Discussion paper:

The impact of stochastic convenience yield on long-term forestry investment decisions

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Abstract

The impact of stochastic convenience yield on long-term forestry investment decisions

This paper investigates whether convenience yield is an important factor in determining optimal decisions for a forestry investment. The Kalman filter method is used to estimate three different models of lumber prices: a mean reverting model, a simple geometric Brownian motion and the two-factor price model due to Schwartz (1997). In the latter model there are two correlated stochastic factors: spot price and convenience yield. The two-factor model is shown to provide a reasonable fit of the term structure of lumber futures prices. The impact of convenience yield on a forestry investment decision is examined using the Schwartz (1997) long-term model which transforms the two-factor price model into a single factor model with a composite price. Using the long-term model an optimal harvesting problem is analyzed, which requires the numerical solution of a Hamilton-Jacobi-Bellman equation. We compare the results for the long-term model to those from single-factor mean reverting and geometric Brownian motion models. The inclusion of convenience yield through the long-term model is found to have a significant impact on land value and optimal harvesting decisions.

1 Introduction

The optimal management of natural resource investments typically depends on the ability of resource owners to interpret and react to volatile commodity prices. Owners of commercial forest land are no exception to this. Landowners are faced with decisions about when to harvest a stand of trees in an environment of highly uncertain timber prices which respond to news about the health of the economy, tariffs and trade barriers, as well as supply side factors such as fire and pests. A long strand of economics literature addresses the dual issues how best to model commodity prices and the determination of optimal resource management decisions under different representations of price. The literature has evolved significantly over the past few decades moving from deterministic models based on versions of Hotelling's rule to stochastic models that draw on finance theory and contingent claims arguments. In addition to stochastic prices, the natural resources literature has investigated the impact of other key uncertain parameters, such as costs, interest rates, and convenience yield, on optimal natural resource management.

The focus of this paper is on lumber prices and optimal decisions in forestry. A number of specifications have been proposed in the literature for modeling stochastic lumber prices, including geometric Brownian motion (GBM), mean reverting processes, jump processes and regime-switching models. For example, Clarke and Reed (1989) and Yin and Newman (1997) solve optimal tree harvesting problems analytically by assuming lumber prices follow GBM. Some researchers including Brazee et al. (1999) have found that mean reversion rather than GBM provides a better characterization of lumber prices. Saphores et al. (2002) find evidence of jumps in Pacific North West stumpage prices in the U.S. and demonstrate at the stand level that ignoring jumps can lead to significantly suboptimal harvesting decisions for old growth timber. A recent insight in the literature suggests that instead of modeling jumps in commodity prices, we may consider regime-switching models, initially proposed by Hamilton (1989), to better capture the main characteristics of lumber prices. Chen and Insley (2008) compare and contrast a two-state regime-switching mean reverting model and

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a traditional mean reverting model. They find that the regime-switching model outperforms the traditional one-factor mean reverting model in terms of fitting prices of market lumber derivatives.

For storable commodities and those that serve as inputs to production, such as lumber and oil, convenience yield[1] plays an important role in price formation. Convenience yield refers to the benefit that producers obtain from holding physical inventories, a benefit not available to individuals holding a futures or forward contract. Convenience yield is expected to be negatively correlated with inventories levels.[2] The seasonal harvesting of trees, as well as the importance of wood products as inputs to other industries, suggest that convenience yield may be important to understanding the dynamics of timber prices.

From a modelling perspective, convenience yield may be viewed as analogous to the dividend obtained from holding a company's stock. Convenience yield helps to explain the relationship between spot prices and futures prices - i.e. the term structure of commodity futures prices. The term structure conveys useful information for hedging or investment decisions, because it synthesizes the information available in the market and reflects the investors' expectations concerning the future. A futures price can be greater or less than the commodity spot price, depending on the relationship between the (net) convenience yield[3] and risk-free interest rate. This is explained by the cost of carry pricing model which expresses forward/futures price as a function of the spot price and the cost of carry.[4] Modelling of convenience yield is important for any analysis of futures prices.

Multi-factor models have been proposed in the literature to describe commodity price dynamics by including stochastic convenience yield to help explain the term structure of commodity futures prices. For example, Gibson and Schwartz (1990) first introduced a two-

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[1] See Working (1948)

[2] See Brennan (1958) and Litzenberger and Rabinowitz (1995).

[3] Net convenience yield is defined as the benefit of holding inventory minus physical storage costs. It is negative if the storage expense is higher. For simplicity, convenience yield mentioned in the rest of this paper refers to net convenience yield.

[4] Cost of carry is defined as the physical storage cost plus the forgone interest. See Pindyck (2001).

factor model, where spot prices are assumed to evolve according to GBM and the convenience yield follows a mean reverting stochastic process. Schwartz (1997) further explores this two-factor model in the context of a term structure model of commodity prices. This model provides a reasonable fit of the term structure of long-term forward prices which are essential for valuing long-term commodity linked investments. The Schwartz (1997) two-factor model has been successfully applied in the modelling of several key commercial commodities, including crude oil and copper. However, to the best of our knowledge no previous work has examined the impact of modeling stochastic convenience yield in an optimal harvesting problem applied to a renewable natural resource such as timber. In this paper, we will investigate the implication of including stochastic convenience yield on the behavior of lumber prices and long-term forestry investment decisions.

The objective of this paper is to further our understanding of the optimal valuation and harvesting of commercial forest land by using lumber future prices to estimate a two-factor model of lumber prices that includes stochastic convenience yield. The approach for estimating the price model uses the Kalman filter as is done in Schwartz (1997). We compare the ability of this two-factor model to match the term structure of lumber futures prices with that of two other simpler models which do not include stochastic convenience yield. These simpler models represent GBM and mean reverting processes. We then use the estimated price models in a multi-rotational optimal tree harvesting problem.

A real options model of the forestry investment valuation is developed assuming a joint stochastic process of lumber prices and convenience yield. The Hamilton-Jacobi-Bellman (HJB) equation characterizing the value of the option to harvest a stand contains three stochastic variables: lumber prices, convenience yield and the stand age. To simplify the solution of the harvesting problem, we use the one-factor model introduced in Schwartz (1998), which retains most of the main characteristics of his two-factor model, especially its ability to fit long-term commodity futures prices. We call this one-factor model the "long-term model". The HJB equation derived using the long run model is solved numerically using the combination of the fully implicit finite difference method, the semi-Lagrangian method

and the penalty method. The optimal harvesting decisions and land value computed for the long-term model are then compared with results using the simple GBM and mean reverting models.

Our main conclusion is that modelling stochastic convenience yield improves our ability to match lumber futures prices and that the long run model provides reasonable estimates of land value and optimal harvesting decisions. The long-run model gives significantly different results than the other simpler models that are used for comparison.

The remainder of the paper is organized as follows. In Section 2 we give a brief description of lumber spot and futures prices. Section 3 describes the two-factor price model as well as the GBM and mean reverting models used for comparison. Section 4 describes the estimation of the price models using the Kalman filter. Section 5 presents the long-term model which is used as an approximation of the two-factor model. Section 6 describes the empirical results of the price model estimation and compares the ability to the different models to match the term structure of futures prices. Section 7 presents the real options model and analysis of an optimal tree harvesting problem. Section 8 concludes.

2   Lumber spot prices and futures prices

Forest products are traded worldwide and Canada is a major player in this market, accounting for 14% of the value of world forest product exports in 2006. There are two types of lumber, softwood and hardwood, with softwood lumber generally used in construction, building and housing purposes. It is also the underlying asset of the lumber futures contract traded in the futures market. There is no single spot market in which a uniform lumber product is traded, and therefore there is no unique lumber spot price. However, there is a single North American market for standard lumber futures contracts. Following the literature[5], the price of the futures contract which is closest to maturity is treated as the lumber spot price.

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[5]See Gibson and Schwartz (1990).

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Item	Max	Min	Mean	Std. Dev.	Skewness	Kurtosis
$ Cdn/MBF	718.1	164.8	423.7	107.6	-0.09	2.50

TABLE 1: *Descriptive statistics for the lumber price time series (as shown in Figure 1), from January 6th, 1995 to April 25th, 2008.*

Lumber futures contracts were first traded on the Chicago Mercantile Exchange (CME) in 1969. The Random Length Lumber futures traded on the CME are for on-track mill delivery of 110,000 board feet (plus or minus 5,000 board feet) of random length 8-foot to 20-foot nominal 2-inch × 4-inch pieces. The delivery contract months for CME Random Length Lumber futures are as follows: January, March, May, July, September and November. The last trading day of each contract is the business day prior to the 16th calendar day of the contract month.

Real spot prices, as approximated by the prices of the lumber futures contract closest to maturity, are shown in Figure 1[6]. These are weekly data, covering the period from January 1995 to April 2008. The original data in U.S. dollars were deflated by the CPI and converted to Canadian dollars. The transformation is made because our real options application is a hypothetical decision problem in Canada's boreal forest. In Figure 1 prior to 2006 there appears to be a tendency to revert to a mean between $400 and $500 (Cdn) per MBF. After 2006 we see a downward progression in price reflecting weak North American lumber markets as well as the impact of a strengthening Canadian dollar. We also observe a significant level of volatility. Summary statistics of the spot lumber prices are reported in Table 1.

There are six lumber futures contracts traded each day on the CME, the first four of which will be used in our analysis as these have the highest trading volumes and can be expected to provide more accurate information. Real weekly prices of the four selected lumber futures prices, ranging from January 1995 to April 2008, are shown in Figure 2. Summary statistics of these four time series are provided in Table 2. The mean price shown in Table 2 is lowest for the shortest maturity contract and rises with contract maturity. Conversely, the largest

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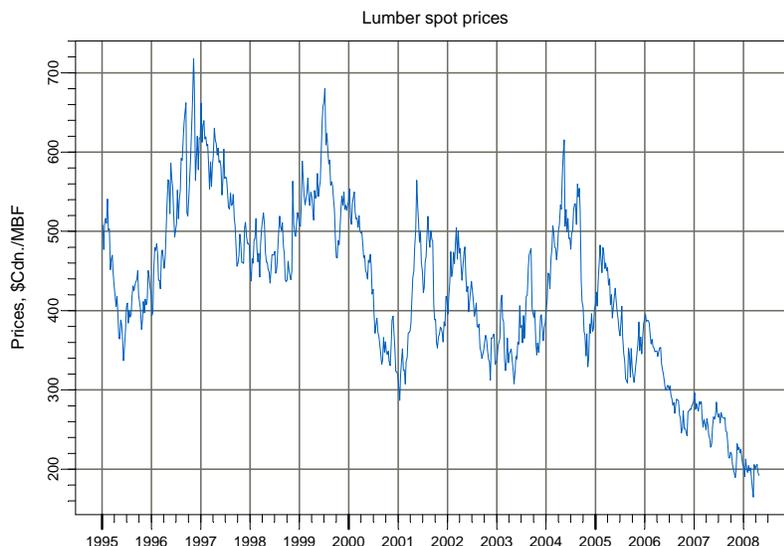[6]Data source: Chicago Mercantile Exchange.

FIGURE 1: *Real prices of lumber futures contract closed to maturity. Weekly data from January 6th, 1995 to April 25th, 2008, \$Cdn./MBF, (MBF $\equiv$ thousand board feet). Nominal prices deflated by the Consumer Price Index, base year = 2005.*

volatilities are for the prices of the short-term contracts, while the volatilities for the two longer term contracts are fairly close. The decreasing pattern of volatilities along the prices curve is often called "the Samuelson effect" in the literature. The term structure of lumber futures shown in Figure 3 provides an illustration of the Samuelson effect. In the diagram the spread of futures prices at 4 (F4) is smaller than at 1 (F1), indicating that the near term prices are more volatile. From this graph, we observe different shapes of the lumber term structure, from backwardation to contango for example.

3   Valuation models

The varying shapes of term structure of lumber futures prices shown in Figure 3 imply the need to model convenience yield in order to capture the main characteristics of lumber spot and futures prices. The two-factor model analyzed in Schwartz (1997) is one of the most
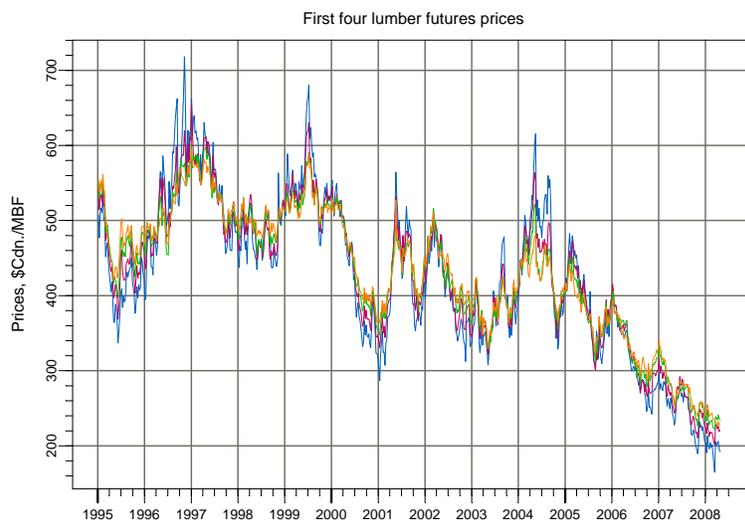
First four lumber futures prices

FIGURE 2: *Real prices of four CME lumber futures,$Cdn./MBF (thousand board feet).*
Weekly data from January 6th, 1995 to April 25th, 2008. Nominal prices deflated by the
Consumer Price Index, base year = 2005.

Item	Number of observations	Mean	Std. Deviation	Maturity(on average)
F1	695	423.7	107.57	1 mon
F2	695	427.5	96.24	3 mon
F3	695	429.6	89.08	5 mon
F4	695	431.5	86.15	7 mon

TABLE 2: *Summary statistics of four chosen CME lumber futures prices, $Cdn./MBF.*
Weekly data from January 6th, 1995 to April 25th, 2008.
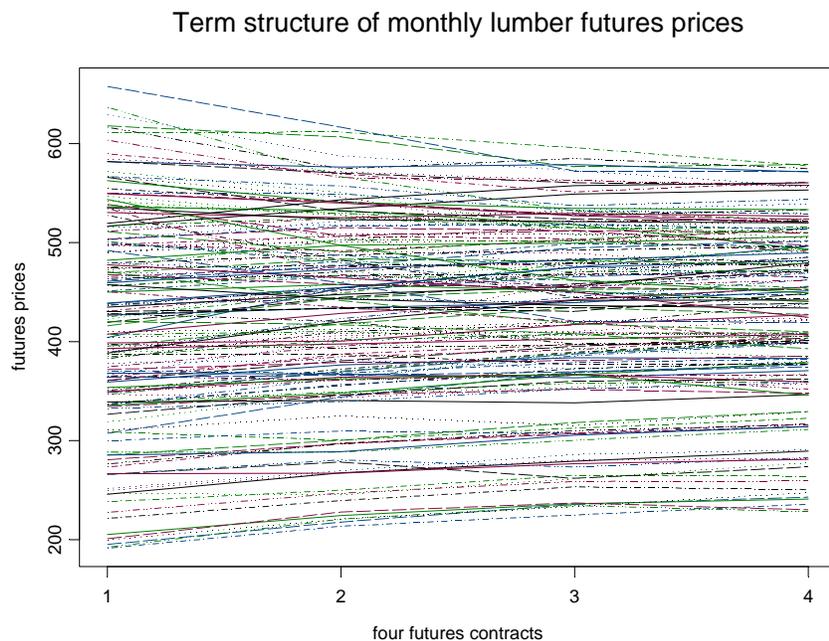
Term structure of monthly lumber futures prices

FIGURE 3: *Term structure of lumber futures,\$Cdn./MBF. Monthly data from January 6th, 1995 to April 25th, 2008.*

popular models in this literature and it has been successfully employed to model several commodities, including crude oil and copper. For the convenience of the reader, in the next section we summarize the Schwartz (1997) two-factor model. We also present two one-factor models, GBM and mean reverting, to be used as comparison with the two-factor model. These one-factor models are also popular in the literature and are simpler to estimate and use in models of investment decisions than the two-factor model. We would like to determine whether the two-factor model does a substantially better job at modelling lumber prices and is therefore worth using despite its increased complexity.

3.1 The Schwartz (1997) two-factor model

The two-factor model analyzed in Schwartz (1997) is based on the model developed in Gibson and Schwartz (1990). Specifically, the spot price S follows a GBM process with a stochastic drift and the net convenience yield δ is formulated as a mean-reverting Ornstein-Uhlenbeck

process. The joint stochastic process of the two state variables in Schwartz (1997) is given by:

$$
\begin{aligned}
dS &= (\mu - \delta)Sdt + \sigma_s S dz_s \\
d\delta &= \kappa(\alpha - \delta)dt + \sigma_\delta dz_\delta \\
dz_s dz_\delta &= \rho dt
\end{aligned}
\tag{1}
$$

where μ is the expected return of spot prices, κ and α represent the mean reversion rate and the long-run equilibrium level of convenience yield respectively, σ_s and σ_δ denote the volatilities of the two state variables, and ρ is the correlation coefficient between the two standard Brownian increments dz_s and dz_δ.

We note in the above specification that μ represents the total expected return from S and it remains constant. As the convenience yield changes the portion of total return that derives from capital gains, $(\mu - \delta)$, and the portion that derives from convenience yield adjusts stochastically while the total return is assumed fixed and determined by the market equilibrium return for that particular asset class.

We expect convenience yield and the commodity price to be positively correlated. Intuitively, when there is excess supply on the market, lumber inventories will rise and the spot price should fall. Convenience yield should also fall since the benefit of owning the commodity is smaller than when the commodity is scarce. A lower convenience yield implies it is more costly to carry commodity inventories. This will tend to drive up the futures price as it becomes more attractive to secure supply in the futures market rather than carrying inventory.

In Equation (1) convenience yield affects S through the correlation coefficient as well as through the drift term. With a positive ρ, a fall in S implies a fall in δ. This lower δ increases the drift rate for S, and hence S is pulled up again. Hence in the model specified in Equation (1), S is characterized by some reversion to the mean, but the mean is not constant.

The specification of convenience yield as a mean reverting process also makes intuitive sense. α represents a long run level that reflects the cost of storing the commodity and a

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benefit conveyed by having immediate access to inventories. δ will vary around α depending on commodity market conditions with $\delta > \alpha$ when markets are buoyant and the reverse when markets are depressed.

The inverse relationship between the level of inventory and the convenience yield prevents the possibility that the net convenience yield goes to infinity. This relationship holds for storable commodities including lumber.

Under the equivalent martingale measure, the risk adjusted processes for the two state variables, the spot price S and convenience yield δ, are expressed as:

$$
\begin{aligned}
dS &= (r - \delta)Sdt + \sigma_s S dz_s^* \\
d\delta &= [\kappa(\alpha - \delta) - \lambda]dt + \sigma_\delta dz_\delta^* \\
dz_s^* dz_\delta^* &= \rho dt
\end{aligned}
\tag{2}
$$

where λ is the market price of convenience yield risk. From the no-arbitrage condition, the risk-adjusted drift of the price process is $r - \delta$. The market price of convenience yield risk has to be incorporated in the risk neutral process of convenience yield, since convenience yield is not traded.

Applying Ito's Lemma, the log spot price $X = \ln S$ in this two-factor model can be derived as:

$$
dX = (\mu - \frac{1}{2}\sigma_s - \delta)dt + \sigma_s dz_s
\tag{3}
$$

The partial differential equation (PDE)[7] characterizing the futures price $F(S, \delta, t, T)$ can be derived using Ito's Lemma and expressed as:

$$
\frac{1}{2}\sigma_s^2 S^2 F_{ss} + (r - \delta)SF_s + \frac{1}{2}\sigma_\delta^2 F_{\delta\delta} + (\kappa(\alpha - \delta) - \lambda)F_\delta + \rho\sigma_s\sigma_\delta SF_{s\delta} - F_t = 0
\tag{4}
$$

subject to boundary condition: $F(S, \delta, T, T) = S$, where T denotes the maturity date of the futures contract. The analytical solution of equation (4) is derived in Jamshidian and Fein (1990) and Bjerksund (1991) and can be expressed as:

$$
F(S, \delta, 0, T) = S \exp\left[A(T) - \delta\frac{1 - e^{-\kappa T}}{\kappa}\right]
\tag{5}
$$

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[7]For detailed derivation of this PDE, see Gibson and Schwartz (1990).

where

$$
\begin{aligned}
A(T) &= (r - \hat{\alpha} + \frac{1}{2}\frac{\sigma_\delta^2}{\kappa^2} - \frac{\sigma_s\sigma_\delta\rho}{\kappa})T + \frac{1}{4}\sigma_\delta^2\frac{1 - e^{-2\kappa T}}{\kappa^3} + (\hat{\alpha}\kappa + \sigma_s\sigma_\delta\rho - \frac{\sigma_\delta^2}{\kappa})\frac{1 - e^{-\kappa T}}{\kappa^2} \\
\hat{\alpha} &= \alpha - \frac{\lambda}{\kappa}
\end{aligned}
\tag{6}
$$

The linear relationship between futures prices and spot prices can be found in the log form of futures prices:

$$
\ln F(S, \delta, 0, T) = \ln S + A(T) - \delta\frac{1 - e^{-\kappa T}}{\kappa}
\tag{7}
$$

Equation (7) will be used for model estimation.

3.2 Single factor models

In order to analyze the impact of incorporating stochastic convenience yield on long-term forestry investment decisions, two single factor models are also estimated and compared in this paper. These one-factor models are the log price mean reverting model analyzed in Schwartz (1997) and a GBM model with a constant convenience yield. Since the two-factor model analyzed in this paper features mean reversion in the commodity's price, it seems reasonable to compare it with a single factor mean reverting model. We also use the GBM model for comparison since it is so widely used and the spot price in two-factor model follows an adjusted GBM process with stochastic convenience yield on the drift term. In this section, these one-factor models are briefly summarized.

3.2.1 The one-factor mean reverting model

This model is the same single factor model as analyzed in Schwartz (1997). The spot prices S are modeled as:

$$
\frac{dS}{S} = \kappa_{MR}[\mu_{MR} - \ln S]dt + \sigma_{MR}dz
\tag{8}
$$

Applying Ito's Lemma, the log spot price $X = \ln S$ follows an Ornstein-Uhlenbeck process:

$$
dX = \kappa_{MR}\Big[(\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}}) - X\Big]dt + \sigma_{MR}dz
\tag{9}
$$

where the mean reverting rate is κ_{MR} and the long-run equilibrium log price level is $\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}}$. The risk-adjusted version of this model can be expressed as:

$$dX = \kappa_{MR}[\alpha^* - X]dt + \sigma_{MR}dz^* \tag{10}$$

where $\alpha^* = \mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}} - \lambda_{MR}$ and λ_{MR} represents the market price of risk.

The corresponding futures price in log form, $\ln F(S, 0, T)$, can be expressed as[8]:

$$\ln F(S, 0, T) = e^{-\kappa_{MR}T}\ln S + (1 - e^{-\kappa_{MR}T})\alpha^* + \frac{\sigma_{MR}^2}{4\kappa_{MR}}(1 - e^{-2\kappa_{MR}T}) \tag{11}$$

This linear relationship between log futures prices and the state variable log spot prices will be used for model estimation.

3.2.2 The GBM model

The GBM model can be expressed as:

$$dS = [\mu_{GBM} - \delta_{GBM}]Sdt + \sigma_{GBM}Sdz \tag{12}$$

where δ_{GBM} refers to the constant convenience yield. Similarly, the log price $X = \ln S$ follows normal distribution which can be expressed as:

$$dX = \left[\mu_{GBM} - \delta_{GBM} - \frac{\sigma_{GBM}^2}{2}\right]dt + \sigma_{GBM}dz \tag{13}$$

The risk-neutral version of this model is:

$$dX = \left[r - \delta_{GBM} - \frac{\sigma_{GBM}^2}{2}\right]dt + \sigma_{GBM}dz^* \tag{14}$$

The conditional mean of X under the equivalent martingale measure is $E_0[X(T)] = (r - \delta_{GBM} - \frac{\sigma_{GBM1}^2}{2})T + X_0$. Its conditional variance is $Var_0[X(T)] = \sigma_{GBM}^2 T$.

Based on the properties of the log normal distribution, the futures price $F(S, 0, T)$ in this model can be expressed as:

$$F(S, 0, T) = e^{\ln S + (r - \delta_{GBM})T} \tag{15}$$

The log futures price can be derived as:

$$\ln F = \ln S + (r - \delta_{GBM})T \tag{16}$$

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[8]See Schwartz (1997).

4   Model estimation: Kalman filter

When state variables are not observable, a practical method for estimating this type of model is by stating the problem in state space form and by using the Kalman filter based on an error prediction decomposition of the log-likelihood function. The Kalman filter is a recursive procedure for estimating unobserved state variables based on observations that depend on these variables (Kalman (1960)). Prediction errors, a by-product of the Kalman filter, can then be used to evaluate the likelihood function and the model parameters are estimated by maximizing this likelihood function.

The state space form consists of a transition equation and a measurement equation. The transition equation describes the dynamics of an unobserved set of state variables. The measurement equation relates the unobserved variables to a vector of observables. In the two-factor model analyzed in this paper, both the lumber spot price and convenience yield are assumed to be unobserved state variables.[9] The lumber spot prices in the single-factor models are also assumed to be unobserved. Futures prices with different maturities observed at different dates serve as observed variables and the measurement equation will specify the relationship between futures prices and the two state variables.

Specifically, the linear Gaussian state space model can be expressed as the following system of equations:

$$x_{t+1} \;=\; d_t + T_t x_t + \eta_t \tag{17}$$

$$y_t \;=\; C_t + Z_t x_t + \epsilon_t \tag{18}$$

where x denotes the vector of unobserved state variables and $y = \ln F$ denotes the observed log futures prices for all the models analyzed in this paper.[10] Equation (17) represents the

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[9] In the commodity literature, since the exact meaning of commodity spot prices like electricity is difficult to pin down, when using Kalman filter to estimate parameters of the model containing spot price dynamics, researchers treat spot prices as unobserved state variable. See Schwartz (1997) for example.

[10] d_t, T_t, C_t, Z_t are terms containing corresponding model parameters which will be specified later in this paper. η_t and ϵ_t denote the disturbances of the two equations.

transition equation of the model, which describes the evolution of the non-observed state vector x_t over time. Equation (18) is the measurement equation describing the vector of observations y_t in terms of the state vector.

Two types of variables used recursively in the Kalman filter algorithm are called priori variables and posteriori variables. Define the observed data set at time t as $Y_t = (y_1, ..., y_t)$. Priori variables refer to the conditional mean, defined as $x_{t|t-1} = E[x_t|Y_{t-1}]$, and conditional variance, defined as $P_{t|t-1} = var[x_t|Y_{t-1}]$, of the state vector x_t based on information available at time $t - 1$. Posteriori variables are the estimates for the mean and variance of the state vector conditional on the information available at time t, denoted as $x_{t|t} = E[x_t|Y_t]$ and $P_{t|t} = var[x_t|Y_t]$ respectively.

The first step of the Kalman filter is to compute one time step ahead priori variables $x_{t|t-1}$ and $P_{t|t-1}$ using the values of posteriori variables at time $t - 1$ via the prediction equations:

$$x_{t|t-1} = d_{t-1} + T_{t-1}x_{t-1|t-1} \tag{19}$$

$$P_{t|t-1} = T_{t-1}P_{t-1|t-1}T'_{t-1} + Var(\eta) \tag{20}$$

Next, with the new observation y_t, the posteriori variables at time t can be updated using updating equations:

$$x_{t|t} = x_{t|t-1} + K_t v_t \tag{21}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}Z'_t K'_t \tag{22}$$

where

$$v_t = y_t - C_t - Z_t x'_{t|t-1} \tag{23}$$

$$F_t = Z_t P_{t|t-1}Z'_t + var(\epsilon) \tag{24}$$

$$K_t = P_{t|t-1}Z'_t F_t^{-1} \tag{25}$$

where v_t is the residual of the measurement equation (18) or prediction error. F_t is the variance of this prediction error, $F_t = var(v_t)$. K_t is the Kalman gain. This process is then repeated until the whole set of observations Y_N has been observed and used in this recursive

process. The resulting estimates of posteriori variables $x_{t|t}$ will be the filtered estimates of the state vector for each observation date t. The smoothed estimates of the state vector can be obtained by using all the information in the observation set Y_N.

Unknown parameters of the state space model can be estimated by maximizing the prediction error decomposition of the log-likelihood function, which is a by-product of the Kalman filter. The sample log-likelihood function is

$$\ln L = \sum_{t+1}^{N} \ln f(v_t) = c - \frac{1}{2} \sum_{t+1}^{N} (\ln |F_t| + v_t' F_t^{-1} v_t) \tag{26}$$

c is a constant and $f(v_t)$ denotes the probability density function of prediction error v_t.

The two-factor model and single-factor models analyzed in this paper are all written in the state space form and the corresponding model parameters are estimated using the Kalman Filter method. The state space form of each model is provided in this section.

4.1 Two-factor model

For the two-factor model, both the stochastic spot price and convenience yield serve as the unobserved state variables $x = [X, \delta]'$, where $X = \ln S$ denotes the log of the spot price. Based on equations (1) and (3), the terms of the transition equation (17) in the state space form can be expressed as:

$$
\begin{aligned}
d_t &= [(\mu - \frac{1}{2}\sigma_s^2)\Delta t, \kappa\alpha\Delta t]' \\
T_t &= \begin{bmatrix} 1 & -\delta_t \\ 0 & 1 - \kappa\delta_t \end{bmatrix}
\end{aligned}
$$

η_t in equation (17) denotes serially uncorrelated disturbances with mean zero, and its variance is expressed as:

$$
Var(\eta_t) = \begin{bmatrix} \sigma_s^2 \Delta t & \rho\sigma_s\sigma_\delta\Delta t \\ \rho\sigma_s\sigma_\delta\Delta t & \sigma_\delta^2\Delta t \end{bmatrix}
$$

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Based on equation (7), the terms of the measurement equation (18) are given as:

$$
\begin{aligned}
C_t &= [A(T_i)] \\
Z_t &= \left[1, -\frac{1 - e^{-\kappa T_i}}{\kappa}\right]
\end{aligned}
$$

$i = 1, ..., N$, where N is the number of futures contracts at each date t. ϵ_t in equation (18) represents a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix. The innovations in the transition equation η_t and those in the measurement equation ϵ_t are assumed to be independent in all the analyzed models in this paper, which means $E[\eta_t \epsilon_t] = 0$.

4.2 One-factor mean reverting model

The spot price in this one-factor mean reverting model is the unobserved state variable, $x = [X]$. Based on equation (9), the terms of the transition equation (17) in the state space form can be expressed as:

$$
\begin{aligned}
d_t &= \left[\kappa_{MR}(\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}})\triangle t\right] \\
T_t &= [1 - \kappa_{MR}\triangle t]
\end{aligned}
$$

η_t in equation (17) denotes serially uncorrelated disturbances with mean zero, and its variance is $\sigma_{MR}^2 \triangle t$.

Based on equation (11), the terms of the measurement equation (18) are given as:

$$
\begin{aligned}
C_t &= \left[(1 - e^{-\kappa_{MR} T_i})\alpha^* + \frac{\sigma_{MR}^2}{4\kappa_{MR}}(1 - e^{-2\kappa_{MR} T_i})\right] \\
Z_t &= [e^{-\kappa_{MR} T_i}]
\end{aligned}
$$

$i = 1, ..., N$. ϵ_t in equation (18) represents a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix.

4.3 GBM model

In this one-factor model, spot price is the unobserved state variable, $x = [X]$. Based on equation (13), the terms of the transition equation (17) in the state space form can be

expressed as:

$$d_t = \left[(\mu_{GBM} - \delta_{GBM} - \frac{\sigma_{GBM}^2}{2})\triangle t\right]$$

$$T_t = [1]$$

η_t in equation (17) denotes serially uncorrelated disturbances with mean zero, and its variance is $\sigma_{GBM}^2 \triangle t$.

Based on equation (16), the terms of the measurement equation (18) are given as:

$$C_t = [(r - \delta_{GBM})T_i]$$

$$Z_t = [1]$$

$i = 1, ..., N$. ϵ_t in equation (18) represents a vector of serially uncorrelated disturbances with zero mean and identity variance-covariance matrix.

5 The Schwartz (1998) one-factor long-term model

Schwartz (1998) develops a one-factor model which is simpler than the two-factor model analyzed in Schwartz (1997) in terms of valuing long-term commodity-related investments, but it closely matches the performance of the two-factor model in terms of fitting the term structure of long term futures prices and the volatilities of all futures contracts. Schwartz (1998) calls it the long-term model. In this section, the one-factor long-term model is summarized.[11] All the parameters in this long-term model are derived from the parameter estimates in the Schwartz (1997) two-factor model.

The motivation for this one-factor long-term model is to match as closely as possible the risk-neutral distribution of the spot prices in the Schwartz (1997) two-factor model. In the risk-neutral world, the spot prices in the two-factor model are lognormally distributed with mean equal to the futures price and variance depending on the volatility of futures returns.[12] Schwartz (1998) applied his one-factor long-term model to oil and was able to

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[11]For the convenience of readers, the derivation of this long-term model is provided in Appendix A.

[12]See Schwartz (1998).

fairly accurately generate long-term futures prices and the term structure of the futures volatilities.

The long-term model uses a composite price, denoted Z, as the single stochastic state variable.[13] Z depends on the two stochastic factors, spot price S and convenience yield δ, as follows:[14]

$$Z(S,\delta) = Se^{[\frac{c-\delta}{\kappa} - \frac{\sigma_\delta^2}{4\kappa^3}]} \tag{27}$$

c is defined as the constant convenience yield used to match the long-term rate of change in the futures prices and is expressed as

$$c = \alpha - \frac{\lambda}{\kappa} - \frac{\sigma_\delta^2}{2\kappa^2} + \frac{\rho\sigma_s\sigma_\delta}{\kappa} \tag{28}$$

Given S, δ, and the model parameters, Z can be calculated based on Equation (27). Z is defined in such a way that the futures prices of this one-factor model $F(Z,T)$ match the long-term futures prices of two-factor model $F(S,\delta,T)$, given the constant convenience yield expressed in Equation (28). It may be noted from Equation (27) that Z is increasing in S and decreasing in δ.

In order to match the volatility of futures returns between the one-factor long term model and the two-factor model, the stochastic differential equation followed by Z is given as:

$$\frac{dZ}{Z} = (r - c)dt + \sigma_F(t)dz \tag{29}$$

where $\sigma_F(t)$ represents the volatility of the futures returns based on the Schwartz (1997) two-factor model and is derived as

$$\sigma_F^2(t) = \sigma_s^2 + \sigma_\delta^2 \frac{(1 - e^{-\kappa t})^2}{\kappa^2} - 2\rho\sigma_s\sigma_\delta \frac{(1 - e^{-\kappa t})}{\kappa} \tag{30}$$

Therefore, the futures price, F, with maturity T and the composite spot price Z, in this one-factor long-term model can be expressed as:

$$F(Z,T) = Ze^{(r-c)T} \tag{31}$$

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[13]In Schwartz (1998), Z is referred to as the shadow price.

[14]This expression is slightly different than the corresponding Equation 17 in Schwartz (1998).

This long-term model devised by Schwartz (1998) is much easier to use in valuing investment opportunities because there is only one stochastic variable, the composite price Z. Schwartz (1997) found that for oil prices the performance of this one-factor model in terms of fitting the long-term futures prices and the term structure of futures volatilities is comparable with that of the two-factor model. We investigate whether the long term model also works for lumber prices.

6   Estimation results

The prices of four lumber futures contracts are used for model estimation and their main characteristics are detailed in Section 2. In this section, the estimation results of one-factor and two-factor models analyzed in this paper are presented and the corresponding model performance is examined. In addition the futures prices implied by the long-term model and the two-factor model are compared to determine whether the former provides a reasonable approximation of the latter.

6.1   The two-factor model

The parameter estimates of the two-factor model, Equation (1), using weekly futures prices are reported in Table 3. The estimate of the correlation coefficient, ρ, is above 0.9 and is statistically different from zero. This result implies that convenience yield is an important factor affecting lumber price dynamics. The positive estimate of ρ is consistent with the theory of storage and helps to explain the mean reverting feature of lumber prices observed in Figure 1. The estimate of the mean reversion rate, κ, for the convenience yield process is high and significant as well. $-ln(0.5)/\kappa$ can be interpreted as the half-life of the time it takes for δ to return to its long run value. With $\kappa = 2.089$ we expect the deviation δ from the long run value will halve in 0.33 years. The estimate of the equilibrium convenience yield level α is not significantly different from zero, which implies that on average, the net convenience yield of lumber is about zero. This result is consistent with the theoretical prediction that

	μ	κ	α	σ_s	σ_δ	ρ	λ
Estimates	-0.124	2.089	-0.142	0.397	0.824	0.934	-0.212
Std. Error	(0.107)	(0.136)	(0.107)	(0.014)	(0.049)	(0.008)	(0.227)
LL	6471.7						

TABLE 3: *Parameter estimates of Schwartz (1997) two-factor model using Kalman filter. LL refers to the value of log-likelihood function. Weekly futures prices, from January 6th, 1995 to April 25th, 2008.*

in equilibrium, the benefit of holding the physical commodity should be equal to the cost of storage, which leads to the zero net convenience yield. The estimate of market price of convenience yield risk, λ, is found to be not significant as well.

Model implied spot prices, S, and market lumber prices proxied by the futures contract closest to maturity, $F1$, are plotted in Figure 4. From this graph it appears that the model implied prices move very closely with the market spot prices.

Figure 5 plots the two model implied state variables, spot prices and convenience yield. In order to make these two series comparable on a single graph, we divide spot prices by 300. The blue line in this figure denotes the adjusted spot prices and red line represents the convenience yield. This figure shows that spot prices and convenience yield tend to move together, confirming the estimation result of a high and positive correlation coefficient ρ. From this figure, we also notice that the net convenience yield can be negative or positive and fluctuates around zero in the range of $[-1, 1]$. Whenever convenience yield exceeds higher than storage cost, net convenience yield is positive. Conversely, if storage cost exceeds convenience yield, net convenience yield will be negative. In the long-run, convenience yield should be approximately equal to storage cost. Summary statistics of net convenience yield are provided in Table 4.

Model estimation errors of both futures prices and log futures prices including Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) expressed in dollars per thousand board feet for four futures contracts are reported in Table 5. The overall average error of
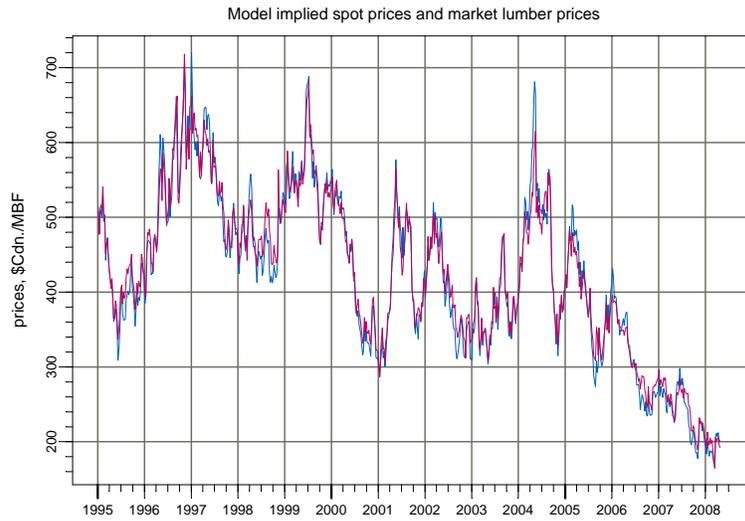
Figure 4: *Plots of model implied (two-factor model) and market spot prices. Blue line: model implied prices. Red line: market prices.*
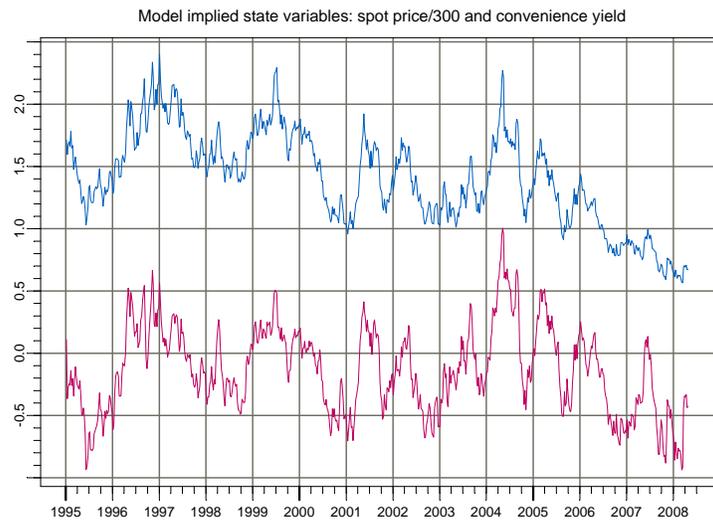

Figure 5: *Blue line: model implied spot prices/300. Red line: model implied convenience yield.*

Item	Max	Min	Mean	Std. Dev.
Net convenience yield	1.00	-0.94	-0.14	0.35

TABLE 4: *Descriptive statistics for model implied net convenience yield.*

Schwartz (1997) two-factor model				
	F1	F2	F3	F4	All
Calibration errors of futures prices					
RMSE	13.664	1.764	4.719	3.600	7.501
MAE	9.701	1.327	3.547	2.711	4.322
Calibration errors of log futures prices					
RMSE	0.031	0.004	0.011	0.009	0.017
MAE	0.023	0.003	0.008	0.006	0.010

TABLE 5: *Average estimation errors of both futures prices and log futures prices of Schwartz (1997) two-factor model expressed as RMSE and MAE of 4 futures contracts, Cdn\$/MBF.*

model implied futures prices from the last column is less than \$8/MBF which is about 1.8% of the mean lumber spot price. The overall average errors of log futures prices expressed in both ways are less than two cents per thousand board feet. It appears that the two-factor model provides a good tracking of the lumber futures time series. Plots of market futures prices and the model implied futures prices for the four futures contracts are shown in Figure 6. Again, the graphs display a reasonably close fit of the model prices versus actual prices.

## 6.2   One-factor mean reverting model

Parameter estimates for the single-factor model are reported in Table 6. From this table we find that all the model parameters are statistically significant except for the market price of risk λ_{MR}. The long-run equilibrium log price level $\mu_{MR} - \frac{\sigma_{MR}^2}{2\kappa_{MR}} = 6.206$ which implies a value for S of \$496 per MBF. The mean reverting rate κ_{MR} is moderate at 0.229. Model implied and market lumber spot prices are plotted in Figure 7. The average error (RMSE)
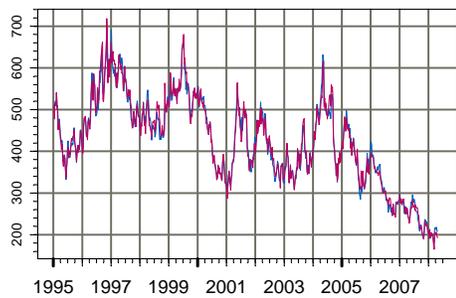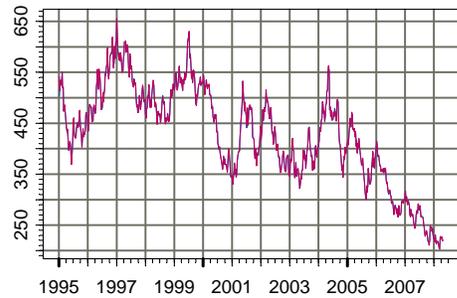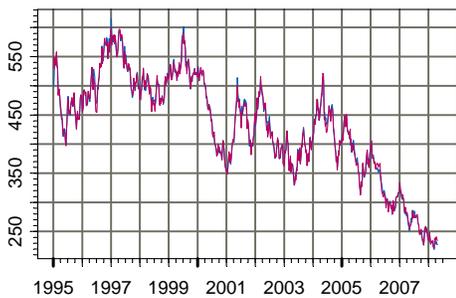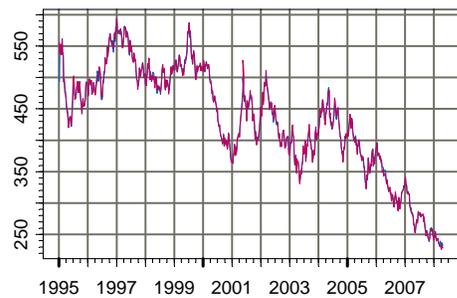
FIGURE 6: *Plots of model implied and market futures prices for the two-factor model and the four chosen futures contracts. Weekly data from January 6th, 1995 to April 25th, 2008. Units are $Cdn per MBF. Blue line: model implied futures prices. Red line: market futures prices.*

	μ_{MR}	κ_{MR}	σ_{MR}	λ_{MR}
Estimates	6.323	0.229	0.231	0.007
Std. Error	(0.297)	(0.031)	(0.009)	(0.284)
LL	5343.2			

TABLE 6: *Parameter estimates of Schwartz (1997) one-factor model using Kalman filter. LL refers to the value of log-likelihood function. Weekly futures prices, from January 6th, 1995 to April 25th, 2008.*

	μ_{GBM}	σ_{MR}	δ_{GBM}
Estimates	-0.065	0.215	-0.027
Std. Error	(0.058)	(0.006)	(0.004)
LL	5145.2		

TABLE 7: *Parameter estimates of GBM model with convenience yield using Kalman filter. LL refers to the value of log-likelihood function. Weekly futures prices, from January 6th, 1995 to April 25th, 2008.*

for all four futures maturities is \$17.8 per MBF which is larger than for the two-factor model. More details are provided in Appendix B.

6.3   The GBM model

Parameter estimates for the GBM model with constant convenience yield are reported in Table 7. The drift term μ_{GBM} is negative, but not statistically significant. The constant convenience yield δ_{GBM} is small in magnitude. Model implied and market lumber spot prices are plotted in Figure 8. The average RMSE for all maturities is \$19.3 per MBF. Details are provided in Appendix B.
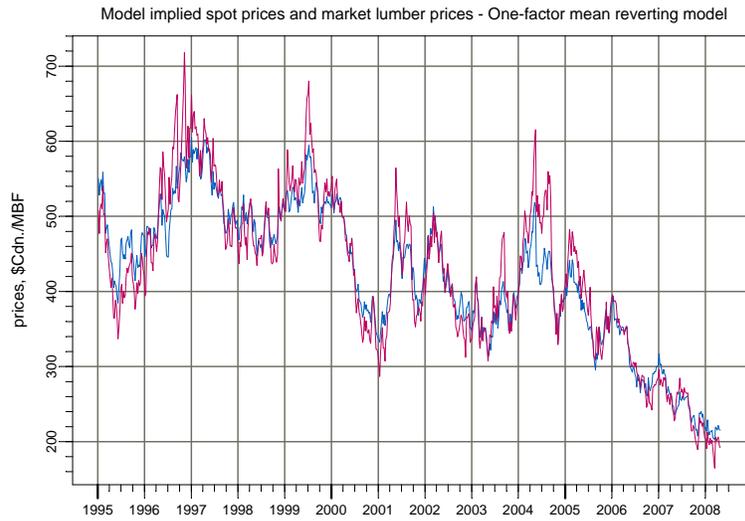
FIGURE 7: *Plots of model implied and market spot prices: one-factor mean reverting model. Blue line: model implied prices. Red line: market prices.*
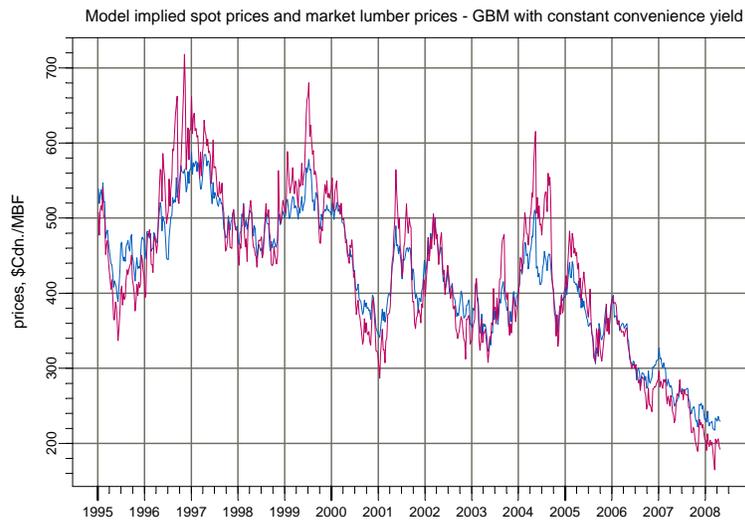

FIGURE 8: *Plots of model implied and market spot prices: one-factor GBM model with constant convenience yield. Blue line: model implied prices. Red line: market prices.*

	Max	Min	Mean	Std. Dev.	Skewness	Kurtosis
Cdn (2005) \$/MBF	558.6	239.2	428.1	77.73	-0.48	2.37

TABLE 8: *Descriptive statistics for the composite spot prices Z of one-factor long-term model.*

6.4 The one-factor long-term model

Since the single factor long-term model proposed in Schwartz (1998) is a mathematical transformation of the two-factor model, the model parameters are the same for the two models. Specifically, the constant convenience yield, c, based on Equation (28) is 0.028. Table 8 shows descriptive statistics for the composite spot price Z of this long-term model. Comparing this table with Table 1, we find that the range of the true spot price is wider than the composite spot price and the composite prices is less volatile than the market spot price. A plot of composite spot prices and model implied spot prices is provided in Figure 9. The composite price shown in this graph is less volatile than the model implied spot price.

For a given maturity T, model implied futures prices of both the two-factor model and the long-term model can be derived based on Equations (5) and (31) respectively. We are interested in the performance of the long-term model in terms of fitting long-term commodity derivatives prices compared to that of the two-factor model. To this end, model implied futures prices with the maturities up to 8 years are calculated for both models. Note that beyond one year there are no actual futures prices that can be used for comparison. The differences expressed in RMSE and MAE of the model implied futures prices with different maturities between the two-factor model and the long-term model are reported in Table 9. The average difference for long term futures contracts (with maturities from 5 years to 8 years) is less than \$9, which is about 2% of the mean futures prices. This result further confirms the close match of these two models in terms of generating long maturity futures prices. Plots of the model implied futures prices with different maturities for these two models are provided and compared in Appendix C.
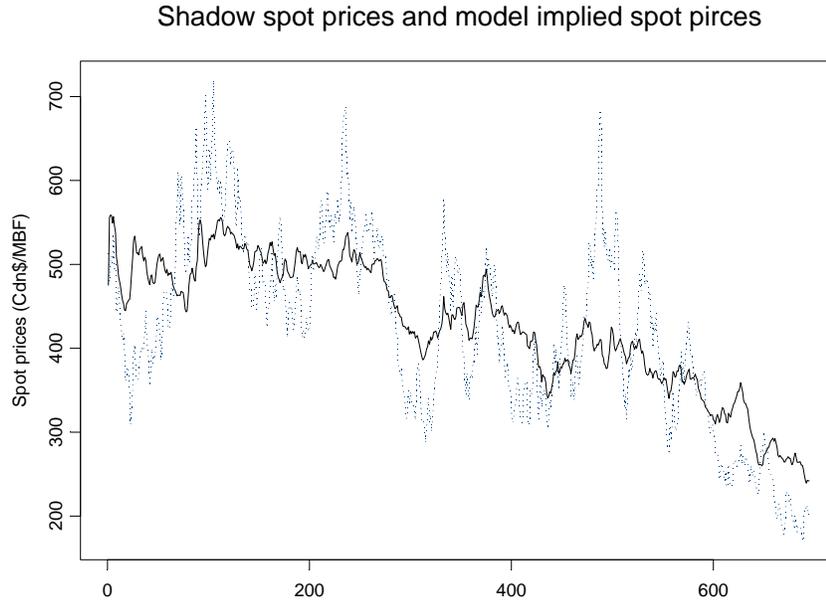
Shadow spot prices and model implied spot pirces

FIGURE 9: *Plots of composite spot prices and model implied spot prices. Solid line: composite spot prices; dotted line: model implied spot prices.*

Differences of futures prices in $/MBF: two-factor model and long-term model				
	F1	F2	F3	F4	F1-F4 average
RMSE	58.83	41.69	29.73	21.52	40.45
MAE	48.43	34.26	24.27	17.35	31.08
	F5	F6	F7	F8	F5-F8 average
RMSE	11.47	8.71	8.75	8.72	9.49
MAE	8.99	8.45	8.59	8.57	8.65
	F9	F10	F11	F12	F9-F12 average
RMSE	8.68	8.63	8.59	8.55	8.61
MAE	8.53	8.49	8.45	8.40	8.47

TABLE 9: *Differences of model implied futures prices with different maturities for two models.*

7   Analysis of a forestry investment

In this section we model the optimal decision of the owner of a stand of trees who seeks to maximize the value of the stand (or land value) by optimally choosing the harvest time. It is assumed that forestry is the best use for this land, so that once the stand is harvested it will be allowed to grow again for future harvesting. Since this is a multi-rotational optimal harvesting problem, it represents a path-dependent option. This is because the value of the option to harvest the stand today depends on the quantity of lumber, which itself depends on the last time when the stand was harvested. The tree harvesting problem is impulse control problem (Phan, 2005). Determining the value of the stand and the optimal harvesting decision requires the numerical solution of a Hamilton-Jacobi-Bellman (HJB) variational inequality.

Some of the details of this timber harvesting problem such as costs and the growth curve for wood volume have been used in other papers including Chen and Insley (2008), Insley and Rollins (2005) and Insley and Wirjanto (Insley and Wirjanto). The latter two papers use a simple one factor mean reverting process for price, while Chen and Insley (2008) examine a regime-switching model.

7.1   Cost, wood volume and price data

We consider a harvesting problem for a hypothetical stand of Jack Pine trees in Ontario's boreal forest assuming that the stand is used for commercial forestry. Values are calculated prior to any stumpage payments or taxes.

Timber volumes and harvesting costs are adopted from Insley and Lei (2007) and are repeated here for the convenience of the reader. Volume and silviculture cost data were kindly provided by Tembec Inc. The estimated volumes reflect 'basic' levels of forestry management which involves $1040 per hectare spent within the first five years on site preparation, planting and tending. These costs are detailed in Table 10. Note that in the Canadian context these basic silviculture expenses are mandated by government regulation for certain stands. We

Item	Cost, $/ha	Age cost incurred
Site preparation	$200	1
Nursery stock	$360	1
Planting	$360	2
First tending	$120	5
Monitoring	$10	35

TABLE 10: *Silviculture costs under a basic regime*

assume that harvesting is not permitted before age 35 once all silvicultural expenditures have been made.

Volumes, estimated by product, are shown in Figure 10 for the basic silvicultural regime.[15] SPF1 and SPF2 are defined as being greater than 12 centimeters at the small end, SPF3 is less than 12 centimeters, and 'other' refers to other less valuable species (poplar and birch). Data used to plot this graph is provided in Insley and Wirjanto (Insley and Wirjanto).

Assumptions for harvesting costs and current log prices at the millgate are given in Table 11. These prices are considered representative for 2003 prices at the millgate in Ontario's boreal forest. Average cost to deliver logs to the lumber mill in 2003 are reported as $55 per cubic meter in a recent Ontario government report (Ontario Ministry of Natural Resources, 2005). From this is subtracted $8 per cubic meter as an average stumpage charge in 2003 giving $47 per cubic meter.[16] It will be noted the lower valued items (SPF3 and poplar/birch) are harvested at a loss. These items must be harvested according to Ontario government regulation. The price for poplar/birch is at roadside, so there is no transportation cost to the mill.

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[15]The yield curves were estimated by Margaret Penner of Forest Analysis Ltd., Huntsville, Ontario for Tembec Inc.

[16]This consists of $35 per cubic meter for harvesting and $12 per cubic meter for transportation. Average stumpage charges are available from the Ontario Ministry of Natural Resources.
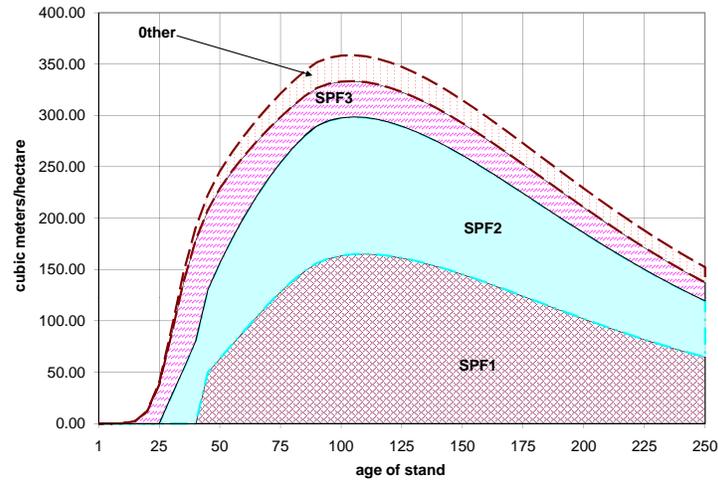
FIGURE 10: *Volumes by product for hypothetical Jack Pine stands in Ontario's boreal forest under basic management*

Harvest and transportation cost	$47
Price of SPF1	$60
Price of SPF2	$55
Price of SPF3	$30
Price of poplar/birch	$20

TABLE 11: *Assumed values for log prices and cost of delivering logs to the mill in $ per cubic meter*

7.2 Optimal harvesting with different price models

Ideally we would solve the optimal harvesting problem using the two-factor model with stochastic price and convenience yield. However this requires the numerical solution of a complex HJB variational inequality with three state variables (price, volume and convenience yield) plus time. We have shown that the performance of the long-term model introduced in Schwartz (1998) is comparable to that of the two-factor model in terms of matching long run futures prices. We therefore analyze the forest investment problem using the long-term model with the composite price as the single stochastic variable. The results from the long-term model will be compared with those from the single factor mean reverting and GBM models. In the following sections, the HJB variational inequality is specified for the three different price models.

7.2.1 The long-term model

In the single-factor long-term model, based on the stochastic process describing the composite price Z in Equation (27), the value of the stand of trees is denoted as $V(Z, \varphi, t)$. At each period the stand owner makes the choice either to harvest the stand immediately or let the trees grow for another period and then reconsider whether or not harvesting should be undertaken. If the stand is harvested the stand owner receives revenue from selling the timber equal to $[(S - C_h)Q(\varphi) + V(Z, 0, t)]$. This it the price of timber, S, less per unit harvesting costs, C_h, times the quantity of timber, $Q(\varphi)$, which is a function of age, $Q = g(\varphi)$. In addition the stand owner receives an asset equal to $V(Z, 0, t)$ which refers to the value of the bare land when the stand is of age zero. If the stand owner chooses to delay harvesting for another period, he receives the value of the land, $V(Z, \varphi, t)$. Using standard no-arbitrage arguments, when it is optimal to delay harvesting (in the continuation region) the value of the stand satisfies the following PDE:

$$V_t + (r - c)ZV_Z + \frac{1}{2}(\sigma_F(t)Z)^2 V_{ZZ} + V_\varphi - rV = 0 \tag{32}$$

where the variance of the futures returns $\sigma_F(t)$ is time dependent and is given in Equation (30). Rewrite the PDE Equation (32) as:

$$HV \equiv rV - (V_t + (r - c)ZV_Z + \frac{1}{2}(\sigma_F(t)Z)^2 V_{ZZ} + V_\varphi) \tag{33}$$

Then the HJB variational inequality characterizing the full optimal harvesting problem can be expressed as:

$$(i) \quad HV \geq 0 \tag{34}$$

$$(ii) \quad V(Z, \varphi, t) - [(S - C_h)Q(\varphi) + V(S, 0, t)] \geq 0$$

$$(iii) \quad HV\left[V(Z, \varphi, t) - [(S - C_h)Q(\varphi) + V(S, 0, t)]\right] = 0$$

Equation (34) implies if the stand of trees is managed optimally either HV, $V(Z, \varphi, t) - [(S - C_h)Q(\varphi) + V(S, 0, t)]$, or both will be equal to zero. If $HV = 0$ and $V(Z, \varphi, t) - [(S - C_h)Q(\varphi) + V(S, 0, t)] > 0$, it is optimal for the investor to continue holding the option by delaying the decision to harvest. In this case growing stand of trees is earning the risk free return and the value of the stand is greater than the payout the owner would receive from harvesting. On the other hand, if $HV < 0$ and $V(Z, \varphi, t) - [(S - C_h)Q(\varphi) + V(S, 0, t)] = 0$, then the value of the stand of trees just equals the value of immediate harvest. The owner is not earning the risk free return from maintaining the standing timber and should harvest the trees. If both parts (i) and (ii) in Equation (34) are equal to zero, either strategy is optimal. Equation (34) may be written more compactly as:

$$min\left\{HV; \left[V(Z, \varphi, t) - [(S - C_h)Q(\varphi) + V(S, 0, t)]\right]\right\} = 0 \tag{35}$$

No analytical solution exists for Equation (35). We solve it numerically, using the combination of the fully implicit finite difference method, the semi-Lagrangian method and the penalty method.[17]

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[17]Details on this approach are provided in Insley and Rollins (2005). An introduction to numerical methods is provided in Wilmott (2006).

7.2.2 Single-factor models

For the one-factor mean reverting model, the value of the stand of trees, $V(S, \varphi, t)$, satisfies the following PDE in the continuation region:

$$V_t + \kappa_{MR}(\mu_{MR} - \lambda_{MR} - \ln S)SV_S + \frac{1}{2}(\sigma_{MR}S)^2 V_{SS} + V_\varphi - rV = 0 \qquad (36)$$

The HJB equation can be expressed as in Equation (35), except that the HV is defined as:

$$HV \equiv rV - (V_t + \kappa_{MR}(\mu_{MR} - \lambda_{MR} - \ln S)SV_S + \frac{1}{2}(\sigma_{MR}S)^2 V_{SS} + V_\varphi)$$

For the GBM model, the value of the stand of trees, $V(S, \varphi, t)$, satisfies the following PDE in the continuation region:

$$V_t + (r - \delta_{GBM})SV_S + \frac{1}{2}(\sigma_{GBM}S)^2 V_{SS} + V_\varphi - rV = 0 \qquad (37)$$

HV in this case is defined as:

$$HV \equiv rV - (V_t + (r - \delta_{GBM})SV_S + \frac{1}{2}(\sigma_{GBM}S)^2 V_{SS} + V_\varphi)$$

As with the long term model, the HJB equations for these single-factor models are solved numerically.

7.3   Results for land value and critical harvesting prices

In this section we present results for each of the lumber price models in terms of the value of the stand of trees (land value) and the critical prices at which it is optimal to harvest. As discussed earlier, the spot price data used to parameterize the models is approximated by the CME random lengths futures price for the nearest maturity date. To value a hypothetical stand of trees in Ontario, the long run equilibrium price (μ_{MR} in Equation (9)) needs to be scaled to reflect Ontario prices at the millgate. Our estimate of price at the millgate in 2003 for SPF1 logs is Cdn.\$60 per cubic meter. In 2003 the average spot price proxied by the price of futures contract closest to maturity was Cdn. \$375 per MBF. We used the ratio of 375/60 as a rough adjustment factor to scale the equilibrium price levels. This rescaling

accounts for transportation costs and milling costs (as well as the conversion from MBF to m^3).

For the one-factor long-term model, the middle curve in Figure 11 shows how the bare land value (a stand age of zero) changes with the composite lumber price, Z. We observe that land value increases with Z. For example, when the composite price is $\$50/m^3$, the land is worth $\$1147$ per hectare. This rises to $\$1559$/ha when the composite price is $\$60$. This makes sense since Z is defined to be increasing in S and decreasing in δ (recall Equation (27)). In line with finance theory, the value of a call option increases with spot price and decreases with the dividend. In our forestry investment problem, the bare land value is like a call option and the convenience yield is like the dividend. Land values for different stand ages are plotted in Figure 12. The land becomes more valuable as the trees mature and as Z increases.

We are more interested in the relationship of land value with spot price S rather than with our constructed composite price. One of the disadvantages of using the long term model is that this relationship is obscured. However we note from Table 4, that net convenience yield fluctuates in the range of $[-1, 1]$. Given the land value estimate for each Z, we can back out what the implied spot price would be when convenience yield is at either $+1$ or -1. This gives us a range for land values versus spot price which are shown as the dashed and dotted curves in Figure 11. For example, when $Z = \$50$, land value is $\$1150$. If $\delta = -1$, the spot price consistent with that Z and land value is $\$31$. If instead $\delta = 1$, the implied spot price must be higher at $\$81$. The logic here is that a higher convenience yield implies that it is more beneficial to hold the harvested lumber rather than trees "on the stump", so the option to harvest is actually worth less. Hence a higher spot price is required to be consistent with a land value of $\$1150$. Figure 11 also implies that given a certain level of convenience yield, land value increases with lumber prices. This can be explained as given a fixed net benefit of holding lumber in stock, the higher the lumber price is, the more the land value is. This result is consistent with the finance theory. Moreover, this graph indicates that the combination of high convenience yield and low spot price will lead to low land value
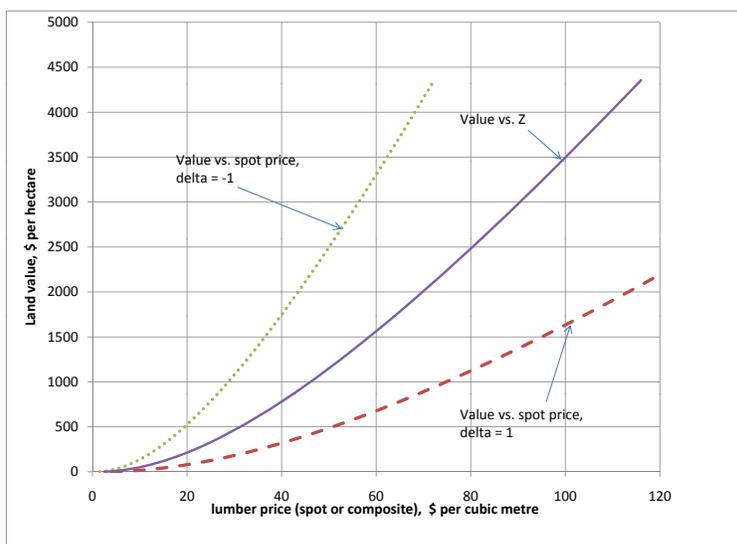
FIGURE 11: *Land values v.s. composite price (Z) or spot price for stand age 0. For land values versus the spot price, δ is set at upper and lower limits of +1 and -1*

and the combination of low convenience yield and high price will generate high land value. This result is also consistent with the finance theory, since with high convenience yield and low spot price, the option is less valuable and vice versa.

Figures 13 and 14 show land value versus the lumber spot price for the mean reverting and GBM models. We observe that for the MR process at a stand age of zero, land value is insensitive to spot price. This follows from the fact that the estimated long run equilibrium price is constant and the speed of mean reversion is a moderate 0.229 (implying a half life to return to this value of three years.) At a stand age of zero, the trees will not be harvested for at least 35 years, so that with this price model we expect to be back at the long run mean by the harvest date.[18] As the stand age increases, land value becomes positively related to spot price, since the stand may be harvested within a few years.

The GBM results are very different from the MR and long run models. At a lumber price

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[18]This result is consistent with the findings in Insley and Rollins (2005) in which a slightly different mean reverting process was used.
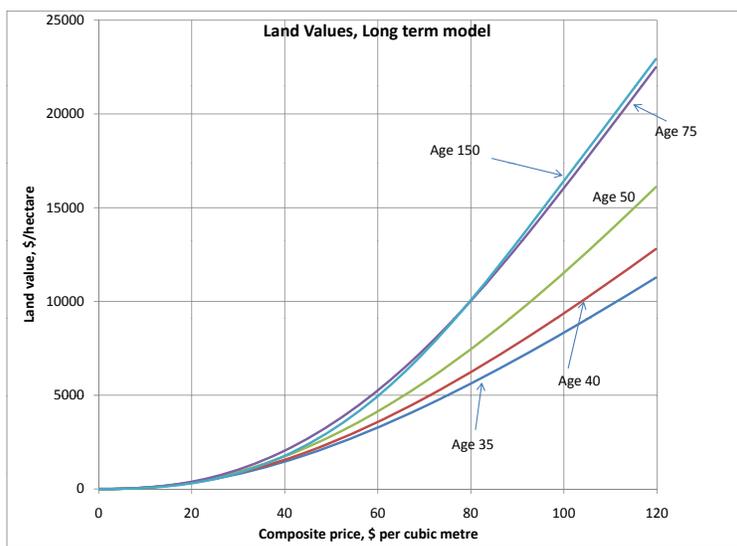
FIGURE 12: *Land values v.s. composite prices for different stand ages.*

of \$50, the GBM model gives a land value of \$199 million per hectare compared to around \$5900 for the MR model. The GBM land values are also much greater than the range given for the long-term model. At a \$50 spot price land value ranges from around \$500 for $\delta = -1$ to around \$2500 for $\delta = 1$. The large land value for GBM is consistent with the estimated parameter values. The risk adjusted drift rate for S in the GBM model is $r - \delta$ which works out to $[0.023 - (-0.027)] = .05$ from the estimates reported in Table 7. This exceeds the assumed riskfree discount rate of 2.3%.

In addition to land value, we are also concerned with critical harvesting prices which indicate when it is optimal to harvest. The middle curve in Figure 15 shows the critical composite price versus stand age for the long-term model. We see the critical Z value is about \$120 per m^3 at age 70 and declines to reach a steady state of around \$90 per m^3. Based on the relationship amongst the spot price, convenience yield, δ, and composite price Z shown in Equation (27), we can calculate a range for the corresponding critical spot prices by substituting in the range of convenience yield. The upper and lower lines in Figure 15 show the corresponding upper and lower bounds of the critical spot prices for the long-term
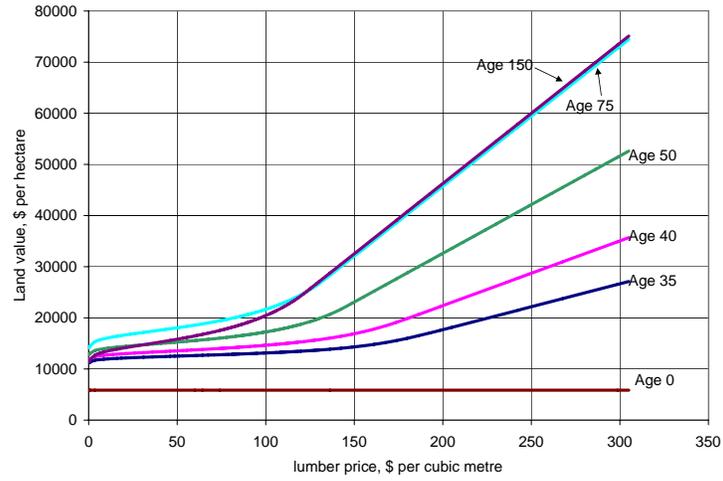
36

FIGURE 13: *One-factor mean reverting model. Land values v.s. lumber spot prices for stands of various ages.*
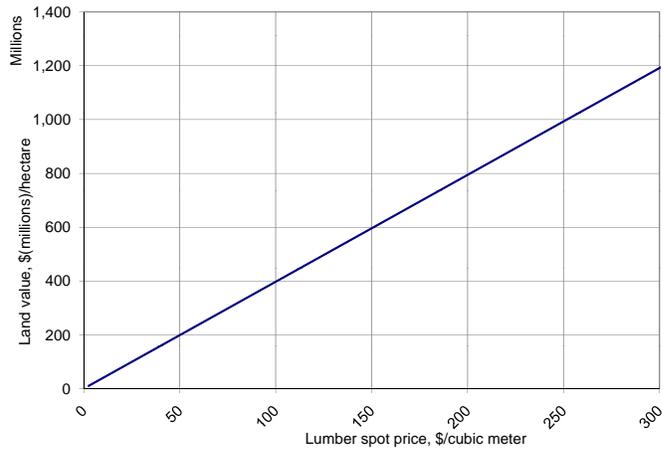

FIGURE 14: *GBM model with constant convenience yield. Land values v.s. lumber spot prices for stand age 0.*

model. The upper line reflects critical S if $\delta = 1$ and the lower line reflects the critical S if $\delta = -1$.

Our long term model is an approximation of the results that would be produced by the two-factor model. For intuition about the upper and lower bounds in Figure 15, we consider the relationship between the convenience yield and the spot price in the two-factor model. Referring to Equation (2), when δ is at its lower bound of -1, this implies the current drift rate of S is relatively high, but it is known that δ will be pulled up quickly in the future to its long run value. In this circumstance the critical prices are relatively low since the future reward for holding harvested lumber will increase while the reward for holding standing trees will decrease. With the high drift rate of S, given this lower bound of convenience yield δ, the land owner should also take advantage of the high future spot prices to harvest at a relatively low critical price. When $\delta = 1$ this implies the expected growth rate in S is relatively low, but it is expected that δ will revert back to its long run mean fairly quickly. This implies that the growth rate of S will increase in the future. Therefore, unless the current spot price is quite high, it is not optimal to harvest. By delaying the harvest the owner of the stand puts off paying harvesting costs and can take advantage of expected future growth in lumber prices.[19]

Figure 16 shows the critical harvesting prices for the MR single-factor model as well as the range of critical prices for the long-term model. Critical prices generated by the mean reverting and long-run models decrease with the stand age. When the trees are young and growing fairly rapidly it makes sense to delay harvesting, so that the critical prices that trigger harvesting are higher. Once tree growth declines we reach an approximate steady state for the critical harvest price.

The critical harvest prices for the MR case lie between the upper and lower bounds for the long run model case. For the MR model there are critical prices defined from age 35 and onward, whereas for the long run model critical prices are defined from age 70 onward. This implies that for the MR model if the spot price hits a very high value it is worthwhile

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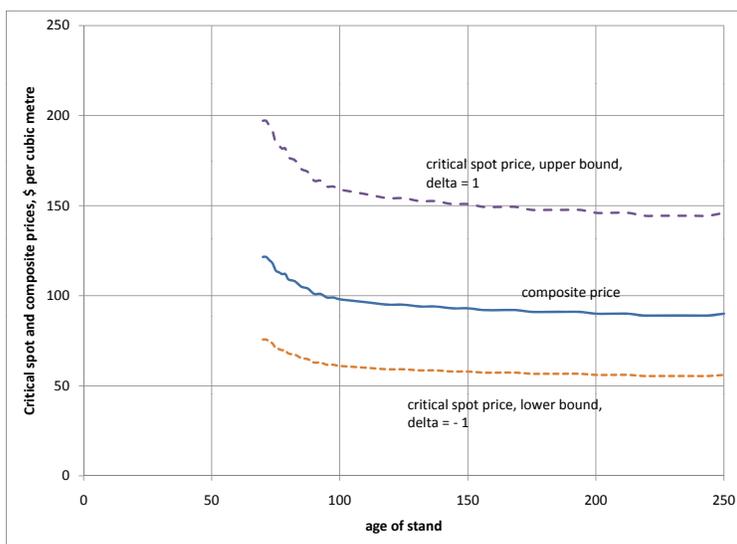[19]We assume people are rational and forward looking in this economy.

FIGURE 15: *Critical composite prices and the calculated range of critical spot prices versus stand ages. Upper bound is associated with $\delta = 1$ and lower bound is associate with $\delta = -1$*

harvesting even though the trees are still very young and growing rapidly. This follows from the assumption of a fixed equilibrium price in this model which make it beneficial to take advantage of any short term price surges. For the long run model, on the other hand, it would never be optimal to harvest before the trees are 70 years of age. In terms of the stochastic process followed by Z, Equation (29), the drift is a small negative number: $r - c = 0.023 - 0.028 = -0.005$. The expected return from holding the trees therefore comes from volume growth rather than any expected upward drift in Z. Hence with the long run model it is not optimal to harvest before age 70 while the volume growth rate is still strongly positive, no matter what the price. Referring to Figure 10 it may observed that volume growth is highest in the years before age 70.

There are no critical prices for the GBM case, implying it would never be optimal to harvest the stand. This is a result of the negative convenience yield which implies a drift rate in the risk neutral world that exceeds the riskless interest rate. (Note that after age 255 it is assumed that wood volume in the stand of trees remains constant.)
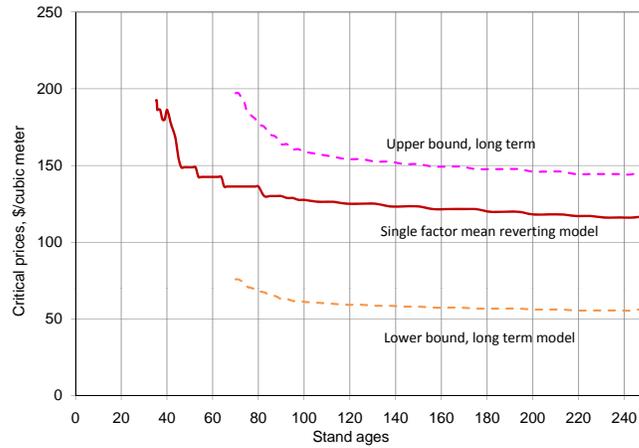
39

FIGURE 16: *Critical spot prices for the MR and GBM models and a range of critical prices for the long-term model.*

In reviewing the results we note that the long run model using the composite price Z gives significantly different land values and critical prices than the other two single factor models. It is interesting that for the GBM model, the parameter values that result from the Kalman filter estimation produce land values that are so different from the other two models. We obtained some land sale data for 2003 in the Ontario region which the timber volume curves apply. The land was marginal agricultural land which was being purchased for reforestation. The average land sale price was around $1100 per hectare. We therefore feel confident in concluding that the GBM results are not reasonable.

The land values given by the MR and long-term model are at least the right order of magnitude. It is significant that the MR model recommends harvesting at much younger stand ages than the long-term model. The composite price in the long-term model is summarizing the long run relationship between convenience yield and spot price. We saw previously that the two-factor model provided a better match of market futures prices than the single factor mean reverting model. The performance of long-term model and two-factor model in

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terms of fitting long-term market data are comparable. In addition economic theory tells us convenience yield is an important consideration for pricing in commodity markets such as lumber. Hence it seems reasonable to have more confidence in the results of the long-run model, than in the simple MR model in which convenience yield is ignored.

8   Concluding remarks

This paper investigates the importance of modeling the stochastic convenience yield of lumber in the context of an optimal tree harvesting problem. Schwartz (1997) proposed a stochastic model of commodity prices with both spot price and convenience yield as stochastic factors. In the first part of the paper, we examine the performance of this two-factor model in terms of its ability to characterize the price of lumber derivatives . The estimation result shows that there is a positive and significant correlation between lumber prices and convenience yield. This two-factor model also provides a good model fit in terms of explaining the dynamics of lumber derivatives.

In the second part of the paper, we examine the impact of stochastic convenience yield on a multi-rotational optimal harvesting problem. The HJB equation characterizing the value of the option to harvest a stand contains three stochastic variables: lumber prices, convenience yield and the stand age. To simplify the solution of the harvesting problem, we use the result of Schwartz (1998) who proposes a one-factor model (called the long-term model) which retains most of the characteristics of his two-factor model, especially its ability to fit long-term futures prices. The HJB equation derived using this one-factor model is solved numerically using the combination of the fully implicit finite difference method, the semi-Lagrangian method and the penalty method. We compare the results of the long term model with two single factor models are common in the literature: a mean reverting model and geometric Brownian motion.

The result shows that including the effect of convenience yield through the long-term model has an important impact on long-term forestry investment decisions. Land values

and critical harvesting prices were significantly different across the three models. The GBM model gave excessive land values. The single factor mean reverting model gave land values of a reasonable order of magnitude, but under MR model harvesting would potentially occur at much earlier stand ages than with the long-term model.

The results for the long-term model also showed that the critical harvesting prices varied significantly depending on the assumed value of the convenience yield. The higher the convenience yield, the higher the spot price that land owner requires to harvest the trees. This follows from the interaction of the convenience yield and the spot price. A high convenience yield today implies a lower convenience yield in the future and also a higher expected growth rate for the spot lumber price. Hence with a high convenience yield, the critical price that induces harvesting is relatively high, and we expect that the stand will be harvested at a later date than for a lower convenience yield.

A natural extension of this research is to solve the HJB equation for the full two-factor problem and compare with the long-run model results. This will be left for future research.

A criticism of both the two-factor model and the long-run model is that the forest owner is required to know what the convenience yield is to formulate his optimal harvesting strategy. Convenience yield is not easily observable, but it can be calculated from futures prices. More informally, one could imagine a forest owner taking into account convenience yield in a more intuitive fashion. If lumber inventories are very low and markets are buoyant, players in the market would be aware that there is a benefit to holding an inventory of logs - i.e. convenience yield is high.

In conclusion, our results demonstrate that convenience yield has an important effect on the optimal harvesting decision and that it is worthwhile using a richer model, such as the long-term model used in this paper, when analyzing forest investment decisions, rather than relying on simple single factor models.

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A Derivation of Schwartz (1998) long-term model

Schwartz (1998) derives the one-factor long-term model based on the basic one-factor GBM model with constant convenience yield. Specifically, the spot price in the basic model follows GBM:

$$dS = (r - c)Sdt + \sigma SdZ \tag{38}$$

where c is the constant convenience yield.[20] Hence the futures price of this basic one-factor model $F(S, T)$ can be derived as:

$$F(S, T) = Se^{(r-c)T} \tag{39}$$

Based on Ito's Lemma, the futures return can be derived as $\frac{dF}{F} = \sigma dZ$. Its volatility is $\frac{Var(\frac{dF}{F})}{dt} = \frac{E[(\frac{dF}{F})^2]}{dt} = \sigma$, which is the same as the volatility of spot prices. The rate of change of the futures price[21] in this model is

$$\frac{\partial F/\partial T}{F} = r - c \tag{40}$$

The futures price of the two-factor model $F(S, \delta, T)$ is given in Equation (5). The rate of change of the futures price in this two-factor model can be derived as:

$$\frac{\partial F/\partial T}{F} = r - \hat{\alpha} + \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\rho\sigma_S\sigma_\delta}{\kappa} + \frac{e^{-2\kappa T}}{2\kappa^2} + [\hat{\alpha}\kappa + \rho\sigma_S\sigma_\delta - \frac{\sigma_\delta^2}{\kappa}]\frac{e^{-\kappa T}}{\kappa} - \delta e^{-\kappa T}$$

As time goes to infinity $T \to \infty$, this rate will converge to:

$$\frac{\partial F/\partial T}{F}\bigg|_{T\to\infty} = r - \hat{\alpha} + \frac{\sigma_\delta^2}{2\kappa^2} - \frac{\rho\sigma_S\sigma_\delta}{\kappa} \tag{41}$$

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[20] All the stochastic processes in this part are expressed in the risk-neutral world.

[21] See Schwartz (1997).

Comparing Equations (40) and (41), if we define the constant convenience yield $c = \hat{\alpha} - \frac{\sigma_\delta^2}{2\kappa^2} + \frac{\rho\sigma_S\sigma_\delta}{\kappa}$ in the long-term model, the rate of change of futures prices in long-term model will converge to that in two-factor model.

With this rate of change $r - c$, the composite price $Z(S, \delta)$ is constructed to match the futures prices of two-factor model $F(S, \delta, T)$ based on the formula for futures prices[22] $F(Z, T) = Ze^{(r-c)T}$. Hence, Z can be derived as:

$$
\begin{aligned}
Z(S, \delta) &= \lim_{T\to\infty} e^{-(r-c)T} F(S, \delta, T) \\
&= Se^{\frac{c-\delta}{\kappa} - \frac{\sigma_\delta^2}{4\kappa^3}}
\end{aligned}
\tag{42}
$$

Given this composite price, Z, expressed in Equation (42), combined with the defined constant convenience yield c, this long-term one-factor model can generate futures prices $F(Z, T)$ which closely match the long-term futures prices in the two-factor model $F(S, \delta, T)$.

Applying Ito's lemma to Equation (5), the futures return in the two-factor model can be derived as:

$$
\frac{dF}{F} = \sigma_S dZ_S - \sigma_\delta \frac{1 - e^{-\kappa T}}{\kappa} dZ_\delta
$$

Hence, the volatility of the futures return for this two-factor model is:

$$
\sigma_F^2(T) = \frac{Var(\frac{dF}{F})}{dt} = \sigma_S^2 + \sigma_\delta^2 \frac{(1 - e^{-\kappa T})^2}{\kappa^2} - 2\rho\sigma_S\sigma_\delta \frac{1 - e^{-\kappa T}}{\kappa}
\tag{43}
$$

Define the stochastic differential equation of composite price Z as:

$$
dZ = (r - c)Z dt + \sigma_F(t)Z dz
\tag{44}
$$

Therefore, the volatility of the futures return in this long-term model is the same as that in two-factor model.

B   Model comparison

This section compares model performances of single-factor models with that of two-factor model in terms of fitting market prices. Model estimation errors including RMSE and MAE

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[22]In this expression, $T \to \infty$ due to the convergence of rate of change to $r - c$.

Single-factor mean reverting model					
	F1	F2	F3	F4	All
Calibration errors of futures prices					
RMSE	31.217	14.065	0.068	9.650	17.787
MAE	23.651	10.860	0.054	7.175	10.435
Calibration errors of log futures prices					
RMSE	0.070	0.032	0.000	0.023	0.040
MAE	0.056	0.026	0.000	0.017	0.025

TABLE 12: *Estimation errors of both futures prices and log futures prices of Schwartz (1997) single-factor model expressed as RMSE and MAE of 4 futures contracts, Cdn$/MBF.*

of the three one-factor models analyzed in this paper are provided here. Plots of model implied futures prices and market futures prices are also shown in this section.

B.1 One-factor mean reverting model

Estimation errors of the one-factor mean reverting model including the Root Mean Square Error (RMSE) and Mean Absolute Error (MAE) are reported in Table 12. Comparing this table with Table 5 we find that except for the third futures contract F3, the errors of the futures contracts expressed in both ways for two-factor model are lower than those for the one-factor mean reverting model. This indicates the better performance of the two-factor model in terms of fitting market lumber derivative prices.

Plots of market futures prices and the model implied futures prices for the four futures contracts are shown in Figure 17. We observe the close match between these two time series. But comparing this figure with Figure 6 we find that except for the futures contract F3, the differences between the two futures prices for the other three futures contracts are higher for the one-factor mean reverting model.
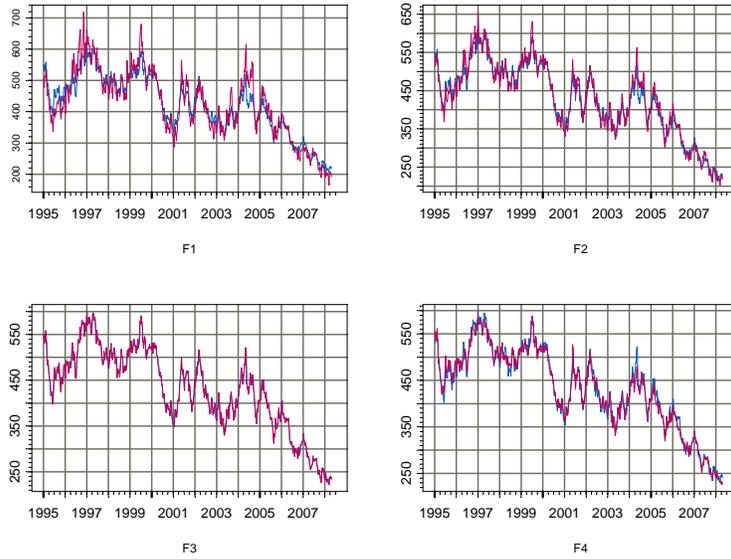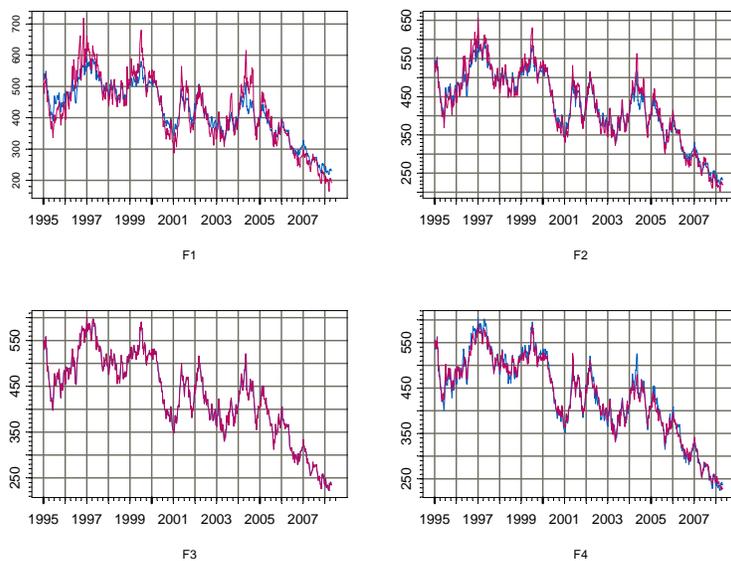
FIGURE 17: *Plots of model implied and market futures prices for the four chosen futures contracts. Weekly data from January 6th, 1995 to April 25th, 2008. Blue line: model implied futures prices. Red line: market futures prices.*

B.2 GBM model

Estimation errors of one-factor GBM model with constant convenience yield are reported in Table 13. We find that except for the third futures contract F3, the errors of the rest futures contracts expressed in both ways for the two-factor model are lower than those for the GBM model. This indicates the better performance of the two-factor model in terms of fitting market lumber derivative prices.

Plots of market futures prices and the model implied futures prices for the four futures contracts are shown in Figure 18. We can also find the close match between these two time series. But comparing this figure with figure 6 we find that except for the futures contract F3, the differences between the two futures prices for the rest three futures contracts are higher for the GBM model.

Single-factor GBM model					
	F1	F2	F3	F4	All
	Calibration errors of futures prices				
RMSE	33.761	15.260	0.000	10.570	19.264
MAE	26.123	12.093	0.000	8.080	11.574
	Calibration errors of log futures prices				
RMSE	0.079	0.036	0.000	0.024	0.045
MAE	0.063	0.029	0.000	0.019	0.028

TABLE 13: *Estimation errors of both futures prices and log futures prices of one-factor GBM model with constant convenience yield expressed as RMSE and MAE of 4 futures contracts, Cdn$/MBF.*

FIGURE 18: *Plots of model implied and market futures prices for the four chosen futures contracts. Weekly data from January 6th, 1995 to April 25th, 2008. Blue line: model implied futures prices. Red line: market futures prices.*
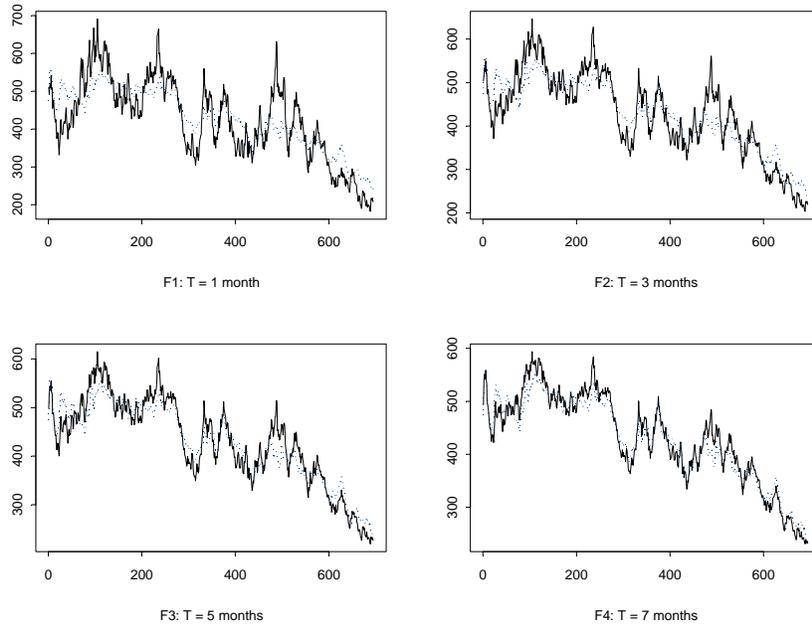
FIGURE 19: *Prices in $/MBF of model implied futures contracts with four short-term maturities for Schwartz (1997) two-factor model and Schwartz (1998) long-term model.*

C   Long-term model performance

Figures 19, 20 and 21 show the model implied futures prices with different maturities for the two-factor model and the long-term model. Comparing these three plots we observe that the differences between the two model implied futures prices are larger for contracts with short-term maturities and smaller for contracts with long-term maturities. This result is consistent with the construction of the long-term model introduced in Schwartz (1998) since the purpose of the long-term model is to match the performance of the two-factor model analyzed in Schwartz (1997) in terms of fitting the long-term futures prices. The discrepancy between these two models in terms of generating the short-term futures prices is not as important in the analysis of a long-term forestry investment.
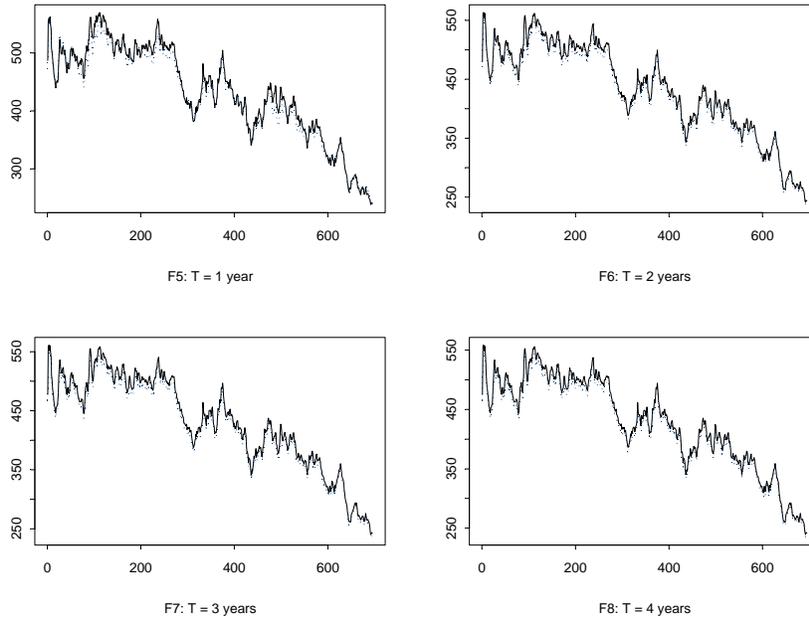
FIGURE 20: *Prices in $/MBF of model implied futures contracts with four mid-term maturities for Schwartz (1997) two-factor model and Schwartz (1998) long-term model.*
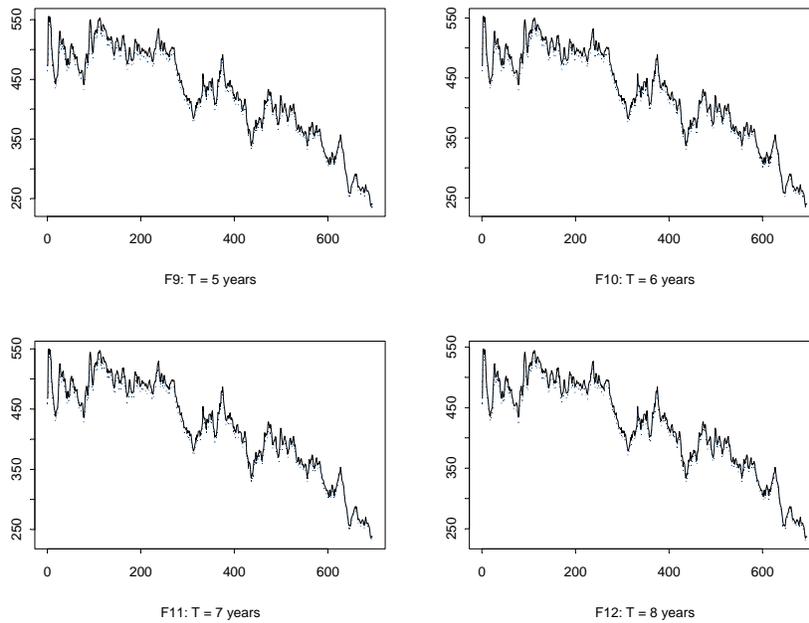

FIGURE 21: *Prices in $/MBF of model implied futures contracts with four long-term maturities for Schwartz (1997) two-factor model and Schwartz (1998) long-term model.*