$). In the parametrization presented, the fully constrained regime is achieved for high levels of i , around 90%—basically an hyperinflation scenario. However, a sizeable amplification effect is also present for moderate inflation rates. For example, when the nominal interest rate is 15% the output response is more than 40% larger than under the Friedman rule.

Figure 3 illustrates the expected income effect identified in Proposition 4. The two panels illustrate how output varies across islands in the two polar regimes. Under the Friedman rule, in panel (a), output per island is independent of the aggregate shock, since the expected income effect is absent. In the fully constrained regime, in panel (b), the output of each island θ is greater after a high aggregate shock.¹⁵

We can then perform some simple comparative statics exercises. In particular, we look at

¹⁵To make the figure easier to read, we consider here a bigger aggregate shock, which reduces the probability of $\theta = 0$ by 0.09 (which generates a 10% output response in the Friedman rule regime).

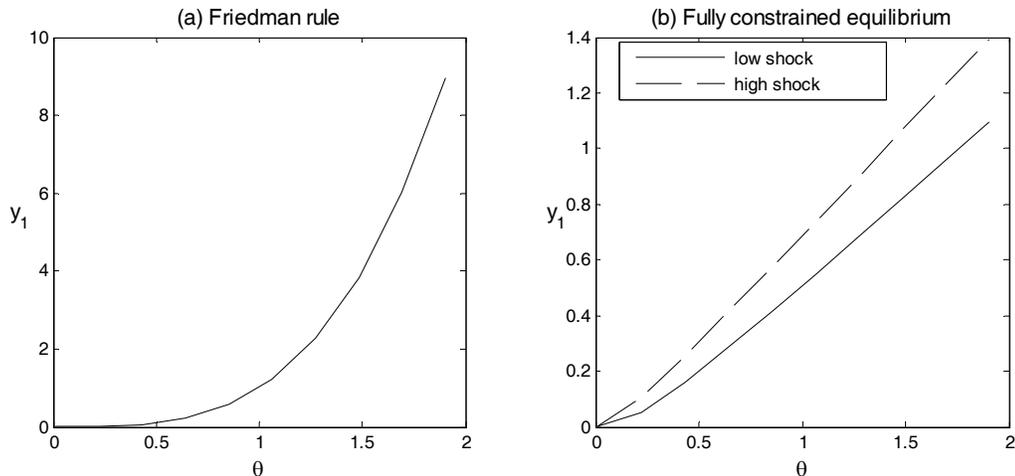


Figure 3: Island-specific output and local productivity shocks

the effect of changing σ_1 and σ_2 on equilibrium volatility. Figure 4 shows the proportional output response for different values of these parameters, plotted against the nominal interest rate i . Panel (a) shows that economies with σ_1 closer to 1, display smaller aggregate volatility. To understand the mechanism behind this figure, notice that the coefficient σ_1 determines the elasticity of demand for local goods in island θ . When σ_1 is closer to 1, differences in productivity θ have a smaller effect on nominal income, $p_1(\theta)y_1(\theta)$, since changes in output are compensated by close to proportional changes in prices.¹⁶ This reduces nominal income volatility across islands, dampening the strength of the expected income effect. The effects of changing σ_2 are shown in panel (b). Increasing agents' risk aversion in period 2 strengthens the agents' self-insurance motive, thus magnifying the expected income effect. Therefore, for higher values of σ_2 , the amplification is more pronounced.¹⁷

As a further exercise, let us relax the assumption that the distribution of θ has positive mass at zero. That assumption was made to ensure that money holdings are always positive at the end of period 1, $m_1(\theta) > 0$ for all θ , so that we could focus our attention on the role of the liquidity constraint in period 2. Consider now the case where the realizations of θ lie in the interval $[\underline{\theta}, \bar{\theta}]$, with $\underline{\theta} > 0$. The equilibrium characterization for this general case is discussed in Appendix B. In particular, suppose θ is still a discrete uniform (with 10 realizations), but

¹⁶The presence of a positive mass of islands with $\theta = 0$, means that some degree of income volatility is present even in the limit case $\sigma_1 = 1$.

¹⁷The example with $\sigma_2 = 2$ shows that the assumption $\sigma_2 \leq 1$ is not necessary for our amplification result. In fact, amplification is stronger for larger values of σ_2 .

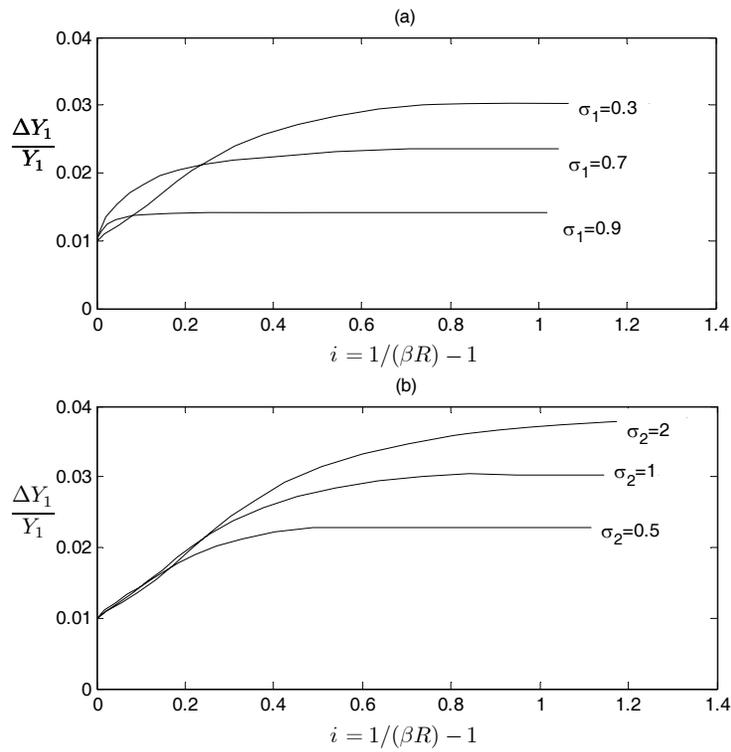


Figure 4: Comparative statics: the output response for different values of σ_1 and σ_2

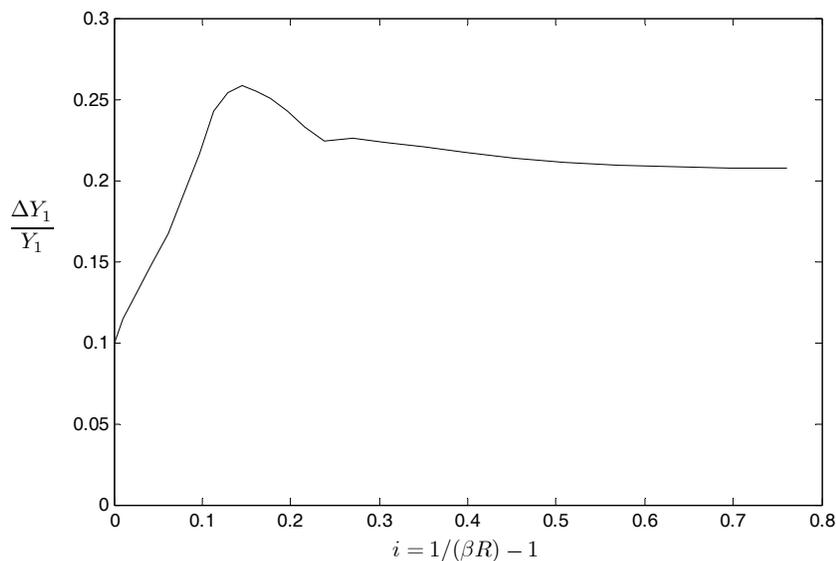


Figure 5: Output response with binding liquidity constraints in period 1

$\underline{\theta} = 0.25$. The remaining parameters and the form of the aggregate shock are unchanged. Figure 5 plots the output response to a positive aggregate shock in this case. Amplification is still present for all $i > 0$. The possibility of a binding liquidity constraint in period 1 generates an interesting non-monotonicity in the relation between the monetary regime and output volatility. For low levels of i , the liquidity constraint in period 1 is not binding, and our effect generates amplification as in the benchmark case. However, when i is high enough, around 12%, the period 1 liquidity constraint starts to bind for consumers in islands with high productivity. This dampens the response of consumption, since some consumers are effectively against a cash-in-advance constraint, reducing the amplification effect. As in the baseline model, there is a cutoff \hat{i} , around 65%, above which changes in i have no further effect on the real allocation. At this point, there is still a sizable degree of amplification.¹⁸

5.3 Countercyclical liquidity premia

Next, let us look at the response of the liquidity premium to aggregate shocks. Our numerical example shows an interesting additional implication of the model: the same mechanism behind

¹⁸In the example presented, for $i \geq \hat{i}$, 40% of the consumers are constrained in period 1. It is also possible to construct examples where, for sufficiently high values of i , all consumers are constrained in period 1. In that case, the amplification effect disappears as i becomes high enough.

the amplification result tends to make the liquidity premium countercyclical.

Let us go back to the extended version of the model presented in Section 3.3, where we introduced an illiquid asset, in zero net supply. Let us assume that the illiquid asset is traded *after* the realization of ζ .¹⁹ Therefore, the price of the illiquid asset is a function of ζ . By combining the appropriate optimality and envelope conditions, we obtain

$$q(\zeta) = \frac{\beta R \int_{\underline{\zeta}}^{\bar{\zeta}} \int_0^{\bar{\theta}} \frac{u'(y_1(\theta, \tilde{\zeta}))}{p_1(\theta, \tilde{\zeta})} dF(\theta|\tilde{\zeta}) dg(\tilde{\zeta})}{\int_0^{\bar{\theta}} \frac{u'(y_1(\theta, \zeta))}{p_1(\theta, \zeta)} dF(\theta|\zeta)}. \quad (27)$$

Consider first an unconstrained economy. An immediate extension of equation (8) shows that $q(\zeta) = 1$, as in the case of no aggregate shocks. Therefore, the liquidity premium is zero and is independent of ζ . In a fully constrained economy, combining the consumers' Euler equations in the three periods and the envelope condition, we get

$$\int_0^{\bar{\theta}} \frac{u'(c(\theta, \zeta))}{p_1(\theta, \zeta)} dF(\theta|\zeta) > \beta R \int_{\underline{\zeta}}^{\bar{\zeta}} \int_0^{\bar{\theta}} \frac{u'(c(\theta, \tilde{\zeta}))}{p_1(\theta, \tilde{\zeta})} dF(\theta|\tilde{\zeta}) dg(\tilde{\zeta}).$$

This implies that the liquidity premium $1/q(\zeta) - 1$ is always positive and, in general, varies with ζ . It is interesting to explore whether we can say anything about the correlation of the liquidity premium and the aggregate shock.

Figure 6 plots the liquidity premium conditional on the two aggregate shocks, for different values of i , and shows that the liquidity premium is countercyclical in our example.²⁰ To understand the underlying mechanism, notice that the liquidity premium is countercyclical whenever the derivative of $1/q(\zeta)$ with respect to ζ is negative. Given (27), this is equivalent to $d \left[\int_0^{\bar{\theta}} \frac{u'(y_1(\theta, \zeta))}{p_1(\theta, \zeta)} dF(\theta|\zeta) \right] / d\zeta < 0$. A useful decomposition is then:

$$\frac{d}{d\zeta} \left[\int_0^{\bar{\theta}} \frac{u'(y_1(\theta, \zeta))}{p_1(\theta, \zeta)} dF(\theta|\zeta) \right] = \int_0^{\bar{\theta}} \frac{u'(y_1(\theta, \zeta))}{p_1(\theta, \zeta)} \frac{\partial f(\theta|\zeta)}{\partial \zeta} d\theta + \int_0^{\bar{\theta}} \frac{\partial \left(\frac{u'(y_1(\theta, \zeta))}{p_1(\theta, \zeta)} \right)}{\partial \zeta} dF(\theta|\zeta). \quad (28)$$

When ζ increases, there are two effects on the liquidity premium, related to the two effects on aggregate output analyzed in Section 4. On the one hand, there is an “own-productivity effect,” represented by the first term of (28), that tends to increase the liquidity premium. If there are more islands with higher productivity, there will be more trade on average and

¹⁹It is easy to show that if we allow agents to trade the illiquid asset *before* the realization of ζ its price would be equal to βR .

²⁰As for Figure 3, we consider here a larger aggregate shock (which reduces the probability of $\theta = 0$ by 0.09), to make the figure easier to read.

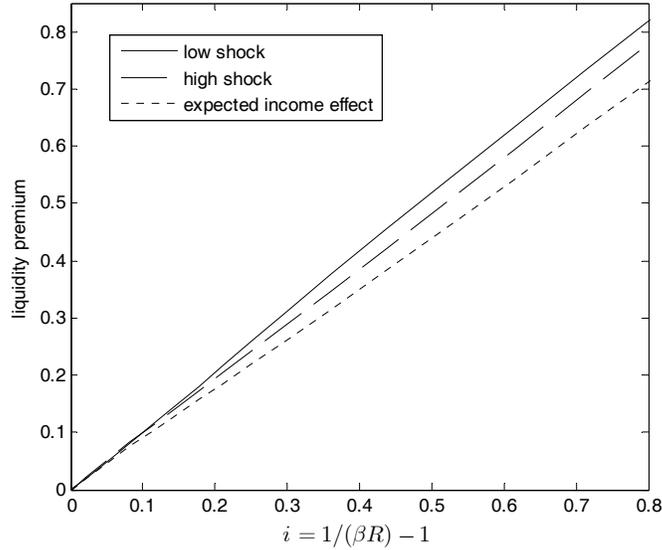


Figure 6: Liquidity premium

agents will need more liquid assets. On the other hand, there is an “expected income” effect, represented by the second term of (28). In a given island θ , consumers expect higher income and reduce their demand for liquidity. This increases consumption and reduces their marginal utility in period 1, driving down the liquidity premium. Notice that the prices $p_1(\theta, \zeta)$ are also adjusting when ζ changes, which makes it hard to derive unambiguous analytical results. However, for all the parameter combinations we have tried, this second effect is positive. In Figure 6, this effect is illustrated by the dotted line labeled “expected income effect,” and is sufficiently strong so as to generate a countercyclical liquidity premium.

6 Extensions

6.1 News shocks

Consider now an economy where the aggregate shock ζ is not observed by the households in period 1. Instead, they all observe a public signal $\xi \in [\underline{\xi}, \bar{\xi}]$, which is drawn at the beginning of each period, together with the aggregate shock ζ , from a continuous distribution with joint density function $g(\zeta, \xi)$.

Take an agent located in an island with productivity θ , his posterior density regarding ζ

can be derived using Bayes' rule:

$$g(\zeta|\xi, \theta) = \frac{f(\theta|\zeta)g(\zeta, \xi)}{\int_{\underline{\zeta}}^{\bar{\zeta}} f(\theta|\tilde{\zeta})g(\tilde{\zeta}, \xi)d\tilde{\zeta}}.$$

The distribution $g(\zeta|\xi, \theta)$ is then used to derive the agent's posterior beliefs regarding $\tilde{\theta}$ in the island where his partner is located

$$F(\tilde{\theta}|\xi, \theta) = \int_{\underline{\zeta}}^{\bar{\zeta}} F(\tilde{\theta}|\zeta)g(\zeta|\xi, \theta) d\zeta.$$

We will make the assumption that $F(\tilde{\theta}|\xi, \theta)$ is non-decreasing in ξ , for any pair $(\theta, \tilde{\theta})$. This means, that conditional on θ , the signal ξ is “good news” for $\tilde{\theta}$, in the sense of Milgrom (1981). We also make the natural assumption that $F(\tilde{\theta}|\xi, \theta)$ is non-decreasing in θ . In period 2, the actual shock ζ is publicly revealed.

In this environment, we study a stationary equilibrium, along the lines of Definition 1, where the distribution of money holdings at the beginning of period 1 is degenerate and prices and allocations depend on idiosyncratic shocks and on the aggregate shocks ξ and ζ . In particular, prices and quantities in period 1, $p_1(\theta, \xi)$ and $y_1(\theta, \xi)$, depend only on θ and ξ , given that ζ is not in the information set of the households in that period. Aggregate output in period 1 becomes

$$Y_1(\zeta, \xi) \equiv \int_0^{\bar{\theta}} y_1(\theta, \xi) dF(\theta|\zeta). \quad (29)$$

We can now look separately at the output response to the productivity shock ζ and to the news shock ξ . In particular, next proposition shows that the output response to ζ is positive both in an unconstrained and in a fully constrained equilibrium, while the output response to the signal ξ is positive only in the fully constrained case.²¹

Proposition 5 *Consider the economy with imperfect information regarding the aggregate shock. In an unconstrained equilibrium $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi = 0$. In a fully constrained equilibrium $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi > 0$.*

This result is not surprising, in light of the analysis in the previous section. Compare the expression for aggregate output under imperfect information (29) with the correspondent expression in the case of full information (21). By definition, the productivity shock ζ affects

²¹The analysis in Appendix B can be easily extended to the case of the economy with aggregate shocks and imperfect information.

the distribution of idiosyncratic shocks $F(\cdot|\zeta)$ in both cases. However, the trading decisions in island θ are affected only by the agents' expectations about that distribution, which, in the case of imperfect information, are driven by the signal ξ . It follows that the effect of ζ is analogous to the own-productivity effect, while the effect of ξ is analogous to the expected income effect. The advantage of an environment with imperfect information, is that these two effects can be disentangled. In particular, our amplification mechanism is captured by the output response to ξ . When liquidity is abundant, as we know from Proposition 1, output in island θ is independent of the economy-wide distribution of productivity and does not respond to ξ . The result that the output response to ξ is positive when liquidity is scarce is a natural extension of Proposition 4. In island θ , trading is higher the more optimistic agents are about trading in all other islands. The only difference is how expectations are formed. The perceived distribution of productivities for an agent in island θ depends now on the signal ξ , instead that on the actual ζ . A positive signal ξ makes both consumers and producers in island θ more optimistic about trading in other islands, even if the underlying ζ is unchanged. This highlights that expectations are at the core of our amplification result.

6.2 Alternative sources of liquidity

There are several alternative ways of introducing a liquid asset in our environment, which lead to formally equivalent models. A sketch of these different approaches can help the interpretation of our results.

Constant money growth. Assume that money pays no interest, and there is a constant money growth rate γ , that is,

$$M_{t+1} = \gamma M_t.$$

Monetary injections take place at the end of period 3, and the government budget constraint is now

$$M_{t+1} = M_t - T_t,$$

where the lump-sum tax/subsidy T_t is time-varying. It is possible to prove that this economy is formally equivalent, in real terms, to our baseline economy. More specifically, given an equilibrium of the economy with interest-bearing notes, there exists an economy with constant money growth that achieves the same equilibrium allocation in real terms. The converse also applies. The argument for this result goes as follows. Take an equilibrium of the baseline

economy, with rate of return R , and equilibrium prices $p_1(\theta), p_2, p_3$. In an economy with constant money growth $\gamma = R^{-1}$, we can construct an equilibrium with an identical allocation in terms of consumption and labor supply, where the prices are time-varying and equal to $\tilde{p}_{1,t}(\theta) = \gamma^t p_1(\theta)$, $\tilde{p}_{2,t} = \gamma^t p_2$, and $\tilde{p}_{3,t} = \gamma^t p_3$. In Appendix C, we provide further details on how this equilibrium is constructed. Here, we just remark that the inflation rate between any pair of periods (s, t) and $(s, t + 1)$ is equal to γ , so that the real rate of return on money is equal to $1/\gamma$ and thus, by construction, to R .

Lucas trees. Consider an economy with a fixed endowment of “trees,” as in Lucas (1978), paying a constant real dividend $d > 0$ in every period 3. The total supply of trees is normalized to 1. In this case, we can derive an equivalence result with the baseline economy by restricting R to the interval $(0, 1/\beta]$. Consider an equilibrium of the economy with interest-bearing notes. Then, there is an equivalent Lucas tree economy where d is equal to the real value of the tax in period 3, T/p_3 , and the household’s endowment in period 3 is equal to $e_3 - d$. In the corresponding equilibrium, consumption and labor supply are identical and the price of trees in terms of consumption goods are given by $Q_1(\theta) = M/p_1(\theta)$, $Q_2 = M/p_2$ and $Q_3 = (M - T)/p_3$. Note that the rate of return of the trees between periods $(3, t)$ and $(3, t + 1)$ is

$$\frac{Q_3 + d}{Q_3} = \frac{M}{M - T},$$

which, by construction, is equal to R . Appendix C describes in detail the equilibrium allocation.

In the Lucas tree version of our model, the analogues of Propositions 2 and 3 can be stated as follows.

Proposition 6 *In the Lucas tree economy, there exist two cutoffs d^C and d^U , with $d^C < d^U$, such that if $d \geq d^U$ there exists an unconstrained equilibrium and if $d \leq d^C$ there exists a fully constrained equilibrium.*

This result gives an additional reason for labeling our two polar cases as “abundant” and “scarce” liquidity. The unconstrained equilibrium arises when the supply of the real asset, captured by the dividend d , is large, and the fully constrained equilibrium arises in the opposite case. Notice that, given the model parameters, it is possible that $d^C \leq 0$. In this case, no Lucas tree economy achieves a fully constrained equilibrium, given that $d > 0$. Whether d^C is positive or negative depends on whether \hat{R} is greater or smaller than 1, and, in general, both cases are possible.

There is a common element among the various approaches considered: the real value of the flow of transfers associated to the liquid asset in period 3. In our baseline economy, this corresponds to the net interest payments on the net money balances $(R - 1)(M - T)/p_3$, which, by the government budget constraint, is equal to T/p_3 . In the constant money growth economy, this is equal to $T_t/\tilde{p}_{3,t}$ and, in the Lucas tree economy, it corresponds to d . Given that local markets are Walrasian, the essential distortion in our economy is the limited ability of agents to do intertemporal trade. A consumer meeting a producer in periods 1 or 2 would be willing to transfer future resources in period 3 in exchange for current consumption. However, as credit contracts are not available, he is not able to promise repayment. The presence of the liquid asset allows him to circumvent this problem, since he can transfer his asset holdings to the producer. The equilibrium value of the liquid asset depends on the value of the real payoffs associated to it. The larger this payoff, the larger the equilibrium value of the asset.²²

7 Concluding Remarks

In this paper we have analyzed how different liquidity/monetary regimes affect the response of an economy to aggregate shocks. When liquid assets are scarce and agents are more likely to be liquidity constrained, the response of the economy is magnified. In this case, a complementarity in agents' trading decisions arises endogenously and amplifies the initial effect of the shock.

Our mechanism is driven by the combination of risk aversion, idiosyncratic uncertainty, and decentralized trade. All three ingredients are necessary for the mechanism to operate. Risk aversion and idiosyncratic risk give rise to an insurance problem. Decentralized trade, together with the anonymity assumption, implies that agents can only self-insure using their money holdings.²³ A nice feature of our setup is that simply by changing the real rate of return on money, we move from an environment in which idiosyncratic risk is perfectly insurable (Friedman rule) to an environment in which idiosyncratic risk is completely uninsurable (fully constrained regime). In this sense, the mechanism identified in this paper speaks more broadly about the effects of uninsurable idiosyncratic risk on aggregate behavior.

If we interpret our liquid asset strictly as a monetary instrument, as we did in our quantitative examples of Section 5, an immediate implication of our model is that high inflation regimes

²²The idea that public liquidity allows consumers to overcome lack of commitment in financial contracts is explored in Woodford (1990) and Holmstrom and Tirole (1998).

²³Reed and Waller (2006) also point out the risk sharing implications of different monetary regimes in a model *à la* Lagos and Wright (2005).

tend to exhibit higher real volatility. Looking at the empirical relation between inflation rates and aggregate volatility, across countries and across time, it is indeed possible to find a positive correlation. Even though it is clearly hard to establish the direction of causality, this correlation is consistent with our mechanism. Our results also show that high inflation is socially costly because it undermines the role of monetary savings as a form of self-insurance. From this point of view, the policy implications of our model are straightforward. Implementing the Friedman rule is optimal and, as a side product, delivers lower aggregate volatility. Notice that this result relies on the availability of lump-sum taxes. The study of optimal monetary policy with distortionary taxation in our setup remains a topic for future research.²⁴

Under a broader interpretation of liquid assets, our paper is related to the literature on the aggregate implications of imperfect risk sharing. This literature has focused on the effects of imperfect risk sharing for capital accumulation, asset prices, and the welfare cost of business cycles. Our result points to a potential effect on the response of output to aggregate shocks. It would be interesting to explore the presence and quantitative significance of our mechanism in a Bewley (1977) type economy with aggregate shocks, such as the one studied in Krusell and Smith (1989).²⁵

Finally, the coordination mechanism identified in our model could have relevant implications for the study of financial markets. In particular, one could replace the trading of goods in our model with trading of risky financial assets, as in recent search models of financial transactions, such as Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2006). This approach could be used to study the behavior of liquidity premia and trading volumes in periods of financial turmoil.

²⁴Aruoba and Chugh (2006) show that, with distortionary taxation, the Friedman rule is not optimal in a model *à la* Lagos and Wright (2005).

²⁵In Krusell and Smith (1989) the entire capital stock of the economy is a liquid asset and the presence of uninsurable idiosyncratic risk has minor effects on aggregate dynamics. To explore our mechanism, it would be interesting to assume that only a fraction of capital income is liquid.

Appendix

A. Proofs for Section 3

Proof of Proposition 1

For each $\theta > 0$ we obtain the first order conditions

$$\begin{aligned} u'(c_1(\tilde{\theta}, \theta))f(\theta) &= \lambda_1(\theta), \\ v'(n(\theta, \tilde{\theta}))f(\theta) &= \theta\lambda_1(\theta), \end{aligned}$$

where $\lambda_1(\theta)$ is the Lagrange multipliers for the resource constraints in island θ . These conditions show that the optimal $c_1(\tilde{\theta}, \theta)$ and $n(\theta, \tilde{\theta})$ are independent of $\tilde{\theta}$. Then, by market clearing $c_1(\tilde{\theta}, \theta) = y_1(\theta, \tilde{\theta})$ for all $\tilde{\theta}$, and we denote this quantity $y_1^{FB}(\theta)$. Let $n^{FB}(\theta)$ denote $n(\theta, \tilde{\theta})$. Then, the resource constraint in island θ becomes $y_1^{FB}(\theta) = \theta n^{FB}(\theta)$. Combining it with the first order conditions above gives (1). Moreover, the first order condition with respect to $c_2(\theta, \tilde{\theta})$ gives $U'(c_2(\theta, \tilde{\theta})) = \lambda_2$, which implies that c_2 is constant across households. The resource constraint in period 2 requires $c_2(\theta, \tilde{\theta}) = e_2$.

Proof of Lemma 1

The first part is an immediate corollary of Proposition 1. The second part follows from applying the implicit function theorem to the planner's optimality condition

$$\theta u'(y_1^{FB}(\theta)) = v'(y_1^{FB}(\theta)/\theta).$$

Proof of Proposition 2

We proved in the text that $\beta R = 1$ is a necessary condition for the existence of an unconstrained equilibrium. Sufficiency will be proved as the last step. First, we prove that any unconstrained equilibrium achieves a first-best allocation. Since (4) holds as an equality for all $(\theta, \tilde{\theta})$, it follows that $c_2(\theta, \tilde{\theta})$ is constant. Then, market clearing requires that $c_2(\theta, \tilde{\theta})$ be equal to e_2 for all $(\theta, \tilde{\theta})$. Substituting in (3) (for a consumer in island θ) and (6), and given that (3) holds as an equality, we obtain

$$u'(c_1(\theta)) = \frac{p_1(\theta)}{p_2} U'(e_2), \quad v'(n(\theta)) = \theta \frac{p_1(\theta)}{p_2} U'(e_2).$$

These two conditions, and market clearing in island θ , imply that

$$\theta u'(y_1(\theta)) = v'(n(\theta)),$$

where $y_1(\theta) = \theta n(\theta)$, which corresponds to the planner optimality condition (1). Therefore, labor supply and consumption in periods 1 and 2 are first-best efficient. Since any consumption allocation in period 3 is consistent with first-best efficiency, this completes the argument.

Next, we prove that equilibrium prices must take the form (9)-(11), with $\kappa \leq \hat{\kappa}$. The consumers' first order conditions immediately imply that (9)-(11) must hold for some $\kappa > 0$. To show that κ is smaller than a cutoff $\hat{\kappa}$ and to derive the value of the cutoff, we need to complete the equilibrium characterization, by deriving consumption in period 3 and money holdings. Substituting the prices (9)-(11) and first-best consumption in periods 1 and 2 in the consumer's budget constraints, and imposing the stationarity conditions $m = M$ and $m_3(\theta, \tilde{\theta}) = M/R$, we obtain

$$m_1(\tilde{\theta}) = M - \kappa u'(y_1^{FB}(\tilde{\theta})) y_1^{FB}(\tilde{\theta}), \tag{30}$$

$$m_2(\theta, \tilde{\theta}) = m_1(\tilde{\theta}) + \kappa u'(y_1^{FB}(\theta)) y_1^{FB}(\theta) - \kappa U'(e_2) e_2, \tag{31}$$

$$M/R = m_2(\theta, \tilde{\theta}) + \kappa U'(e_2) e_2 + \kappa(e_3 - c_3(\theta, \tilde{\theta})) - T. \tag{32}$$

For each pair $(\theta, \tilde{\theta})$, conditions (30)-(32), together with the government budget constraint, can be solved to derive $m_1(\theta, \tilde{\theta})$, $m_2(\theta, \tilde{\theta})$ and $c_3(\theta, \tilde{\theta})$. In particular, we obtain

$$c_3(\theta, \tilde{\theta}) = e_3 - u'(y_1^{FB}(\tilde{\theta}))y_1^{FB}(\tilde{\theta}) + u'(y_1^{FB}(\theta))y_1^{FB}(\theta), \quad (33)$$

which is independent of κ and strictly positive, given our assumption that e_3 is large. It remains to check that $m_1(\theta, \tilde{\theta})$ and $m_2(\theta, \tilde{\theta})$ are non-negative for all pairs $(\theta, \tilde{\theta})$. From (30) and (31), this requires that

$$\kappa \max_{\tilde{\theta}} \left[u'(y_1^{FB}(\tilde{\theta}))y_1^{FB}(\tilde{\theta}) \right] \leq M, \quad (34)$$

$$\kappa \max_{\theta, \tilde{\theta}} \left[u'(y_1^{FB}(\tilde{\theta}))y_1^{FB}(\tilde{\theta}) - u'(y_1^{FB}(\theta))y_1^{FB}(\theta) + U'(e_2)e_2 \right] \leq M. \quad (35)$$

Lemma 1 together with the assumption $cu''(c)/u'(c) \leq 1$ guarantees that $u'(y_1^{FB}(\theta))y_1^{FB}(\theta)$ is increasing in θ . Notice that $y_1^{FB}(0) = 0$ and $p_1(0) = \kappa u'(y_1^{FB}(0)) = \infty$, so $p_1(0)y_1^{FB}(0)$ is not well defined. We adopt the natural convention that $p_1(0)y_1^{FB}(0) = 0$. It follows that a necessary and sufficient condition for (34) and (35) is $\kappa \leq \hat{\kappa}$ where $\hat{\kappa}$ is defined in (12).

Finally, we prove that the condition $R = 1/\beta$ is sufficient for an unconstrained equilibrium to exist. To do so, we construct an equilibrium with prices (9)-(11). From the argument above, we know that labor supply and consumption in periods 1 and 2 must be at their first-best level, and consumption in period 3 must be equal to (33). Choosing any $\kappa \leq \hat{\kappa}$ ensures that money holdings are non-negative. It is straightforward to check that this allocation satisfies market clearing and that it is individually optimal, given that it satisfies the first order conditions (3)-(6).

Proof of Corollary 1

The statement follows immediately from the condition $\kappa \leq \hat{\kappa}$, the definition of $\hat{\kappa}$ (12) and the pricing condition (10).

Preliminary Results for Proposition 3

In order to prove Proposition 3, it is useful to prove several preliminary lemmas. These results will also be useful to prove Proposition 4.

The following lemmas allow us to establish that the system of functional equations (18)-(19) has a unique solution, $(p_1(\cdot), y_1(\cdot))$. To do so, we define a fixed point problem for the function $x(\cdot)$. Recall from the text that $x(\theta) \equiv p_1(\theta)y_1(\theta)$. To save on notation, in the lemmas we drop the period index and use $p(\theta)$ and $y(\theta)$.

Notice that, in an island where $\theta = 0$, output is zero and, from condition (18), the price $p(0)$ may be infinity. Hence, nominal income in island 0 may be not well defined. The natural solution is to set $x(0) = 0$. Moreover, non-negativity of consumption in period 2 requires that $x(\theta) \leq e_2$ for all θ . Therefore, we restrict attention to the set of measurable, bounded functions $x : [0, \bar{\theta}] \rightarrow [0, e_2]$ that satisfy $x(0) = 0$, which we denote by X .

Lemma 4 *Given $\theta > 0$ and a function $x \in X$, there exists a unique pair (p, y) which solves the system of equations*

$$u'(y) - p \int_0^{\bar{\theta}} U'(e_2 - py + x(\tilde{\theta})) dF(\tilde{\theta}) = 0, \quad (36)$$

$$v'\left(\frac{y}{\theta}\right) - \theta p \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}) + py) dF(\tilde{\theta}) = 0. \quad (37)$$

The pair (p, y) satisfies $py \in [0, e_2]$.

Proof. We proceed in two steps, first we prove existence, then uniqueness.

Step 1. Existence. For a given $p \in (0, \infty)$, it is easy to show that there is a unique y which solves (36) and a unique y which solves (37), which we denote, respectively, by $y^D(p)$ and $y^S(p)$. Finding a solution to (36)-(37), is equivalent to finding a p that solves

$$y^D(p) - y^S(p) = 0. \quad (38)$$

It is straightforward to prove that $y^D(p)$ and $y^S(p)$ are continuous on $(0, \infty)$. We now prove that they satisfy four properties: (a) $py^D(p) < e_2$ for all $p \in (0, \infty)$, (b) $y^S(p) < \theta\bar{n}$ for all $p \in (0, \infty)$, (c) $\limsup_{p \rightarrow 0} y^D(p) = \infty$, and (d) $\limsup_{p \rightarrow \infty} py^S(p) = \infty$. Notice that $x(0) = 0$ with positive probability, so the Inada condition for U can be used to prove property (a). Similarly, to prove property (b), we can use the assumption $\lim_{n \rightarrow \bar{n}} v'(n) = \infty$. To prove (c) notice that (a) implies $\limsup_{p \rightarrow 0} py^D(p) \leq e_2$. If $\limsup_{p \rightarrow 0} py^D(p) = e_2$, then, we immediately have $\limsup_{p \rightarrow 0} y^D(p) = \infty$. If, instead, $\limsup_{p \rightarrow 0} py^D(p) < e_2$, then there exists a $K \in (0, e_2)$ and an $\epsilon > 0$ such that $py^D(p) < K$ for all $p \in (0, \epsilon)$. Since U' is decreasing, this implies that $U'(e_2 - py^D(p) + x(\tilde{\theta}))$ is bounded above by $U'(e_2 - K) < \infty$ for all $p \in (0, \epsilon)$, which implies

$$\lim_{p \rightarrow 0} p \int_0^{\bar{\theta}} U'(e_2 - py^D(p) + x(\tilde{\theta})) dF(\tilde{\theta}) = 0.$$

Using (36), this requires $\lim_{p \rightarrow 0} u'(y^D(p)) = 0$ and, hence, $\lim_{p \rightarrow 0} y^D(p) = \infty$. To prove property (d), suppose, by contradiction, that there exist a $K > 0$ and a $P > 0$, such that $py^S(p) \leq K$ for all $p \geq P$. Then $U'(e_2 - x(\tilde{\theta}) + py^S(p))$ is bounded below by $U'(e_2 + K) > 0$ for all $p \in (P, \infty)$, which implies

$$\lim_{p \rightarrow \infty} p \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}) + py^S(p)) dF(\tilde{\theta}) = \infty. \quad (39)$$

Moreover, since $0 \leq py^S(p) \leq K$ for all $p \geq P$, it follows that $\lim_{p \rightarrow \infty} y^S(p) = 0$ and thus

$$\lim_{p \rightarrow \infty} v'(y^S(p)/\theta) < \infty. \quad (40)$$

Using equation (37), conditions (39) and (40) lead to a contradiction, completing the proof of (d). Properties (a) and (d) immediately imply $\limsup_{p \rightarrow \infty} (py^S(p) - py^D(p)) = \infty$, while (b) and (c) imply $\limsup_{p \rightarrow 0} (y^D(p) - y^S(p)) = \infty$. It follows that there exists a pair (p', p'') , with $p' < p''$, such that $y^D(p') - y^S(p') > 0$ and $y^D(p'') - y^S(p'') < 0$. By the intermediate value theorem there exists a p which solves (38). Property (a) immediately implies that $py \in [0, e_2]$, where $y = y^D(p) = y^S(p)$.

Step 2. Uniqueness. Let \hat{p} be a zero of (38), and $\hat{y} = y^D(\hat{p}) = y^S(\hat{p})$. To show uniqueness, it is sufficient to show that $dy^D(p)/dp - dy^S(p)/dp < 0$ at $p = \hat{p}$. Applying the implicit function theorem gives

$$\left[\frac{dy^D(p)}{dp} \right]_{p=\hat{p}} = \frac{\int_0^{\bar{\theta}} U'(\tilde{c}_2^D) dF(\tilde{\theta}) - \hat{p}\hat{y} \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta})}{u''(\hat{y}) + \hat{p}^2 \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta})},$$

where $\tilde{c}_2^D = e_2 - \hat{p}\hat{y} + x(\tilde{\theta})$ and

$$\left[\frac{dy^S(p)}{dp} \right]_{p=\hat{p}} = \frac{\int_0^{\bar{\theta}} U'(\tilde{c}_2^S) dF(\tilde{\theta}) + \hat{p}\hat{y} \int_0^{\bar{\theta}} U''(\tilde{c}_2^S) dF(\tilde{\theta})}{v''(\hat{y}/\theta)/\theta^2 - \hat{p}^2 \int_0^{\bar{\theta}} U''(\tilde{c}_2^S) dF(\tilde{\theta})}.$$

where $\tilde{c}_2^S = e_2 - x(\tilde{\theta}) + \hat{p}\hat{y}$. Using (36)-(37), the required inequality can then be rewritten as

$$\begin{aligned} \frac{v''(\hat{y}/\theta)}{\theta^2} \left(\frac{u'(\hat{y})}{\hat{p}} - \hat{p}\hat{y} \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta}) \right) - \frac{v'(\hat{y}/\theta)}{\theta\hat{p}} \left(u''(\hat{y}) + \hat{p}^2 \int_0^{\bar{\theta}} U''(\tilde{c}_2^D) dF(\tilde{\theta}) \right) \\ + \hat{p} \int_0^{\bar{\theta}} U''(\tilde{c}_2^S) dF(\tilde{\theta}) (u'(\hat{y}) + \hat{y}u''(\hat{y})) > 0. \end{aligned}$$

The first two terms on the left-hand side are positive. The assumption that u' has elasticity smaller than or equal to 1 implies that also the last term is positive, completing the argument. ■

Lemma 5 *Given a function $x \in X$, for any $\theta > 0$ let $(p(\theta), y(\theta))$ be the unique pair solving the system (36)-(37) and define $z(\theta) \equiv p(\theta)y(\theta)$. The function $z(\theta)$ is monotone increasing.*

Proof. Define the two functions

$$\begin{aligned} h_1(z, y; \theta) &\equiv u'(y)y - z \int_0^{\bar{\theta}} U' \left(e_2 - z + x(\tilde{\theta}) \right) dF(\tilde{\theta}), \\ h_2(z, y; \theta) &\equiv v' \left(\frac{y}{\theta} \right) \frac{y}{\theta} - z \int_0^{\bar{\theta}} U' \left(e_2 - x(\tilde{\theta}) + z \right) dF(\tilde{\theta}), \end{aligned}$$

which correspond to the left-hand sides of (36) and (37) multiplied, respectively, by y and y/θ . Lemma 4 ensures that for each $\theta > 0$ there is a unique positive pair $(z(\theta), y(\theta))$ which satisfies

$$\begin{aligned} h_1(z(\theta), y(\theta); \theta) &= 0, \\ h_2(z(\theta), y(\theta); \theta) &= 0. \end{aligned}$$

Applying the implicit function theorem, gives

$$z'(\theta) = \frac{\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \theta} - \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial \theta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (41)$$

To prove the lemma it is sufficient to show that $z'(\theta) > 0$ for all $\theta \in (0, \bar{\theta}]$. Using z and y as shorthand for $z(\theta)$ and $y(\theta)$, the numerator on the right-hand side of (41) can be written as

$$-\frac{y}{\theta^2} \left[v' \left(\frac{y}{\theta} \right) + v'' \left(\frac{y}{\theta} \right) \frac{y}{\theta} \right] [u'(y) + u''(y)y],$$

and the denominator can be written, after some algebra, as

$$\begin{aligned} [u'(y) + u''(y)y] z \int_0^{\bar{\theta}} U'' \left(e_2 - x(\tilde{\theta}) + z \right) dF(\tilde{\theta}) + \\ + \left[v' \left(\frac{y}{\theta} \right) + v'' \left(\frac{y}{\theta} \right) \frac{y}{\theta} \right] \frac{z}{\theta} \int_0^{\bar{\theta}} U'' \left(e_2 - z + x(\tilde{\theta}) \right) dF(\tilde{\theta}) + \frac{y^2}{z\theta^2} \left[u''(y)v' \left(\frac{y}{\theta} \right) \theta - u'(y)v'' \left(\frac{y}{\theta} \right) \right]. \end{aligned} \quad (42)$$

The assumption that u' has elasticity smaller than 1 ensures that both numerator and denominator are negative, completing the proof. ■

We can now define a map T from the space X into itself.

Definition 2 *Given a function $x \in X$, for any $\theta > 0$ let $(p(\theta), y(\theta))$ be the unique pair solving the system (36)-(37). Define a map $T : X \rightarrow X$ as follows. Set $(Tx)(\theta) = p(\theta)y(\theta)$ if $\theta > 0$ and $(Tx)(\theta) = 0$ if $\theta = 0$.*

The following lemmas prove monotonicity and discounting for the map T . These properties will be used to find a fixed point of T . In turns, this fixed point will be used to construct the equilibrium in Proposition 3.

Lemma 6 *Take any $x^0, x^1 \in X$, with $x^1(\theta) \geq x^0(\theta)$ for all θ . Then $(Tx^1)(\theta) \geq (Tx^0)(\theta)$ for all θ .*

Proof. For each $\tilde{\theta} \in [0, \bar{\theta}]$ and $\alpha \in [0, 1]$, define

$$x(\tilde{\theta}, \alpha) \equiv x^0(\tilde{\theta}) + \alpha \Delta(\tilde{\theta}),$$

where $\Delta(\tilde{\theta}) \equiv x^1(\tilde{\theta}) - x^0(\tilde{\theta}) \geq 0$. Notice that $x(\tilde{\theta}, 0) = x^0(\tilde{\theta})$ and $x(\tilde{\theta}, 1) = x^1(\tilde{\theta})$. Fix a value for θ and define the two functions

$$\begin{aligned} h_1(z, y; \alpha) &\equiv yu'(y) - z \int_0^{\bar{\theta}} U' \left(e_2 - z + x(\tilde{\theta}, \alpha) \right) dF(\tilde{\theta}), \\ h_2(z, y; \alpha) &\equiv v' \left(\frac{y}{\theta} \right) \frac{y}{\theta} - z \int_0^{\bar{\theta}} U' \left(e_2 - x(\tilde{\theta}, \alpha) + z \right) dF(\tilde{\theta}). \end{aligned}$$

Applying Lemma 4, for each $\alpha \in [0, 1]$ we can find a unique positive pair $(z(\alpha), y(\alpha))$ that satisfies

$$\begin{aligned} h_1(z(\alpha), y(\alpha); \alpha) &= 0, \\ h_2(z(\alpha), y(\alpha); \alpha) &= 0. \end{aligned}$$

We are abusing notation in the definition of $h_1(\cdot, \cdot; \alpha)$, $h_2(\cdot, \cdot; \alpha)$, $z(\alpha)$, and $y(\alpha)$, given that the same symbols were used above to define functions of θ . Here we keep θ constant throughout the proof, so no confusion should arise. Notice that, by construction, $(Tx^0)(\theta) = z(0)$ and $(Tx^1)(\theta) = z(1)$. Therefore, to prove our statement it is sufficient to show that $z'(\alpha) \geq 0$ for all $\alpha \in [0, 1]$.

Applying the implicit function theorem, we obtain

$$z'(\alpha) = \frac{\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \alpha} - \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial \alpha}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (43)$$

Using z and y as shorthand for $z(\alpha)$ and $y(\alpha)$, the numerator on the right-hand side of (43) can be written as

$$\begin{aligned} [u'(y) + u''(y)y]z \int_0^{\bar{\theta}} U'' \left(e_2 - x(\tilde{\theta}, \alpha) + z \right) \Delta(\tilde{\theta}) dF(\tilde{\theta}) + \\ + \frac{z}{\theta} \left[v' \left(\frac{y}{\theta} \right) + v'' \left(\frac{y}{\theta} \right) \frac{y}{\theta} \right] \int_0^{\bar{\theta}} U'' \left(e_2 - z + x(\tilde{\theta}, \alpha) \right) \Delta(\tilde{\theta}) dF(\tilde{\theta}). \end{aligned}$$

The denominator takes a form analogous to (42). Again, the assumption that u' has elasticity smaller than 1, ensures that both the numerator and the denominator are negative, completing the argument. ■

Before proving the discounting property, it is convenient to restrict the space X to the space \tilde{X} of functions bounded in $[0, \bar{z}]$ for an appropriate $\bar{z} < e_2$. The following lemma shows that the map T maps \tilde{X} into itself, and that any fixed point of T in X must lie in \tilde{X} .

Lemma 7 *There exists a $\bar{z} < e_2$, such that if $x \in X$ then $(Tx)(\theta) \leq \bar{z}$ for all θ .*

Proof. Set $\bar{x}(0) = 0$ and $\bar{x}(\theta) = e_2$ for all $\theta > 0$. Setting $x(\cdot) = \bar{x}(\cdot)$ and $\theta = \bar{\theta}$, equations (36)-(37) take the form

$$\begin{aligned} u'(y) &= p[F(0)U'(e_2 - py) + (1 - F(0))U'(2e_2 - py)], \\ v'(y/\bar{\theta}) &= \bar{\theta}p[F(0)U'(e_2 + py) + (1 - F(0))U'(py)]. \end{aligned}$$

Let (\hat{p}, \hat{y}) denote the pair solving these equations, and let $\bar{z} \equiv \hat{p}\hat{y}$. Since $F(0) > 0$ and U satisfies the Inada condition, $\lim_{c \rightarrow 0} U'(c) = \infty$, inspecting the first equation shows that $\bar{z} < e_2$. Now take any $x \in X$. Since $x(\theta) \leq \bar{x}(\theta)$ for all θ , Lemma 6 implies that $(Tx)(\theta) \leq (T\bar{x})(\theta)$. Moreover, Lemma 5 implies that $(T\bar{x})(\theta) \leq (T\bar{x})(\bar{\theta}) = \bar{z}$. Combining these inequalities we obtain $(Tx)(\theta) \leq \bar{z}$. ■

Lemma 8 *There exists a $\delta \in (0, 1)$ such that the map T satisfies the discounting property: for any $x^0, x^1 \in \tilde{X}$ such that $x^1(\theta) = x^0(\theta) + a$ for some $a > 0$, the follow inequality holds*

$$|(Tx^1)(\theta) - (Tx^0)(\theta)| \leq \delta a \text{ for all } \theta \in [0, \bar{\theta}].$$

Proof. Proceeding as in the proof of Lemma 6, define

$$x(\tilde{\theta}, \alpha) \equiv x^0(\tilde{\theta}) + \alpha \Delta(\tilde{\theta}),$$

where now $\Delta(\tilde{\theta}) = a$ for all $\tilde{\theta}$. After some algebra, we obtain

$$z'(\alpha) = \frac{\left(1 + \frac{yu''(y)}{u'(y)}\right)A + \left(1 + \frac{nv''(n)}{v'(n)}\right)B}{\left(1 + \frac{yu''(y)}{u'(y)}\right)A + \left(1 + \frac{nv''(n)}{v'(n)}\right)B + \frac{nv''(n)}{v'(n)} - \frac{yu''(y)}{u'(y)}} a, \quad (44)$$

where y and n are shorthand for $y(\alpha)$ and $y(\alpha)/\theta$ and

$$\begin{aligned} A &= -\frac{z(\alpha) \int_0^{\bar{\theta}} U''(e_2 - x(\tilde{\theta}, \alpha) + z(\alpha)) dF(\tilde{\theta})}{\int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}, \alpha) + z(\alpha)) dF(\tilde{\theta})}, \\ B &= -\frac{z(\alpha) \int_0^{\bar{\theta}} U''(e_2 - z(\alpha) + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta})}{\int_0^{\bar{\theta}} U'(e_2 - z(\alpha) + x(\tilde{\theta}, \alpha)) dF(\tilde{\theta})}. \end{aligned}$$

Now, given that $z(\alpha)$ and $x(\tilde{\theta}, \alpha)$ are both in $[0, \bar{z}]$ and $\bar{z} < e_2$, and given that U has continuous first and second derivatives on $(0, \infty)$, it follows that both A and B are bounded above. We can then find a uniform upper bound on both A and B , independent of α and of the functions x^0 and x^1 chosen. Let C be this upper bound. Given that $u''(y) \leq 0$, then

$$\left(1 + \frac{yu''(y)}{u'(y)}\right)A + \left(1 + \frac{nv''(n)}{v'(n)}\right)B \leq \left(2 + \frac{nv''(n)}{v'(n)}\right)C.$$

Therefore, (44) implies

$$z'(\alpha) \leq \left(1 + \frac{nv''(n)/v'(n) - yu''(y)/u'(y)}{(2 + nv''(n)/v'(n))C}\right)^{-1} a.$$

Recall that $\underline{\sigma} > 0$ is a lower bound for $-yu''(y)/u'(y)$. Then

$$\frac{nv''(n)/v'(n) - yu''(y)/u'(y)}{(2 + nv''(n)/v'(n))C} \geq \frac{-yu''(y)/u'(y)}{2C} \geq \frac{\underline{\sigma}}{2C}.$$

Setting

$$\delta \equiv \frac{1}{1 + \underline{\sigma}/(2C)} < 1,$$

it follows that

$$z'(\alpha) \leq \delta a$$

for all $\alpha \in [0, 1]$. Integrating both sides of the last inequality over $[0, 1]$, gives $z(1) - z(0) \leq \delta a$. By construction $(Tx^1)(\theta) = z(1)$ and $(Tx^0)(\theta) = z(0)$, completing the proof. ■

Proof of Proposition 3

We first uniquely characterize prices and allocations in a fully constrained equilibrium. Next, we will use this characterization to prove our claim. The argument in the text and the preliminary results above show that if there exists an equilibrium with $m_2(\theta, \tilde{\theta}) = 0$ for all θ and $\tilde{\theta}$, then $p_1(\theta)$ and $y_1(\theta)$ must solve the functional equations (18)-(19). To find the equilibrium pair $(p_1(\theta), y_1(\theta))$ we first find a fixed point of the map T defined above (Definition 2). Lemmas 6 and 8 show that T is a map from a space of bounded functions into itself and satisfies the assumptions of Blackwell's theorem. Therefore, a fixed point exists and is unique. Let x denote the fixed point, then Lemma 4 shows that we can find two functions $p_1(\theta)$ and $y_1(\theta)$ that satisfy (36)-(37). Since $x(\theta)$ is a fixed point of T we have $x(\theta) = p_1(\theta)y_1(\theta)$, and substituting in (36)-(37) shows that (18)-(19) are satisfied. Therefore, in a fully constrained equilibrium $p_1(\theta)$ and $y_1(\theta)$ are uniquely determined, and so is labor supply $n(\theta) = y_1(\theta)/\theta$. Moreover, from the budget constraint and the market clearing condition in period 2, consumption in period 2 is uniquely determined by $c_2(\theta, \tilde{\theta}) = e_2 - p_1(\tilde{\theta})y_1(\tilde{\theta}) + p_1(\theta)y_1(\theta)$. The price p_2 is equal to 1, as argued in the text. From the consumer's budget constraint in period 3 and the government budget constraint we obtain $c_3 = e_3$. Combining the Euler equations (3) and (5) and the envelope condition (7), p_3 is uniquely pinned down by

$$\frac{1}{p_3} = \beta R \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta}). \quad (45)$$

Finally, equilibrium money holdings are $m_1(\theta) = M - p_1(\theta)y_1(\theta)$, $m_2(\theta, \tilde{\theta}) = 0$, and $m_3(\theta, \tilde{\theta}) = M/R$.

Define the cutoff

$$\hat{R} \equiv \frac{1}{\beta} \frac{U'(c_2(\bar{\theta}, \underline{\theta}))}{\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta})}.$$

The only optimality condition that remains to be checked is the Euler equation in period 2, (4). Given the definition of $c_2(\theta, \tilde{\theta})$, Lemma 5 implies that it is an increasing function of θ and a decreasing function of $\tilde{\theta}$. It follows that a necessary and sufficient condition for (4) to hold for all $(\theta, \tilde{\theta})$ is

$$U'(c_2(\bar{\theta}, \underline{\theta})) \geq \frac{1}{p_3}. \quad (46)$$

Substituting the expression (45) for $1/p_3$ on the right-hand side, this condition is equivalent to $R \leq \hat{R}$. Therefore, if an unconstrained equilibrium exists, since $c_2(\theta, \tilde{\theta})$ is uniquely determined, condition (46) implies that $R \leq \hat{R}$, proving necessity. If $R \leq \hat{R}$, the previous steps show how to construct a fully constrained equilibrium, proving sufficiency.

Proof of Lemma 2

The second part of the Lemma follows immediately from Lemma 5. For the first part, we start from the same functions $h_1(z, y; \theta)$ and $h_2(z, y; \theta)$ defined in the proof of Lemma 5 and apply the implicit function theorem to get

$$y'(\theta) = \frac{\frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial \theta} - \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial \theta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (47)$$

To complete the proof of the lemma it is sufficient to show that $y'(\theta) > 0$ for all $\theta \in (0, \bar{\theta}]$. Using z and y as shorthand for $z(\theta)$ and $y(\theta)$, the numerator on the right-hand side of (47) can be written as

$$\frac{y}{\theta^2} \left[v' \left(\frac{y}{\theta} \right) + v'' \left(\frac{y}{\theta} \right) \frac{y}{\theta} \right] \left[z \int_0^{\bar{\theta}} U'' \left(e_2 - z + x(\tilde{\theta}) \right) dF(\tilde{\theta}) - \int_0^{\bar{\theta}} U' \left(e_2 - z + x(\tilde{\theta}) \right) dF(\tilde{\theta}) \right],$$

showing that it is negative. Finally, the denominator is equal to (42) and is negative as we have argued in the proof of Lemma 5, completing the argument.

B. Derivations and proofs for Sections 4 and 5

Equilibrium with aggregate shocks

General characterization. In a stationary equilibrium with aggregate shocks the optimality conditions (3)-(6), take the following form:

$$u'(c_1(\tilde{\theta}, \zeta)) \geq \frac{p_1(\tilde{\theta}, \zeta)}{p_2(\zeta)} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}, \zeta)) dF(\theta|\zeta) \quad (m_1(\tilde{\theta}, \zeta) \geq 0) \text{ for all } \tilde{\theta}, \zeta, \quad (48)$$

$$U'(c_2(\theta, \tilde{\theta}, \zeta)) \geq \frac{p_2(\zeta)}{p_3} \quad (m_2(\theta, \tilde{\theta}, \zeta) \geq 0) \text{ for all } \theta, \tilde{\theta}, \zeta, \quad (49)$$

$$1 = \beta R p_3 \int_{\underline{\zeta}}^{\bar{\zeta}} \int_0^{\bar{\theta}} \frac{u'(c_1(\theta, \zeta))}{p_1(\theta, \zeta)} dF(\theta) dG(\zeta), \quad (50)$$

$$v'(n(\theta, \zeta)) = \frac{\theta p_1(\theta, \zeta)}{p_2(\zeta)} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta) \text{ for all } \theta, \zeta, \quad (51)$$

where (50) is derived by substituting the envelope condition (7) in the analog of (5). Notice that condition (50) shows that, with i.i.d. shocks, p_3 is independent of the aggregate shock ζ . An equilibrium is given by prices and allocations that satisfy (51) to (50), together with the market clearing conditions and the budget constraints:

$$\begin{aligned} m_1(\tilde{\theta}, \zeta) + p_1(\tilde{\theta}, \zeta) c_1(\tilde{\theta}, \zeta) &= M, \\ m_2(\theta, \tilde{\theta}, \zeta) + p_2(\zeta) c_2(\theta, \tilde{\theta}, \zeta) &= m_1(\tilde{\theta}, \zeta) + p_1(\theta, \zeta) y_1(\theta, \zeta), \\ M/R + p_3 c_3(\theta, \tilde{\theta}, \zeta) &= p_3 e_3 + m_2(\theta, \tilde{\theta}, \zeta) + p_2(\zeta) e_2 - T. \end{aligned}$$

To compute an equilibrium it is sufficient to find prices and quantities solving the system formed by (51), the market clearing condition $c_1(\theta, \zeta) = \theta n(\theta, \zeta)$ for all θ, ζ , and equations

$$\begin{aligned} u'(c_1(\theta, \zeta)) &= \max \left\{ u' \left(\frac{M}{p_1(\theta, \zeta)} \right), \frac{p_1(\theta, \zeta)}{p_2(\zeta)} \int_0^{\bar{\theta}} U'(c_2(\tilde{\theta}, \theta, \zeta)) dF(\tilde{\theta}|\zeta) \right\} \text{ for all } \theta, \zeta, \\ c_2(\theta, \tilde{\theta}, \zeta) &= \min \left\{ \frac{M}{p_2(\zeta)} - \frac{p_1(\tilde{\theta}, \zeta)}{p_2(\zeta)} c_1(\tilde{\theta}, \zeta) + \frac{p_1(\theta, \zeta)}{p_2(\zeta)} c_1(\theta, \zeta), U'^{-1} \left(\frac{p_2(\zeta)}{p_3} \right) \right\} \text{ for all } \theta, \tilde{\theta}, \zeta, \\ &\int_0^{\bar{\theta}} \int_0^{\bar{\theta}} c_2(\theta, \tilde{\theta}, \zeta) dF(\theta) dF(\tilde{\theta}) = e_2 \text{ for all } \zeta, \end{aligned}$$

and

$$\frac{1}{p_3} = \beta R \int_{\underline{\zeta}}^{\bar{\zeta}} \int_0^{\bar{\theta}} \frac{u'(c_1(\theta, \zeta))}{p_1(\theta, \zeta)} dF(\theta|\zeta) dG(\zeta).$$

This system is written in general form, allowing for cases where the constraints $m_1(\theta, \zeta) \geq 0$ and $m_2(\theta, \theta, \zeta) \geq 0$ are binding only for a subset of, respectively, Θ and Θ^2 . Therefore, we can use it to compute equilibria for: (i) economies with $R \in (\hat{R}, 1/\beta)$ where the liquidity constraint in period 2 is non-binding for some pairs $(\theta, \tilde{\theta})$, and (ii) economies where the assumption $F(0) > 0$ is relaxed, allowing for a binding liquidity constraint in period 1 for some θ . This system is used to compute all the equilibria in Section 5.

Unconstrained and fully constrained equilibria. The characterization of an unconstrained equilibrium is straightforward, thanks to Lemma 1, which shows that the first-best allocation in period 1 is independent of the distribution $F(\cdot|\zeta)$. Let us discuss the construction of a fully constrained equilibrium for the economy with aggregate shocks (under the usual assumption $F(0|\zeta) > 0$). Now, a fully constrained equilibrium is one where $m_2(\theta, \tilde{\theta}, \zeta) = 0$ for all $\theta, \tilde{\theta}$ and ζ . In such an equilibrium $p_2(\zeta)$ is independent of ζ and equal to M/e_2 , as in the case of no aggregate shocks. Therefore, we can define a system of functional equations analogous to (18)-(19), for each ζ . Proceeding as in the proof of Proposition 3, we can then find equilibrium functions $p_1(\cdot, \zeta)$ and $y_1(\cdot, \zeta)$ for each value of ζ separately. The distribution of ζ only matters for the determination of p_3 and for the cutoff \hat{R} , which is now

$$\hat{R} \equiv \frac{1}{\beta} \frac{\min_{\zeta} \{U'(c_2(\bar{\theta}, \underline{\theta}, \zeta))\}}{\int_{\underline{\zeta}}^{\bar{\zeta}} \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta}, \zeta)) dF(\theta) dF(\tilde{\theta}) dG(\zeta)},$$

where $c_2(\theta, \tilde{\theta}, \zeta) = e_2 - p_1(\tilde{\theta}, \zeta)y_1(\tilde{\theta}, \zeta) + p_1(\theta, \zeta)y(\theta, \zeta)$.

Proof of Proposition 4

The proof proceeds in three steps. The first two steps prove that, for each θ , the nominal income in island θ is increasing with the aggregate shock ζ , that is, $x(\theta, \zeta)$ is increasing in ζ , where $x(\theta, \zeta) \equiv p_1(\theta, \zeta)y_1(\theta, \zeta)$. Using this result, the third step shows that $y_1(\theta, \zeta)$ is increasing in ζ . Consider two values ζ^I and ζ^{II} , with $\zeta^{II} > \zeta^I$. Denote, respectively, by T_I and T_{II} the maps defined in Definition 2 under the distributions $F(\theta|\zeta^I)$ and $F(\theta|\zeta^{II})$. Let x^I and x^{II} be the fixed points of T_I and T_{II} . That is, $x^I(\theta) \equiv x(\theta, \zeta^I)$ and $x^{II}(\theta) \equiv x(\theta, \zeta^{II})$ for all θ . Again, to save on notation, we drop the period index for y_1 .

Step 1. Let the function x^0 be defined as $x^0 = T_{II}x^I$. In this step, we want to prove that $x^0(\theta) > x^I(\theta)$ for all $\theta > 0$. We will prove it pointwise for each θ . Fix $\theta > 0$ and define the functions

$$\begin{aligned} h_1(z, y; \zeta) &\equiv yu'(y) - z \int_0^{\bar{\theta}} U'(e_2 - z + x^I(\tilde{\theta})) dF(\tilde{\theta}|\zeta), \\ h_2(z, y; \zeta) &\equiv v' \left(\frac{y}{\theta} \right) \frac{y}{\theta} - z \int_0^{\bar{\theta}} U'(e_2 - x^I(\tilde{\theta}) + z) dF(\tilde{\theta}|\zeta), \end{aligned}$$

for $\zeta \in [\zeta^I, \zeta^{II}]$. Lemma 4 implies that we can find a unique pair $(z(\zeta), y(\zeta))$ that satisfies

$$\begin{aligned} h_1(z(\zeta), y(\zeta); \zeta) &= 0, \\ h_2(z(\zeta), y(\zeta); \zeta) &= 0. \end{aligned}$$

Once more, we are abusing notation in the definition of $h_1(\cdot, \cdot; \zeta)$, $h_2(\cdot, \cdot; \zeta)$, $z(\zeta)$, and $y(\zeta)$. However, as θ is kept constant, there is no room for confusion. Notice that $z(\zeta^I) = x^I(\theta)$, since x^I is a fixed point of T_I , and $z(\zeta^{II}) = x^0(\theta)$, by construction. Therefore, to prove our statement we need to show that $z(\zeta^{II}) > z(\zeta^I)$. It is sufficient to show that $z'(\zeta) > 0$ for all $\zeta \in [\zeta^I, \zeta^{II}]$. Applying the implicit function theorem gives

$$z'(\zeta) = \frac{\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial \zeta} - \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial \zeta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}. \quad (52)$$

Notice that $x^I(\tilde{\theta})$ is monotone increasing in $\tilde{\theta}$, by Lemma 5, and U is strictly concave. Therefore, $U'(e_2 - z + x^I(\tilde{\theta}))$ is decreasing in $\tilde{\theta}$ and $U'(e_2 - x^I(\tilde{\theta}) + z)$ is increasing in $\tilde{\theta}$. By the properties of first-order stochastic dominance, $\int_0^{\bar{\theta}} U'(e_2 - z + x^I(\tilde{\theta})) dF(\tilde{\theta}|\zeta)$ is decreasing in ζ and $\int_0^{\bar{\theta}} U'(e_2 - x^I(\tilde{\theta}) + z) dF(\tilde{\theta}|\zeta)$ is increasing in ζ . This implies that $\partial h_1 / \partial \zeta > 0$ and $\partial h_2 / \partial \zeta < 0$. Using y as shorthand for $y(\zeta)$, the numerator on the right-hand side of (52) is, with the usual notation,

$$[u'(y) + yu''(y)] \frac{\partial h_2}{\partial \zeta} - \frac{1}{\theta} \left[v' \left(\frac{y}{\theta} \right) + v'' \left(\frac{y}{\theta} \right) \frac{y}{\theta} \right] \frac{\partial h_1}{\partial \zeta}.$$

The denominator is the analogue of (42). Once more, the assumption that $yu''(y)/u'(y) \geq -1$ ensures that both numerator and denominator are negative, completing the argument.

Step 2. Define the sequence of functions (x^0, x^1, \dots) in X , using the recursion $x^{j+1} = T_{II}x^j$. Since, by step 1, $x^0 \geq x^I$ (where by $x^0 \geq x^I$ we mean $x^0(\theta) \geq x^I(\theta)$ for all $\theta > 0$) and, by Lemma 6, T_{II} is a monotone operator, it follows that this sequence is monotone, with $x^{j+1} \geq x^j$. Moreover, T_{II} is a contraction by Lemmas 6 and 8, so this sequence has a limit point, which coincides with the fixed point x^{II} . This implies that $x^{II} \geq x^0$ and, together with the result in step 1, shows that $x^{II} > x^I$, as we wanted to prove.

Step 3. Fix $\theta > 0$ and, with the usual abuse of notation, define the functions

$$\begin{aligned} h_1(z, y; \zeta) &\equiv yu'(y) - z \int_0^{\bar{\theta}} U'(e_2 - z + x(\tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta), \\ h_2(z, y; \zeta) &\equiv v' \left(\frac{y}{\theta} \right) \frac{y}{\theta} - z \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}, \zeta) + z) dF(\tilde{\theta}|\zeta). \end{aligned}$$

Notice the difference with the definitions of h_1 and h_2 in step 1, now $x(\tilde{\theta}, \zeta)$ replaces $x^I(\tilde{\theta})$. The functions $z(\zeta)$ and $y(\zeta)$ are defined in the usual way. Applying the implicit function theorem, we get

$$y'(\zeta) = \frac{\frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial \zeta} - \frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial \zeta}}{\frac{\partial h_1}{\partial z} \frac{\partial h_2}{\partial y} - \frac{\partial h_2}{\partial z} \frac{\partial h_1}{\partial y}}.$$

To evaluate the numerator, notice that

$$\begin{aligned} \frac{\partial h_1}{\partial z} &= - \int_0^{\bar{\theta}} U'(e_2 - z + x(\tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta) + z \int_0^{\bar{\theta}} U''(e_2 - z + x(\tilde{\theta}, \zeta)) dF(\tilde{\theta}|\zeta) < 0, \\ \frac{\partial h_2}{\partial z} &= - \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}, \zeta) + z) dF(\tilde{\theta}|\zeta) - z \int_0^{\bar{\theta}} U''(e_2 - x(\tilde{\theta}, \zeta) + z) dF(\tilde{\theta}|\zeta) \leq \\ &\leq - \int_0^{\bar{\theta}} \left[U'(e_2 - x(\tilde{\theta}, \zeta) + z) + (e_2 - x(\tilde{\theta}, \zeta) + z) U''(e_2 - x(\tilde{\theta}, \zeta) + z) \right] dF(\tilde{\theta}|\zeta) \leq 0, \end{aligned}$$

where the last inequality follows from the assumption that U' has elasticity less than or equal to 1 (this is the only place where this assumption is used). Furthermore, notice that

$$\frac{\partial h_1}{\partial \zeta} = -z \int_0^{\bar{\theta}} U''(e_2 - z + x(\tilde{\theta}, \zeta)) \frac{\partial x(\tilde{\theta}, \zeta)}{\partial \zeta} dF(\tilde{\theta}|\zeta) - z \int_0^{\bar{\theta}} U'(e_2 - z + x(\tilde{\theta}, \zeta)) \frac{\partial f(\tilde{\theta}|\zeta)}{\partial \zeta} d\tilde{\theta} > 0$$

where the first element is positive from steps 1 and 2, and the second element is positive because ζ leads to a first order stochastic increase in $\tilde{\theta}$ and $U'(e_2 - z + x(\tilde{\theta}, \zeta))$ is decreasing in $\tilde{\theta}$. A similar reasoning shows that

$$\frac{\partial h_2}{\partial \zeta} = z \int_0^{\bar{\theta}} U''(e_2 - x(\tilde{\theta}, \zeta) + z) \frac{\partial x(\tilde{\theta}, \zeta)}{\partial \zeta} dF(\tilde{\theta}|\zeta) + z \int_0^{\bar{\theta}} U'(e_2 - x(\tilde{\theta}, \zeta) + z) \frac{\partial f(\tilde{\theta}|\zeta)}{\partial \zeta} d\tilde{\theta} < 0.$$

Putting together the four inequalities just derived shows that the numerator is negative. The denominator takes the usual form, analogous to (42), and is negative. This completes the proof.

C. Results and proofs for Section 6

Proof of Proposition 5

From expression (29) it is immediate to obtain

$$\begin{aligned}\frac{\partial Y_1(\zeta, \xi)}{\partial \zeta} &= \int_0^{\bar{\theta}} y_1(\theta, \xi) \frac{\partial f(\theta|\zeta)}{\partial \zeta} d\theta, \\ \frac{\partial Y_1(\zeta, \xi)}{\partial \xi} &= \int_0^{\bar{\theta}} \frac{\partial y_1(\theta, \xi)}{\partial \xi} dF(\theta|\zeta).\end{aligned}$$

In the case of an unconstrained equilibrium, with $R = 1/\beta$, the analogue of Proposition 2 can be easily derived, showing that $\partial y_1(\theta, \xi)/\partial \xi = 0$ and $\partial y_1(\theta, \xi)/\partial \theta > 0$. These properties imply that $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi = 0$.

Consider a fully constrained equilibrium, with $R < \hat{R}$. For each value of ξ , the functions $p_1(\theta, \xi)$ and $y_1(\theta, \xi)$ can be derived solving the following system of functional equations, analogous to (23)-(24):

$$\begin{aligned}u'(y_1(\theta, \xi)) &= p_1(\theta, \xi) \int_0^{\bar{\theta}} U'(e_2 - p_1(\theta, \xi)y_1(\theta, \xi) + p_1(\tilde{\theta}, \xi)y_1(\tilde{\theta}, \xi))dF(\tilde{\theta}|\xi, \theta), \\ v'\left(\frac{y_1(\theta, \xi)}{\theta}\right) &= \theta p_1(\theta, \xi) \int_0^{\bar{\theta}} U'(e_2 - p_1(\tilde{\theta}, \xi)y_1(\tilde{\theta}, \xi) + p_1(\theta, \xi)y_1(\theta, \xi))dF(\tilde{\theta}|\xi, \theta).\end{aligned}$$

The only formal difference between these and (23)-(24) is that the distribution $F(\tilde{\theta}|\xi, \theta)$ depends also on θ . However, this does not affect any of the steps of Proposition 3 (there is only a minor difference in the proof of the analogue of Lemma 5, details available on request). Therefore, this system has a unique solution for each ξ . Next, following the steps of Lemma 2 and Proposition 4, we can show that $y_1(\theta, \xi)$ is increasing in θ and ξ . This implies that $\partial Y_1(\zeta, \xi)/\partial \zeta > 0$ and $\partial Y_1(\zeta, \xi)/\partial \xi > 0$.

Mapping with alternative models of liquidity supply

Constant money growth. Let $p_1(\theta), p_2, p_3, m_1(\theta), m_2(\theta, \tilde{\theta}), m_3(\theta, \tilde{\theta})$, and $c_1(\theta), c_2(\theta, \tilde{\theta}), c_3(\theta, \tilde{\theta})$ be prices, money balances, and consumption levels in a stationary equilibrium of the economy with interest-bearing notes. To construct the corresponding equilibrium of the constant money growth economy, let $M_0 = M$, $\gamma = R^{-1}$ and $T_t = T/\gamma^t$. As in the text, set the prices

$$\begin{aligned}\tilde{p}_{1,t}(\theta) &= \gamma^t p_1(\theta), \\ \tilde{p}_{s,t} &= \gamma^t p_s, \text{ for } s = 2, 3.\end{aligned}$$

Take the same consumption functions of the economy with interest-bearing notes and set the money balance functions as follows

$$\tilde{m}_{1,t}(\theta) = \gamma^t m_1(\theta), \quad \tilde{m}_{2,t}(\theta, \tilde{\theta}) = \gamma^t m_2(\theta, \tilde{\theta}), \quad \tilde{m}_{3,t}(\theta, \tilde{\theta}) = M_{t+1}.$$

The household's budget constraints, for a household beginning with M_t on date t , are now

$$\begin{aligned}\tilde{m}_{1,t}(\tilde{\theta}) + \tilde{p}_{1,t}(\tilde{\theta})c_1(\tilde{\theta}) &= M_t, \\ \tilde{m}_{2,t}(\theta, \tilde{\theta}) + \tilde{p}_{2,t}c_2(\theta, \tilde{\theta}) &= \tilde{m}_{1,t}(\tilde{\theta}) + \tilde{p}_{1,t}(\theta)y_1(\theta), \\ \tilde{m}_{3,t}(\theta, \tilde{\theta}) + \tilde{p}_{3,t}c_3(\theta, \tilde{\theta}) &= \tilde{p}_{3,t}y_3 + \tilde{m}_{2,t}(\theta, \tilde{\theta}) + \tilde{p}_{2,t}y_2 - T_t.\end{aligned}$$

Some algebra shows that they are satisfied. Moreover, all intertemporal prices are unchanged, as $\tilde{p}_{1,t}(\theta)/\tilde{p}_{2,t} = p_1(\theta)/p_2$, $\tilde{p}_{2,t}/\tilde{p}_{3,t} = p_2/p_3$ and

$$\frac{\tilde{p}_{3,t}}{\tilde{p}_{1,t+1}(\theta)} = \frac{1}{\gamma} \frac{\tilde{p}_{3,t}}{\tilde{p}_{1,t}(\theta)} = R \frac{p_3}{p_1(\theta)}.$$

Substituting these prices in the household's first order conditions shows that the allocation is individually optimal, completing the equilibrium construction.

Lucas trees. Starting with an equilibrium of the baseline economy, set the prices as in the text:

$$Q_1(\theta) = \frac{M}{p_1(\theta)}, \quad Q_2 = \frac{M}{p_2}, \quad Q_3 = \frac{M - T}{p_3}.$$

Let h_1, h_2 and h_3 denote holdings of the real asset and set their values to

$$h_1(\theta) = \frac{m_1(\theta)}{M}, \quad h_2(\theta, \tilde{\theta}) = \frac{m_2(\theta, \tilde{\theta})}{M}, \quad h_3(\theta, \tilde{\theta}) = 1.$$

The household's budget constraints, for a household beginning with 1 unit of the real asset on date t , are:

$$\begin{aligned} Q_1(\tilde{\theta})h_1(\tilde{\theta}) + c_1(\tilde{\theta}) &= Q_1(\tilde{\theta}), \\ Q_2h_2(\theta, \tilde{\theta}) + c_2(\theta, \tilde{\theta}) &= Q_2h_1(\tilde{\theta}) + Q_2(y_1(\theta)/Q_1(\theta)), \\ Q_3h_3(\theta, \tilde{\theta}) + \tilde{c}_3(\theta, \tilde{\theta}) &= e_3 - d + (Q_3 + d)(h_2(\theta, \tilde{\theta}) + e_2/Q_2). \end{aligned}$$

Some algebra shows that these constraints are satisfied. Moreover, it can be checked that all the household optimality conditions are satisfied, given that all intertemporal prices are unchanged.

Proof of Proposition 6

To prove these two results, we exploit the mapping between the baseline economy and the Lucas tree economy derived above. Consider the baseline economy with interest-bearing notes. Let $\hat{\kappa}$ be defined as in Proposition 2, for a given value of T . Proposition 2 ensures that for any $\kappa \leq \hat{\kappa}$ there exists an unconstrained equilibrium of the baseline economy with $R = 1/\beta$. Define the cutoff d^U as

$$d^U \equiv \frac{T}{\hat{\kappa}}.$$

Then, for any $d \geq d^U$, we can find the correspondent unconstrained equilibrium in a Lucas tree economy with $d = T/\kappa$.

For the second part, recall that Proposition 3 ensures that we can find a fully constrained equilibrium of the baseline economy for any $R \leq \hat{R}$. In that case, using (45) and the normalization $M = e_2$, real balances in period 3 are:

$$\frac{M}{p_3} = e_2 R \phi,$$

where

$$\phi \equiv \beta \int_0^{\bar{\theta}} \int_0^{\bar{\theta}} U'(c_2(\theta, \tilde{\theta})) dF(\theta) dF(\tilde{\theta}).$$

Notice that ϕ is a constant which is equal in all fully constrained equilibria (in particular, it is independent of R). Using the government budget constraint, this implies that for each $R \leq \hat{R}$ the equilibrium real value of the tax T is given by $T/p_3 = (R - 1)\phi e_2$. Define

$$d^C \equiv (\hat{R} - 1)\phi e_2.$$

For any $d \leq d^C$, we can find the correspondent fully constrained equilibrium in a Lucas tree economy, with $d = (R - 1)\phi e_2$.

References

- [1] Acemoglu, Daron and Fabrizio Zilibotti, “Was Prometheus Unbounded by Chance? Risk, Diversification and Growth,” *Journal of Political Economy*, 1997, 105, 709-751.
- [2] Aruoba, S. Borogan and Sanjay K. Chugh, “Optimal Fiscal and Monetary Policy When Money is Essential,” FRB International Finance Discussion Paper No. 880, 2006.
- [3] Bencivenga, Valerie and Bruce Smith, “Financial Intermediation and Endogenous Growth,” *Review of Economic Studies*, 1991, 58, 195-209.
- [4] Berentsen, Aleksander, Camera, Gabriele and Christopher Waller, “The Distribution Of Money Balances And The Nonneutrality Of Money,” *International Economic Review*, 2005, 46, 465-486.
- [5] Bernanke, Ben and Mark Gertler, “Agency Costs, Net Worth, and Business Fluctuations,” *American Economic Review*, 1989, 79, 14-31.
- [6] Bewley, Truman, “The Permanent Income Hypothesis: A Theoretical Formulation,” *Journal of Economic Theory*, 1977, 16, 252-292.
- [7] Cochrane, John, “Asset Pricing Program Review: Liquidity, Trading and Asset Prices,” Winter 2005 NBER reporter.
- [8] Craig, Ben and Guillaume Rocheteau, “Inflation and Welfare: a Search Approach,” 2007, mimeo.
- [9] Diamond, Peter, “Aggregate Demand Management in Search Equilibrium,” *Journal of Political Economy*, 1982, 90, 881-94.
- [10] Diamond, Peter, “Money in Search Equilibrium,” *Econometrica*, 1984, 52, 1-20.
- [11] Duffie, Darrell, Nicolae Garleanu and Lasse Heje Pedersen, “Over-the-counter markets,” *Econometrica*, 2005, 73, 1815-1847.
- [12] Holmstrom, Bengt and Jean Tirole, “Private and Public Supply of Liquidity,” *Journal of Political Economy*, 1998, 106, 1-40.

- [13] Krishnamurthy, Arvind, “The Bond/Old-Bond Spread,” *Journal of Financial Economics*, 2002, 66, 463-506.
- [14] Kiyotaki, Nobuhiro and John H. Moore, “Credit Cycles,” *Journal of Political Economy*, 1997, 105, 211-248.
- [15] Kiyotaki, Nobuhiro and John H. Moore, “Liquidity, Business Cycles and Monetary Policy,” 2001, mimeo, Princeton University.
- [16] Kiyotaki, Nobuhiro and Randall Wright, “On Money as a Medium of Exchange,” *Journal of Political Economy*, 1989, 97, 927-954.
- [17] Krusell, Per and Anthony A. Smith, Jr., “Income and Wealth Heterogeneity in the Macroeconomy,” *The Journal of Political Economy*, 1998, 106, 867-896.
- [18] Lagos, Ricardo and Randall Wright, “A Unified Framework of Money Theory and Policy Analysis,” *Journal of Political Economy*, 2005, 113, 463-484.
- [19] Lagos, Ricardo and Guillaume Rocheteau, “Search in Asset Markets,” Federal Reserve Bank of Minneapolis Staff Report 375, 2006.
- [20] Longstaff, Francis, “The Flight-to-Liquidity Premium in U.S. Treasury Bond Prices,” *Journal of Business*, 2004, 77, 511-526.
- [21] Lucas, Robert and Edward Prescott, “Equilibrium Search and Unemployment,” *Journal of Political Economy*, 1974, 7, 188-209.
- [22] Lucas, Robert, “Asset Prices in an Exchange Economy,” *Econometrica*, 1978, 46, 1429–1444.
- [23] Lucas, Robert, “Inflation and Welfare,” *Econometrica*, 2000, 68, 247-274.
- [24] Milgrom, Paul, “Good News and Bad News: Representation Theorems and Applications,” *Bell Journal of Economics*, 1981, 12, 380-91.
- [25] Parker, Jonathan and Bruce Preston, “Precautionary Saving and Consumption Fluctuations,” *American Economic Review*, 2005, 95, 1119-43.
- [26] Reed, Robert and Christopher Waller, “Money and Risk Sharing,” *Journal of Money, Credit, and Banking*, 2006, 38, 6, 1599-1618.

- [27] Rocheteau, Guillaume and Randall Wright, “Money in Competitive Equilibrium, in Search Equilibrium, and in Competitive Search Equilibrium,” *Econometrica*, 2005, 73, 175-202.
- [28] Shi, Shouyong, “A Divisible Search Model of Fiat Money,” *Econometrica*, 1997, 65, 75-102.
- [29] Woodford, Michael, “Public Debt as Private Liquidity,” *American Economic Review*, 1990, 77, 2, 93-98.