# Public vs. Private Offers in the Market for Lemons

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**Abstract**

We analyze a version of Akerlof's market for lemons in which a sequence of buyers make offers to a long-lived seller endowed with a single unit for sale. We consider both the case in which previous offers are observable and the case in which they are not. When offers are observable, trade may only occur in the first period, so that the resulting inefficiency may be worse than in the static model. In the unobservable case, trade occurs with probability one eventually.

## 1 Introduction

In this paper, we examine the relationship between the outcome of trade with asymmetric information and the observability of past offers. Search models typically assume that successive potential buyers never learn anything over time about the offering of the seller, so that the distribution of offers faced by the seller is stationary. On the other hand, bargaining models usually assume that potential buyers observe the entire public history, including past offers that were rejected. This affects the buyers' beliefs about the quality of the good that is being put on sale, and therefore the offers that are submitted.

While most markets characterized by adverse selection fall between those two extremes, they widely vary in this respect. In the housing market, potential buyers typically know the time on market, as well as the list price that is quoted by the seller. Buyers of second-hand cars do not usually have any reliable information regarding what offers and how many offers have been turned down by the seller. Employers may obtain verifiable information about the duration of unemployment of potential employees, but much less evidence regarding offers they may have rejected in the meantime. In yet other markets, it is up to the seller to decide ex ante whether his decisions will be public or private.

We consider two variants of Akerlof's market for lemons. In both variants, a sufficiently patient single seller with private information faces a sequence of (short-run) potential buyers who submit offers. Buyers know how long the item has been put on sale. In the first variant, buyers also know the offers that were rejected, while in the second those offers remain unobservable. All that is known by a potential buyer is that all the previous offers must have been turned down.

Results contrast sharply. In the case of public offers, only the first potential buyer submits a serious offer. That is, only the first offer is accepted with positive probability. In case this offer is turned down by the seller, as does occur if his valuation is high enough, all later offers are losing offers: from that point on, the seller rejects all equilibrium offers. Therefore, merely allowing for trade over time does not solve the adverse selection problem. In fact, it may worsen it. Not so, however, in the case of private offers. In every equilibrium, trade occurs with probability one eventually, and there always exist equilibria in which trade occurs in finite time.

This striking contrast can be understood as follows. Consider the case of public offers. Suppose that an out-of-equilibrium high offer is submitted. Because turning it down is interpreted as very favorable news by the next buyer, whose offer following such a deviation will be serious, the seller accepts this high offer only if his valuation is especially low. That is, the perspective of selling at a higher price tomorrow exacerbates the adverse selection problem today, making the high offer unattractive to the seller. While this intuition helps explain why it is equilibrium behavior for all potential buyers but the first one to submit losing offers, it is worth pointing out that we show that this is the *unique* equilibrium.

Consider next the case of private offers. For the sake of contradiction, suppose that from some point on, all equilibrium offers are losing. Because offers are unobservable, the behavior of future bidders is not affected by a deviation by the current bidder. Therefore, a potential buyer would strictly prefer an offer slightly below his worst-case valuation for the good to his equilibrium offer, as the former always implies a strictly positive profit conditional on trade, and it is necessarily accepted by the seller with some positive probability.

In terms of actual payoffs, comparisons are less clear-cut. For instance, the dynamic game with public offers may actually be more or less efficient than the static game, depending on the parameters. As we argue, the low probability of trade in the dynamic model is driven by competition among potential buyers. Somewhat paradoxically, if there is a unique, long-lived and equally patient potential buyer, the good is traded with probability one in the unique equilibrium outcome. To shed more light on the relationship between the static game and the infinite-horizon game, we provide a detailed analysis of the game with finite, but arbitrary, horizon.

While it is possible to explicitly solve for equilibrium strategies in the case of private offers, the case of public offers is more complex, and we provide only a partial characterization of the equilibrium strategies. We show by means of specific example that equilibrium multiplicity can occur, and we prove that, quite generally, all potential buyers but the first and the last ones must use mixed strategies. Explicit solutions are available in some special cases: (i) the case of two periods, (ii) the case of two values, (iii) some particular ranges of parameters.

## 1.1 Related Literature

Our contribution is related to three strands of literature. First, several authors have already considered dynamic versions of Akerlof's model. Second, our model shares many features with the bargaining literature. Finally, a pair of papers have investigated the difference between public and private offers in the framework of Spence's educational signaling model.

Janssen and Roy (2002) consider a dynamic, competitive durable good setting, with a fixed set of sellers. They prove that all trade for *all* qualities of the good occurs in finite time. While there are several inessential differences between their model and ours, the critical difference lies in the market mechanism. In their model, the price in every period must clear the market. That is, by definition, the market price must be at least as large as the good's expected value to the buyer conditional on trade, with equality if trade occurs with positive probability (this is condition (ii) of their equilibrium definition).[1] This expected value is derived from the equilibrium strategies when such trade occurs with positive probability, and is assumed to be at least as large as the lowest unsold value even when no trade occurs in a given period (this is condition (iv) of their definition). This immediately implies that the price exceeds the valuation to the lowest quality seller, so that trade must occur eventually, if not in a given period. Also related are Taylor (1999), Hendel and Lizzeri (1999), Blouin (2003) and Hendel, Lizzeri and Siniscalchi (2005).

In the bargaining literature, the closest paper is Evans (1989), which shows that, with correlated values, the bargaining may result in an impasse when the buyer is too impatient relative to the seller. Our assumption of short-run buyers is less general, since it implies that the buyer is actually myopic.[2] However, Evans considers the case of binary values. Moreover, there is no gain from trade in case of a low value. In our set-up, Proposition 1 holds instead quite generally, and trade is always strictly efficient. In his appendix, Vincent (1989) provides another example of equilibrium in which bargaining breaks down. As in Evans, there are only two possible values in his object. While there are gains of trade for both values in his case, it is not known whether his example admits other equilibria, potentially exhausting all gains of trade eventually. Other related contributions include Cramton (1984), Gul and Sonnenschein (1988) and Vincent (1990). Other related contributions include Cramton (1984), Gul and Sonnenschein (1988), Vincent (1990), and Deneckere and Liang (2006).

Nöldeke and van Damme (1990) and Swinkels (1999) develop an analogous distinction in Spence's signalling model. Both consider a discrete-time version of the model, in which education is acquired continuously and a sequence of short-run firms submit offers that the worker can either accept or reject. Nöldeke and van Damme considers the case of public offers, while Swinkels focuses mainly on the case of private offers. Nöldeke and van Damme shows that there is a unique equilibrium outcome that satisfies the never a weak best response requirement, and that

[1]More precisely, equality obtains whenever there is a positive measure of goods' qualities traded, since there is a continuum of sellers in their model.

[2]There is no difficulty in generalizing Proposition 1 to the case of an impatient, but not myopic buyer, but we feel that there is no much gain from such generality.

the equilibrium outcome converges to the Riley outcome as the time interval between consecutive periods shrinks. With private offers, Swinkels proves that the sequential equilibrium outcome is unique, and shows that, in contrast to the public case, it involves pooling in the limit. While the set-up is rather different, the logic driving these results is similar to ours, at least with public offers. Indeed, in both papers, when offers are observable, firms (buyers) are deterred from submitting mutually beneficial offers because rejecting such an offer sends a strong signal to future firms (buyers) that is so attractive that only very low types would prefer to accept the offer immediately.

# 2 The model

We consider a dynamic game between a single seller, with one unit for sale, and a countable infinity of potential buyers, or buyers for short. Time is discrete, and indexed by $n = 1, \ldots, \infty$. At each time $n$, one buyer makes an offer for the unit. Each buyer makes an offer only at one time, and we refer to buyer $n$ as the buyer who makes an offer in period $n$, provided the seller has accepted no previous offer. After observing the offer, the buyer either accepts or turns down the offer. If the offer is accepted, the game ends. If the offer is turned down, a period elapses and it is the next buyer's turn to submit an offer.

The (reservation) value of the unit is seller's private information. The reservation value to the seller is $c(x)$, where the random variable $x$ is determined by nature and uniformly distributed over the interval $[\underline{x}, 1]$, $\underline{x} \in [0, 1)$. We interpret $x$ as an index, such as the quality of the good. The valuation of the unit to buyers is common to all of them, and is denoted $v(x)$. Buyers do not observe the realization of $x$, but its distribution is common knowledge.

We assume that $c$ is strictly increasing, positive and twice differentiable, with bounded derivatives. We assume that $v$ is positive, strictly increasing and differentiable, with bounded derivative. We set $M_{c'} = \sup|c'|$, $M_{c''} = \sup|c''|$, $M_{v'} = \sup|v'|$, $M = \max(M_{c'}, M_{c''}, M_{v'})$, and $m = \inf|v'|$.

Observe that the assumption that $x$ is uniformly distributed is with little loss of generality, since few restrictions are imposed on the functions $v$ and $c$.[3]

We assume that gains from trade are always positive with $\nu := \inf_x \{v(x) - c(x)\} > 0$. In examples and extensions, we shall often restrict attention to the case in which $v(x) = x$ and $c(x) = \alpha x$, with $\underline{x} > 0$, i.e. the reservation value to the seller is a fraction $\alpha \in (0, 1)$ of the valuation $x$ to the buyers. The seller is impatient, with discount factor $\delta < 1$. We are particularly interested in the case in which $\delta$ is sufficiently large. To be specific, we set $\bar{\delta} = 1 - m/3M$, and will always assume $\delta > \bar{\delta}$. In each period in which the seller owns the unit, he derives a per-period gross surplus of $(1 - \delta)\, c(x)$. Therefore, the seller can always guarantee a gross surplus of $c(x)$ by never selling the unit.

[3] In particular, our results are still valid if the distribution of $x$ has a bounded density, bounded away from zero.

Buyer $n$ submits an offer $p_n$ that can take any real value. An outcome of the game is a triple $(x, n, p_n)$, with the interpretation that the realized index is $x$, and that the seller accepts buyer $n$'s offer of $p_n$ (implying that he rejected all previous offers). The case $n = \infty$ corresponds to the outcome in which the seller rejects all offers (set $p_\infty$ equal to zero). The seller's von Neumann-Morgenstern utility function over outcomes is his net surplus:

$$\sum_{i=1}^{n-1} (1-\delta)\,\delta^{i-1} c(x) + \delta^{n-1} p_n - c(x) = \delta^{n-1}\left(p_n - c(x)\right),$$

when $n < \infty$, and zero otherwise. An alternative formulation that is equivalent to the one above is that the seller derives no per-period gross surplus from owning the unit, but incurs a production cost of $c(x)$ at the time he accepts the buyer's offer. It is immediate that this interpretation yields the same utility function.

Buyer $n$'s utility is $v(x) - p_n$ in the outcome $(x, n, p_n)$, and zero otherwise. We define the players' expected utility over lotteries of outcomes, or payoff for short, in the standard fashion. We allow for mixed strategies on the part of all players.

We consider both the case in which offers are public, or observable, and the case in which previous offers are private, or not observable. It is worth pointing out that the results for the case in which offers are public would also hold for any information structure (about previous offers) in which each buyer $n > 1$ observes the offer made by buyer $n - 1$.

A history $h^{n-1} \in H^{n-1}$ in case no agreement has been reached at time $n$ is a sequence $(p_1, \ldots, p_{n-1})$ of offers that were submitted by the buyers and rejected by the seller (we set $H_0$ equal to $\{\varnothing\}$). A behavior strategy for the seller is a sequence $(\sigma_S^n)$, where $\sigma_S^n$ is a probability transition from $[\underline{x}, 1] \times H^{n-1} \times \mathbb{R}$ in to $\{\texttt{Accept}, \texttt{Decline}\}$ mapping the realized valuation $v$, the history $h^{n-1}$, and buyer $n$'s offer $p_n$ into a probability of acceptance. In the public case, a strategy for buyer $n$ is a probability transition $\sigma_B^n$ from $H^{n-1}$ to $\mathbf{R}$.[4] In the private case, a strategy for buyer $n$ is a probability distribution $\sigma_B^n$ over $\mathbf{R}$.

Observe that, whether offers are public or private, the seller's optimal strategy must be of the cut-off type. That is, if $\sigma_S^n(x, h^{n-1}, p_n)$ assigns positive probability to `Accept` for some $v$, then $\sigma_S^n(x', h^{n-1}, p_n)$ assigns probability one to `Accept`, for all $x' > x$. The proof of this skimming property can be found in Fudenberg and Tirole (Chapter 10, Lemma 10.1), for instance. The *infimum* of such valuations $x$ is called the *marginal valuation* (at history $(h^{n-1}, p_n)$ given the strategy profile). Since the specification of the action of the seller with marginal valuation does not affect payoffs, we also identify equilibria which only differ in this regard. For definiteness, in all formal statements, we shall follow the convention that the seller with marginal valuation rejects the offer. For conciseness, we shall omit to specify that some statements only hold 'with probability one'. For instance, we shall say that the seller accepts the offer when he does so with

[4]That is, for each $h^{n-1} \in H^{n-1}$, $\sigma_B^n(h^{n-1})$ is a probability distribution over $\mathbf{R}$, and the probability $\sigma_B^n(\cdot)[A]$ assigned to any Borel set $A \subset \mathbf{R}$ is a measurable function of $h^{n-1}$

probability one. Standard arguments also establish that buyers never submit any offer that is strictly larger than $c(1) = \bar{c}$, the highest possible reservation value to the seller.

We use the perfect bayesian equilibrium concept. In the public case, we will compute the belief of buyer $n$ after a (possibly out-of-equilibrium) history $h^{n-1}$ under the assumption that the seller rejected previous offers out of rational purposes.[5] Thus, the belief of buyer $n$ over $x$ is the uniform distribution over some interval $[\underline{x}_n, 1]$.

In the private case, the only information sets which are reached with probability zero are associated with stages for which the probability is one that the seller will have accepted some earlier offer. In such a stage, we assume that buyer $n$'s belief assigns probability 1 to $x = 1$.

Given some (perfect bayesian) equilibrium, a buyer's offer is *serious* if it is accepted by the seller with positive probability. An offer is *losing* if it is not serious. Clearly, the specification of losing offers in a equilibrium is to a large extent arbitrary. Therefore, statements about uniqueness are understood to be made up to the specification of the losing offers. Finally, an offer is a *winning* offer if it is accepted with probability one.

We briefly sketch here the static version of the dynamic game described above: there is only one potential buyer, who submits a take-it-or-leave-it offer. The game then ends whether the offer is accepted or rejected, with payoffs specified as before (with $n = 1$). The model considered by Akerlof (1970) is not quite the static version of this game, as the market mechanism adopted there is Walrasian. Much closer is the second variant analyzed by Wilson (1980), although he considers a continuum of buyers. Clearly, the seller accepts any offer $p$ provided $p > c(x)$. Therefore, the buyer offers $c(x^*)$, where $x^*$ maximizes

$$\int_{\underline{x}}^{x} (v(t) - c(x))\, dt,$$

over $x \in [\underline{x}, 1]$. More generally, given $t \in [\underline{x}, 1)$, let $x^*(t)$ denote the marginal valuation given the optimal offer when the distribution is uniform over $[t, 1]$. Observe that $x^*(t) > t$ for all $t \in [\underline{x}, 1)$.

# 3 The two-buyer case

We here solve the two-buyer case.

## 3.1 Observable offers

## 3.2 Hidden offers

Equilibrium behavior entails some indeterminacy. Whereas buyer 2's behavior is uniquely determined, there is a continuum of strategies for buyer 1 that support equilibrium and yield different

[5] This is well-defined as long as a seller with $x = 1$ would find it optimal to reject all offers along $h^{n-1}$. If this is not the case, the belief of buyer $n$ is assumed to assign probability 1 to $x = 1$.

payoffs. As we argue later, this indeterminacy is linked to the linearity of $v(\cdot)$.

**Proposition 3.1** *At any equilibrium, buyer 1's payoff is zero, and buyer 2's payoff is positive. The strategy of buyer 2 assigns probability* $(2\alpha-1)/2\alpha\delta$ *to the price offer* $\alpha \times \frac{\underline{x}}{\beta}$*, and probability* $1-(2\alpha-1)/2\alpha\delta$ *to the price offer* $\alpha \times \frac{\alpha \underline{x}}{\beta}$*.*

Since buyer 2 is the last buyer, type $t$ of the seller accepts the lower (higher) offer iff $t \geq \tilde{x}_2 := \alpha\frac{\underline{x}}{\beta}$, resp. $t \geq \overline{x}_2 := \frac{\underline{x}}{\beta}$.

**Proof.** We let an equilibrium be given, and first prove the assertion on equilibrium payoffs. Denote $\underline{x}_2$ the marginal type of buyer 1's lowest offer: thus, buyer 2's belief on the seller's type is concentrated on the interval $[\underline{x}_2, 1]$. Note that $\underline{x}_2 < 1$, since buyer 1 would have to offer a price of $\alpha$ to attract type 1, and would then incur a loss. When offering a price in $(\alpha\underline{x}_2, \underline{x}_2)$, buyer 2 obtains a positive payoff. His equilibrium payoff is a fortiori positive. In particular, the marginal type, say $x_{\min}$, of buyer 2's lowest equilibrium offer, is bounded away from $\underline{x}_2$. In this simple two-stage context, we make an observation of more general breadth. For $x \in [\underline{x}, x_{\min}]$, and denoting by $\tilde{p}_2$ the (random) price offer of buyer 2, the minimal offer $p_1(x)$ that type $x$ will accept is given by

$$p_1(x) - \alpha x = \delta\left(\mathbf{E}\left[\tilde{p}_2\right] - \alpha x\right),$$

since $\tilde{p}_2 \geq \alpha x$ with probability 1.

As a function of the marginal type $x$, buyer 1's payoff thus writes

$$\begin{aligned}\pi_1(x) &= (x-\underline{x})\left(\frac{x+\underline{x}}{2} - p_1(x)\right) \\ &= (x-\underline{x})\left(\frac{\underline{x}}{2} - \delta\mathbf{E}[\tilde{p}_2] + x\left(\frac{1}{2} - \alpha(1-\delta)\right)\right).\end{aligned}$$

Since $\delta > \dfrac{\alpha}{2\alpha-1}$, $\pi_1$ is strictly convex on $[\underline{x}, x_{\min}]$. Hence, buyer 1 makes no offer with marginal type in $(\underline{x}, x_{\min})$. Since $\underline{x}_2 < x_{\min}$, one must have $\underline{x}_2 = \underline{x}$: buyer 1 submits a losing offer with positive probability. His equilibrium payoff is therefore zero.

We next argue by contradiction that none of the buyers' equilibrium strategies $\sigma_1, \sigma_2$ is pure. If $\sigma_1$ were pure, it would assign probability 1 to a losing offer. Hence buyer 2 would be facing the "static", monopsonic, problem: he would submit the offer $\alpha^2\underline{x}/\beta$ with probability one, thereby attracting all types up to $\tilde{x}_2 = \alpha\underline{x}/\beta$. Assume more generally that $\sigma_2$ is pure, and assigns probability 1 to some price $\alpha\tilde{x}$. This readily implies, as above, that buyer 1 submits no offer with marginal type in $(\underline{x}, \tilde{x})$. We now show that buyer 1 would also not submit offers with marginal types in $(\tilde{x}, 1]$. Indeed, for such $x \in (\tilde{x}, 1]$, the minimal price that buyer 1 must offer so as to attract type $x$, is simply $p_1(x) = \alpha x$. As a function of $x \in (\tilde{x}, 1]$, buyer 1's payoff thus writes $\pi_1(x) = (x - \underline{x})\left((1/2 - \alpha)\, x + \underline{x}/2\right)$, hence is quadratic and strictly concave. A type $x > \tilde{x}$ associated with an equilibrium offer would maximize $\pi_1$ over $(\tilde{x}, 1)$, and satisfy $\pi_1(x) = 0$. Hence, both factors $x - \underline{x}$ and $(1/2 - \alpha)\, x + \underline{x}/2$ would vanish – a contradiction. We are now left with only two possible equilibrium offers for buyer 1: a losing offer, and the offer $\alpha\tilde{x}$, with marginal type $\tilde{x}$. Facing any such strategy, the distribution of types below $\tilde{x}$, as faced by buyer 2, is still uniform over $[\underline{x}, \tilde{x}]$. Since $\pi_2(\tilde{x}) > 0$, and since $p_1(\tilde{x}) = p_2(\tilde{x})$, this implies $\pi_1(\tilde{x}) > 0$ – a contradiction.

We now prove that buyer 2's strategy assigns positive probability to exactly two points. We denote by $G_1$ the c.d.f. of the marginal type of buyer 1's offer. We also set $F_2(x) = \int_{\underline{x}}^{x} G_1(t)dt$: $F_2(x)$ it the (unconditional) probability that the seller os of type $t \leq x$, and has rejected buyer 1's offer – it is continuous and convex over $[\underline{x}, 1]$. Thus, buyer 2's payoff when offering $\alpha x$, is equal to

$$\pi_2(x) = \int_{\underline{x}}^{x} (t - \alpha x) dF_2(t).$$

We proceed in two steps. First, we prove that, given any two (marginal types of) equilibrium offers $x_{21} < x_{22}$ of buyer 2, $\sigma_1$ assigns positive probability to offers with marginal type in $(x_{21}, x_{22})$. Assume to the contrary that $G_1$ is constant over $[x_{21}, x_{22})$. This implies that $\pi_2$ is strictly concave over $(x_{21}, x_{22})$, and continuous at both endpoints. Hence $\pi_2(x_{21}) \neq \pi_2(x_{22})$ – a contradiction.

Second, we prove that, given any two types $x_{11}, x_{12} \in (\underline{x}, 1]$ associated with equilibrium offers of buyer 1, $\sigma_2$ assigns probability 0 to offers with marginal type in $(x_{11}, x_{12})$. We argue as follows. Sice buyer 1's equilibrium payoff is zero, one has

$$p_1(x = \frac{x + \underline{x}}{2}, \text{ for } x = x_{11}, x_{12}.$$

On the other hand, $p_1(x) = \alpha x + \mathbf{E}[(\tilde{p}_2 - \alpha x)1_{\tilde{p}_2 \geq \alpha x}]$ is a convex function. For any price $p$, set $\phi_p(x) := \alpha x + \mathbf{E}[(\tilde{p}_2 - \alpha x)1_{\tilde{p}_2 \geq p}]$. Plainly, $p_1(x) \geq \phi_p(x)$ for each $x$, with equality if $p = \alpha x$. Consider any $p$ in the interval $(\alpha x_{11}, \alpha x_{12})$. For such $p$'s, the two affine maps $\frac{x + \underline{x}}{2}$ and $\phi_p(x)$ coincide at the interior point $x = p/\alpha$, and $\frac{x + \underline{x}}{2} \geq \phi_p(x)$ everywhere. Hence, $\phi_p(x) = p_1(x)$ for all $x \in [x_{11}, x_{12}]$. Thus, the probability that $\tilde{p}_2$ fall in the interval $[\alpha x_{11}, p)$ is zero, and our claim follows.

We now combine these two observations. Assume that the support of the probability distribution $\sigma_2$ contains at least three points, with marginal types $\underline{x} < x_{21} < x_{22} < x_{23}$ – that is, any neighborhood of resp. $\alpha x_{21}$, $\alpha x_{22}$ and $\alpha x_{23}$ receives positive probability under $\sigma_2$. Hence, buyer 1's strategy assigns positive probability to offers with marginal type in some interval $[x_{21} + \varepsilon, x_{22} - \varepsilon]$, and to offers with marginal type in $[x_{22} + \varepsilon, x_{23} - \varepsilon]$. By the second observation above, this implies in turn that $\sigma_2$ cannot assign positive probability to offers with marginal type in $(x_{22}-\varepsilon, x_{22}+\varepsilon)$ – this is the desired contradiction.

To summarize our findings so far:

- buyer 2's equilibrium strategy assigns positive probability to some offers $\alpha x_{21} < \alpha x_{22}$, with marginal types $x_{21}$ and $x_{22}$;
- buyer 1's equilibrium strategy assigns positive probability to losing offers and to offers with marginal type in $[x_{21}, x_{22}]$, and probability zero to other offers.

It is also readily checked that $\sigma_1$ must assign probability zero to the offers whose marginal types are $x_{21}$ and $x_{22}$.

The lowest type $x_{21}$ must coincide with $\alpha \underline{x}/\beta$, the static optimum. Indeed, for any offer with marginal type in $[\underline{x}, x_{21}]$, buyer 2 faces a uniform distribution of type. Hence, if $x_{21} > \alpha\underline{x}/\beta$, buyer 2 would gain from lowering his offer. Assume now that $x_{21} < \alpha\underline{x}/\beta$. If buyer 2 would increase his lower offer to $\alpha^2\underline{x}/\beta$, he would attract all types up to $\alpha\underline{x}/\beta$, and would face a more favorable distribution of types than in the static case. Hence his profit would be higher than when offering $\alpha x_{21}$.

We complete the characterization of buyer 2's strategy by looking into buyer 1's equilibrium conditions. The price $p_1(x)$ that buyer 1 must offer in order to attract a type $x \in (x_{21}, x_{22})$ is given by

$$p_1(x) = \alpha x + \delta y(\alpha x_{22} - \alpha x). \tag{1}$$

Since $\sigma_1$ assigns positive probability to offers with marginal type in $(x_{21}, x_{22})$, one has $p_1(x) = \frac{x+\underline{x}}{2}$, for some $x \in (x_{21}, x_{22})$, and $p_1(x) \geq \frac{x+\underline{x}}{2}$ for all $x \in (x_{21}, x_{22})$. Since $p_1(\cdot)$ is affine, the equality $p_1(x) = \frac{x+\underline{x}}{2}$ holds throughout the interval $(x_{21}, x_{22})$. Identification with (10) yields

$$\alpha(1-\delta y) = \frac{1}{2} \text{ and } \frac{\underline{x}}{2} = \delta\alpha y x_{22}.$$

This completes the proof of Proposition 3.1. ■

When facing $\sigma_{\acute{e}^*}$, any offer with marginal type $\underline{x}$, or in $[x_{21}, x_{22}]$ yields a zero payoff, and any other offer yields a negative payoff. Thus, given any such a strategy $\sigma_1$, the pair $(\sigma_1, \sigma_2^*)$ is an equilibrium, if and only if $\sigma_2^*$ is a best reply to $\sigma_1$. A necessary and sufficient condition for this to hold is that

$$\pi_2(\tilde{x}_2) = \pi_2(\bar{x}_2) \text{ and } \pi_2(x) \leq \pi_2(\bar{x}_2). \text{ for each } x \geq \bar{x}_2.$$

Integration by parts yields

$$\begin{aligned} \pi_2(x) &= \int_{\underline{x}}^{x} (t - \alpha x) dF_2(t) \\ &= \pi_2(\tilde{x}_2) + (1-\alpha)\left(xF_2(x) - \tilde{x}_2 F_2(\tilde{x}_2)\right) - \int_{\tilde{x}_2}^{x} F_2(t)dt. \end{aligned}$$

The function $F_2$ is convex and continuous over $[_tildex_2, 1]$, with $F_2(\tilde{x}_2) \leq \tilde{x}_2 - \underline{x}$ and $F_2(x) = F_2(\tilde{x}_2) + (x - \tilde{x}_2)$ for $x \geq \tilde{x}_2$. Conversely, any such function is associated with some strategy of buyer 1. Among these functions, a necessary and sufficient equilibrium condition is thus

$$(1-\alpha)\left(xF_2(x) - \tilde{x}_2 F_2(\tilde{x}_2)\right) \leq \int_{\tilde{x}_2}^{x} F_2(t)dt$$

for all $x \geq \tilde{x}_2$, with equality for $x = \tilde{x}_2$. Plainly, there is a continuum of such strategies. Not surprisingly, these conditions do not depend on the discount factor.

Among these strategies, it turns out that there is a unique one that assigns positive probability, $1-\lambda$ and $\lambda$, to exactly two offers, with marginal type $\underline{x}$ and $\overline{x}_1 \in (\tilde{x}_2, \bar{x}_2)$. Indeed, for given $\lambda$ and $\bar{x}_1$, the strategy of buyer 1 defines an equilibrium iff $\pi_2(\tilde{x}_2) = \pi_2(\bar{x}_2)$ and $\pi_2'(\tilde{x}_2) = 0$. Elementary manipulations show that these conditions imply

$$\lambda = \frac{-2+\alpha+\sqrt{\alpha}\sqrt{4+\alpha}}{2\alpha} \text{ and } \bar{x}_1 = \underline{x}\left(1 + \frac{1-\alpha}{\alpha\lambda}\right) \in (\tilde{x}_2, \bar{x}_2).$$

We denote $\sigma_1^*$ the corresponding strategy.

In all equilibria, buyer 1's payoff is zero, while buyer 2's payoff, $\pi_2(\tilde{x}_2)$, is proportional to the weight $1-y$ assigned to losing offers in buyer 1's strategy. The (necessary) condition $\pi_2'(\bar{x}_2)$ writes $(1-\alpha)(1-y) = \dfrac{1-\alpha}{\alpha} - \beta \displaystyle\int_{\tilde{x}_2}^{\bar{x}_2} dF_2(t)$. Therefore, the higher buyer 2's payoff, the lower $\displaystyle\int_{\tilde{x}_2}^{\bar{x}_2} dF_2(t)$ – the probability that teh seller is of some type $t \in [\tilde{x}_2, \bar{x}_2]$ and rejects buyer 1's offer.

On the other hand, the social surplus from having type $t$ accepting an offer is $(1-\alpha)t$. Observe that any type $t \leq \tilde{x}_2$ sells for sure to buyer 2, if not to buyer 1. The probability that type $t > \tilde{x}_2$ trades with buyer 2, conditional on not selling to buyer 1, is independent, both of $t$ and on the equilibrium. Neglecting delay, the social surplus is therefore decreasing w.r.t. $\displaystyle\int_{\tilde{x}_2}^{\bar{x}_2} dF_2(t)$: maximization of the social surplus amounts here to optimizing buyer 2's payoff. It can be checked that in this respect, the optimal equilibrium is $(\sigma_1^*, \sigma_2^*)$.

# 4 Observable offers

## 4.1 Main result

In this section, we maintain the assumption that offers are public. Observe that the equation

$$\int_{x}^{1} \left(v\left(t\right) - \bar{c}\right) dt = 0$$

admits either no or exactly one solution $x$ in $[\underline{x}, 1)$ since its integrand is strictly positive (negative) above (below) the unique root of $v(t) = \bar{c}$. Obviously, this solution $x_1$ exists if and only if:

$$\int_{\underline{x}}^{1} (v(t) - \bar{c})\, dt < 0,$$

that is, if it is unprofitable for the first buyer to submit an offer that is accepted with probability one by the seller. More generally, given $1 =: x_0 > x_1 > \cdots > x_k$, define $x_{k+1}$ as the unique solution in $[\underline{x}, x_k)$, if any, of the equation

$$\int_{x}^{x_k} (v(t) - c(x_k))\, dt = 0.$$

Clearly, $x_k - x_{k+1}$ is bounded away from zero, hence this process must eventually stop. The resulting finite sequence $\{x_k\}_{k=0}^{K}$, $x_k \subset [\underline{x}, 1]$, all $k$, is easy to compute for special functions $v$ and $c$. For instance, if $c(x) = \alpha v(x) = \alpha x$, with $\alpha > 1/2$, one has $x_k = (2\alpha - 1)^k$. The sequence $\{x_k\}$ plays an important role in Proposition 4.1.

**Proposition 4.1** *Assume that $x_K > \underline{x}$, and $\delta > \bar{\delta}$. There is a (essentially) unique equilibrium, which is independent of $\delta$. On the equilibrium path, the first buyer submits the offer $c(x_K)$, which the seller accepts if and only if $x < x_K$. If the offer is rejected, all buyers $n > 1$ submit a losing offer.*

For each $n > 1$, buyer $n$'s strategy offers $c(x_k)$, after any history $h^{n-1}$ with marginal type $x \in (x_{k+1}, x_k]$.

**Proof**: The proof is by induction over $K$. The proof for $K = 0$ is in most respects identical to the proof of the induction step, and we therefore provide only the latter. We let an equilibrium be given and assume that, for every $n \geq 1$ and after any history $h^{n-1}$ on the equilibrium path such that $\underline{x}_n = x$, buyer $n$ offers $c(x_l)$ whenever $x \in (x_{l+1}, x_l]$ for some $l < k$. We now prove that the same conclusion holds for $l = k$. The proof is broken into the following four steps:

- whenever $\underline{x}_n = x \in (x_{k+1}, x_k)$, no equilibrium offer of buyer $n$ is accepted by some type $s > x_k$;

- whenever $\underline{x}_n = x \in (x_{k+1}, x_k]$, if an equilibrium offer of buyer $n$ is accepted by $s = x_k$, then all subsequent offers are losing ones; besides, if $\underline{x}_n = x_k$, the unique equilibrium offer of buyer $n$ is $c(x_k)$;

- whenever $\underline{x}_n = x \in (x_{k+1}, x_k]$ is close enough to $x_k$, the unique equilibrium offer of buyer $n$ is $c(x_k)$, which the seller accepts if and only his type is at most $x_k$;

- whenever $\underline{x}_n = x \in (x_{k+1}, x_k]$, the unique equilibrium offer of buyer $n$ is $c(x_k)$, which the seller accepts if and only his type is at most $x_k$.

Step 1: If buyer $n$ submits an offer $p(s)$ with marginal type $s \in (x_{l+1}, x_l]$ for some $l < k$, the following offer is $c(x_l)$ by the induction hypothesis. Hence, $p(s)$ must solve

$$p(s) - c(s) = \delta(c(x_l) - c(s)),$$

so that buyer $n$'s payoff is

$$\frac{1}{1-x} \int_x^s (v(t) - \delta c(x_l) - (1-\delta)c(s))\, dt.$$

As a function of $s$, the integral is twice differentiable over the interval $(x_{l+1}, x_l]$, with first and second derivatives given by

$$v(s) - \delta c(x_l) - (1-\delta)c(s) - (1-\delta)c'(s)(s-x)$$

and

$$v'(s) - 2(1-\delta)c^{''}(s)(s-x).$$

Since $(2M_{c'} + M_{c''})(1-\delta) < m$, buyer $n$'s payoff is strictly convex over $(x_{l+1}, x_l]$. Since buyer $n$'s payoff is negative for $s = x_l$, the claim follows.

Step 2: We argue by contradiction. We thus assume that, for some $n$ and $h^{n-1}$ with $\underline{x}_n = x \in (x_{k+1}, x_k)$, an equilibrium offer $p_n$ by buyer $n$ with marginal type $x_k$ is eventually followed, with positive probability, by a serious offer. This implies $p_n > c(x_k)$. Let $\bar{p}$ be the supremum of all such offers (with marginal type $x_k$), where the supremum is taken over all $n$ and $h^{n-1}$.

Note that, for given any buyer $n$ and history $h^{n-1}$, the offer $p_n(x)$ with marginal type $x \leq x_k$, does not exceed either $\bar{p}$. Indeed, denoting $p^*$, the supremum of all such offers, and $\tau_x$ the first buyer which submits an offer, which type $x$ accepts, one has $p_n(x) - c(x) = \mathbf{E}\left[\delta^{\tau - x_n}(p_{\tau_x} - c(x)) | \tau_x > n\right] \leq \delta \max(\bar{p}, p^*) - c(x))$, hence $p^* \leq \bar{p}$.

Consider a buyer and a history – still denoted $n$ and $h^{n-1}$ – who submits an equilibrium offer $p_n$ with marginal type $x_k$, and such that $p_n > (1-\delta)c(x_k) + \delta\bar{p}$. If instead, buyer $n$ deviates and submits a serious offer $p(s)$ with marginal type $s < x_k$, then $p(s)$ does not exceed

$$p(s) \leq (1-\delta)c(s) + \delta\bar{p} \leq (1-\delta)c(x_k) + \delta\bar{p}.$$

By choosing $s$ close enough to $x_k$, buyer $n$'s payoff, $\frac{1}{1-x}\int_x^s \{v(t) - p(s)\}\, dt$, is thus higher than the equilibrium payoff, $\frac{1}{1-x}\int_x^{x_k} \{v(t) - p_n\}\, dt$ – a contradiction.

We turn to the second assertion, and let $n$ and $h^{n-1} \in H^{n-1}$ be given, with $\underline{x}_n = x_k$. Since $\underline{x} < x_K$, there must exist, along $h^{n-1}$, a buyer who submitted a serious offer with marginal type $x_k$. As we just proved, any such offer is necessarily followed by losing offers. In particular, buyer $n$'s equilibrium offer is $c(x_k)$.

Step 3: Let $n$ and $h^{n-1} \in H^{n-1}$ be given, with $\underline{x}_n = x < x_k$. Consider a potential offer $p(s)$, with marginal type $s$. Obviously, $p(s) \geq c(s)$. Observe also that by Step 2, $p(s) - c(s)$ converges to zero as $s$ increases to $x_k$. Hence, buyer $n$'s payoff, $\frac{1}{1-x}\int_x^s \{v(t) - p(s)\}\, dt$, is at most $\frac{1}{1-x}\int_x^s \{v(t) - c(s)\}\, dt$, and the difference converges to zero, as $s$ increases to $x_k$. The latter integral, as a function of $s$, is differentiable, with derivative $v(s) - c(s) - c'(s)(s-x)$, which is positive as soon as $s - x < \frac{m}{M_{c'}}$. Thus, for $x$ close enough to $x_k$, the upper bound, $\frac{1}{1-x}\int_x^s \{v(t) - c(s)\}\, dt$, is increasing over $[x, x_k]$. Hence, for such $x$, buyer $n$'s equilibrium offer must be $c(x_k)$.

Step 4: Again, we argue by contradiction. We assume that, for some $n$ and $h^{n-1}$ with $\underline{x}_n > x_{k+1}$, buyer $n$ assigns positive probability to serious offers with marginal type below $x_k$. Among all such $n$ and $h^{n-1}$, let $\tilde{x} \in (x_{k+1}, x_k)$ be the supremum of $\underline{x}_n$.

Consider now any $n$ and $h^{n-1}$ with $\underline{x} = x < \tilde{x}$. By definition of $\tilde{x}$, any offer $p(s)$ with marginal type $s > \tilde{x}$ is followed by an offer $c(x_k)$ from the next buyer, so that $p(s)$ must satisfy

$$p(s) - c(s) = \delta(c(x_k) - c(s)),$$

and buyer $n$'s payoff writes

$$\frac{1}{1-x}\int_x^s \{v(t) - \delta c(x_k) - (1-\delta)c(s)\}\, dt.$$

As in Step 1, the integral is strictly convex in $s$. Therefore, the marginal type of any equilibrium offer is either equal to $x_k$, or lies in the interval $[x, \tilde{x}]$. In the former case, buyer $n$'s offer is $c(x_k)$, and his payoff is positive since $\tilde{x} > x_{k+1}$. In the latter case, buyer $n$'s payoff is at most $(\tilde{x} - x)\,(v(\tilde{x}) - c(x))$, which is arbitrarily close to zero, provided $x$ is close enough to $\tilde{x}$. As a consequence, for $x < \tilde{x}$ close to $\tilde{x}$, the unique equilibrium offer of buyer $n$ is $c(x_k)$, with marginal type $x_k$. This contradicts the definition of $\tilde{x}$.

■

For completeness, let us briefly comment on the knife-edge case in which $\underline{x} = x_K$. Then as long as the marginal valuation is $\underline{x}$, any randomization over the offers $\{c(x_K), c(x_{K-1})\}$ is optimal, the payoff of either offer being zero. Because $\underline{x} = x_K$, equilibrium considerations do not uniquely 'pin down' the mixture, as is done in the proof above for the case $\underline{x} < x_K$ in which the marginal valuation is $x_k$, $k \leq K$, after an equilibrium offer that is serious. Indeed, the only reason why the equilibrium (as opposed to the equilibrium outcome) for the case $\underline{x} < x_K$ is not unique is that nothing pins down the behavior when the marginal valuation is $x_k$, $k \leq K$, *following an out-of-equilibrium offer*. Beyond this indeterminacy, the case $\underline{x} = x_K$ is identical to the case $\underline{x} < x_K$; in particular, along the equilibrium path, the seller will reject all offers provided $t \geq x_{K-1}$.

The comparison to the static case is immediate: if $\underline{x}$ is sufficiently close to $x_K$, then the probability of agreement is arbitrarily small, and the outcome is more inefficient than in the static case. On the other hand, if $\underline{x}$ is sufficiently small relative to $x_K$, then the probability of agreement is larger than in the static case, as it must be that $x^* (\underline{x}) < x_K$, since the payoff from offering $c(x_K)$ can be chosen arbitrarily close to zero.

## 4.2 Patient Single buyer

Proposition 1 assumes that each buyer makes only one offer. However, its proof goes through with a single, long-lived buyer provided the buyer's discount factor is small enough, fixing the seller's discount factor. On the other hand, the result is no longer valid if the long-lived buyer and seller share the same discount factor $\delta < 1$. In that case, we know from Vincent (1989) that there exists a (generically) unique Perfect Bayesian Equilibrium, and that, in this equilibrium, bargaining ends after a finite sequence of serious offers, one of which is accepted. Furthermore, Vincent exhibits an example with binary values in which delay does not vanish as the time interval between successive offers tends to zero.[6]

For sake of illustration, we describe here the equilibrium in the case in which $c(x) = \alpha v(x) = \alpha x$. Define the sequences:

$$x_0 = 0,\ x_{n+1} = \frac{\alpha}{1-\alpha} + \frac{\delta x_n^2}{x_{n-1}};\ z_0 = 1,\ z_n = \prod_{k=1}^{n} \frac{x_k - 1}{x_k}.$$

We show in appendix that there exists a unique equilibrium of the game with a long-lived buyer with discount factor $\delta$. With probability one, agreement is reached in finite time. If the marginal valuation is $s \in [z_{n+1}, z_n)$, the buyer offers

$$p = (1-\delta)(1-\alpha)\frac{x_n^2}{x_n - 1}s + \delta^n \alpha,$$

which the seller accepts if and only if:

$$t < \frac{x_n}{x_n - 1}s.$$

The expected payoff of the buyer is then:

$$\frac{1}{2}\left((1-\delta)(1-\alpha)\frac{x_n^2}{x_n-1} - 1\right)s^2 + \delta^n \alpha s - \frac{2\alpha - 1}{2}\delta^n.$$

We solve here for the case in which $F$ is the uniform distribution. All proofs are gathered in Appendix.

[6] In fact, Vincent (1989) proves this result more generally for the case in which the buyer is at least as patient as the seller.

In fact, the maximal number of offers in equilibrium, $N$, converges as $\delta \to 1$, so that, as the time interval between successive offers tends to zero, agreement is immediate, contrasting with the binary example studied by Vincent (1989). [To see this, observe that, for all $n$, the value of $x_n$ tends to a well-defined limit strictly larger than one, and therefore $z_{n+1} - z_n$ tends to a strictly positive limit; given $\underline{x}$, it then follows that, for $\delta$ large enough, the duration of bargaining is independent of $\delta$.]

### 4.3 Finite Horizon

Proposition 1 implies that all buyers but the first one submit losing offers. Yet when the game has finite horizon, this conclusion is blatantly false when $\underline{x} < x_1$. In particular, if the seller rejects all previous offers with positive probability, the last buyer must submit a serious offer. Indeed, his problem reduces then to the static case, for some specific $\underline{x}$. Whether agreement is reached with probability one before the last buyer, or the last buyer submits a serious bid, the qualitative conclusions of the finite horizon game seem to cast some doubt on the pertinence of Proposition 1. Therefore, the analysis of the game of the finite-horizon is not only an important extension that includes the static benchmark as a special case, but also a robustness test: as the length of the horizon increase, do the equilibria of the game with finite-horizon converge to the infinite-horizon equilibrium? For simplicity, we state our results here only for the case in which $v(x) = x$ and $c(x) = \alpha x$, with $\underline{x} > 0$, only their extension to the case of general functions is immediate.

**Proposition 2**: The equilibrium strategies of the finite horizon game converge pointwise to the equilibrium strategies in the infinite-horizon game, as the length goes to infinity. In particular, the Perfect Bayesian equilibrium is essentially unique, and is in pure strategy. In that equilibrium, the strategy of buyer $i$ is associated with thresholds $0 < s_i^0 < s_i^1 < \cdots < s_i^{k_i} = 1$. The strategy $\sigma_i$ has the following form:

- if $\underline{v}_i \leq s_i^0$, buyer $i$ offers a price $b_i \underline{v}_i$, and attracts all types up to $c_i \underline{v}_i$ for some $c_i \geq 1$ and $b_i \geq \alpha$. Thus, $\sigma_i(\underline{v}_i) = c_i \underline{v}_i$.
- if $\underline{v}_i \in (s_i^k, s_i^0)$, $\sigma_i(\underline{v}_i) = s_{i-1}^{l_k}$ for some $l_k$: buyer $i$ offers a price which does not depend on the specific value of $\underline{v}_i$ in that interval, and attracts all types up to $s_{i-1}^{l_k}$.

# 5 Unobservable offers

In this section, we maintain the assumption that all offers are unobservable, or private.

For the sake of completeness, let's first sketch a proof that an equilibrium exists. If no other buyer ever submits an offer above $\bar{c}$, it is suboptimal for a player to submit such an offer. Hence, for the purpose of equilibrium existence, we can limit the set of mixed (or behavior) strategies to the set $\mathcal{M}([0, \bar{c}])$ of probability distributions over the interval $[0, \bar{c}]$, endowed with the weak-$\star$

topology. The set of strategy profiles is thus the countable product $\mathcal{M}([0,\bar{c}])^{\mathbf{N}}$. It is compact and metric when endowed with the product topology. Since the random outcome of buyer $n$'s choice is not known to the seller unless he has rejected the first $n-1$ offers, the buyer $n$'s payoff function is not the usual multi-linear extension of the payoff induced by pure profiles. We however follow the standard proof. Let any buyer $n$ be given, and denote $x(p,\mu)$ the marginal type for the offer $p$, given a strategy profile. It is jointly continuous in $p$ and $\mu$. Hence, the set $B_n(\mu) \subset \mathcal{M}([0,\bar{c}])$ of best replies of buyer $n$ to the strategy profile $\mu$, is convex-valued and upper hemi-continuous in $\mu$. Using a fixed-point theorem, the existence of a (Nash) equilibrium follows.

We let an equilibrium $\sigma$ be given. The main result is the following.

**Proposition 5.1** *Trade eventually occurs, with probability one.*

**Proof:** Given $x \in [\underline{x}, 1]$, let $F_n(x)$ denote the (unconditional) probability that the seller is of type $t \le x$ and has rejected all offers submitted by buyers $i = 1, \ldots, n-1$. Suppose for the sake of contradiction that trade does not occur with probability one eventually, i.e. $\lim_{n\to\infty} F_n(x) \neq 0$ for some $x < 1$. In particular, the probability that the seller will accept buyer $n$'s offer, conditional on having rejected the previous ones, converges to zero as $n$ increases, hence the successive buyers' equilibrium payoffs also converge to zero.

Let $F = \lim_{n\to\infty} F_n$. Choose $x$ such that $F(x) > 0$ and

$$\int_{\underline{x}}^{x} \left( v(t) - c(x) - \frac{\nu}{2} \right) dF(t) > 0. \tag{2}$$

Note that $\dfrac{F(x) - F_n(x)}{F_n(x)}$ is the probability that type $x$ will accept an offer from some buyer beyond $n$ (conditional on having rejected all previous offers). Since $F(x) > 0$, this probability converges to zero, and the offer $p_n(s)$ with marginal type $s$ thus converges to $c(x)$. As a result, $p_n(s) \le c(s) + \frac{\nu}{2}$ for all $n$ large and, using (2), buyer $n$'s equilibrium payoff is bounded away from zero − a contradiction. ■

Thus, offers that are accepted by seller's types arbitrarily close to one are eventually submitted.

Proposition 5.1 holds irrespective of $\delta$. We now assume, without further notice, that $\delta > \bar{\bar{\delta}}$.

We prove below that type 1 actually accepts an offer in finite time with probability one. Thus, there is a buyer who eventually offers $\bar{c}$, which the seller accepts, irrespective of his type. We introduce some notation. As before, given some equilibrium, $F_n(x)$ denotes the (unconditional) probability that the unit is of index $t \le x$ and that the seller has rejected all offers submitted by buyers $i = 1, \ldots, n-1$. Set $\underline{x}_n := \inf \{x : F_n(x) > 0\}$.

Buyer $n$'s strategy is a probability distribution over offers in $[c(\underline{x}), \bar{c}]$. We denote by $P_n$ its support[7] and $T_n$ the corresponding (closed) set of marginal types. That is, if buyer $n$'s strategy

[7] That is, the smallest closed set with probability one.

has finite support, $x \in T_n$ if and only if there exists an offer $p_n$ submitted by buyer $n$ with positive probability that the seller accepts if and only if his type is less than or equal to $x$.

**Proposition 5.2** *There exists $n \geq 1$, with $1 \in T_n$. Let $N_0$ denote the first of these stages. Then $T_n \subset \{\underline{x}_{N_0}, 1\}$, for all $n \geq N_0$.*

*For each $n > N_0$, buyer $n$'s equilibrium payoff is zero.*

It is convenient to discuss here the case where $\underline{x} > x_1$. Note that a buyer who is called to make an offer, gets a positive payoff when offering $\bar{c}$. Hence, it follows from Proposition 5.2 that the unique equilibrium outcome of the game is such that the first buyer offers $\bar{c}$, which the seller accepts. For such values of $\underline{x}$, the equilibrium outcome is the same in both versions of the game (but it differs from that of the static case).

From now on, we will assume $\underline{x} \leq x_1$. Buyer $N_0$'s equilibrium payoff is then also zero.

Thus, from stage $N_0$ on, all equilibrium offers are either winning offers, or losing offers. By Proposition 5.1, one of these buyers eventually submits the winning offer $\bar{c}$. However, equilibrium behavior after buyer $N_0$ is essentially indeterminate. Indeed, as the next result states, any equilibrium $\sigma$ can be modified into an equilibrium where trade occurs in bounded time, or into an equilibrium in which all buyers trade with positive probability.

**Proposition 5.3** *For every equilibrium $\sigma$,*

- *There is an equilibrium $\tilde{\sigma}$, with $\tilde{\sigma}_B^n = \sigma_B^n$ for all $n < N_0$, and $\underline{x}_n = 1$ for some $n \geq N_0$.*
- *There is an equilibrium $\tilde{\sigma}$, with $\tilde{\sigma}_B^n = \sigma_B^n$ for all $n < N_0$, and $\underline{x}_n = \underline{x}_{N_0}$ for all $n \geq N_0$.*

Moreover, infinitely many such equilibria exist. All these equilibria are payoff-equivalent, for the seller and the buyers alike.

Denote $\underline{N}_0$ the last buyer $n$ who submits a serious offer below $\bar{c}$ with positive probability. Thus, all buyers $n = \underline{N}_0 + 1, \ldots, N_0 - 1$ submit only losing offers.

In the first phase of the equilibrium, all buyers trade with positive probability.

**Proposition 5.4** *No buyer $n \leq \underline{N}_0$ uses a pure strategy, except possibly buyer 1. All buyers $n \leq \underline{N}_0$ submit a serious offer with positive probability.*

Without further assumptions, it is difficult to establish additional structural properties on equilibrium strategies. If however the valuation function $v$ is concave, then all equilibrium strategies have finite support.

**Proposition 5.5** *Assume that $v$ is (strictly) concave over $(\underline{x}, 1)$, and let $\sigma = (\sigma_n)_{n \geq 1}$ be an equilibrium. Then the support of the probability distribution $\sigma_n$ is a finite set, for every buyer $n \geq 1$.*

In the public case, competition between buyers prevent buyers $n \geq 2$ from getting positive payoffs. However, the first buyer buyer gets a positive payoff (independent of $\delta$). We do not know whether a similar zero-payoff result holds in the private offers version. In any case, all equilibrium profits are tiny and vanish rapidly, as the seller becomes infinitely patient.

**Proposition 5.6** *The expected payoff to buyer $n$ is at most*

$$\frac{2}{m}\underline{x}_{n+1}^2(1-\delta)^2\left(v\left(\underline{x}_{n+1}\right)-c\left(\underline{x}_{n+1}\right)\right)^2.$$

Thus, all equilibrium profits are bounded by $2m^{-1}(1-\delta)^2 \sup|v-c|$. Due to the term $(1-\delta)^2$, the discounted sum of all buyers' profits converges to zero, as the seller becomes more patient. In addition, we prove that buyers with positive payoff are increasingly rare, as the seller becomes more patient.

**Proposition 5.7** *Let $n_1 < n_2$ be two buyers with positive equilibrium profit. One has*

$$n_2 - n_1 \geq \frac{\delta}{1-\delta}\ln\left(1+\frac{m}{2M_{c'}}\right).$$

The efficiency of the trading mechanism is directly related to the delay in trade – that is, to $\mathbf{E}\left[\delta^\tau\right]$, where $\tau$ is the first buyer offering $\bar{c}$. As we now show, any equilibrium must exhibit a non-trivial, but finite, delay.

**Proposition 5.8** *There exist constants $0 < c_1 < c_2 < 1$ and $C > 0$ such that, for all $\delta \geq \bar{\delta}$ and all equilibrium:*

- *$N_0 \leq C/(1-\delta)$;*
- *$c_1 \leq \mathbf{E}\left[\delta^\tau\right] \leq c_2$.*

$c_1 > 0$ implies that the discounted delay remains finite. $c_2 < 1$ implies that the discounted delay is non-trivial, even as the seller becomes more patient.

Non-trivial delay makes it difficult to compare equilibrium payoffs of the seller across versions. However that the profit of the seller is bounded away from zero in the private case. By contrast, the seller's payoff is zero if $\underline{x} = x_{K+1}$, and depends continuously on $\underline{x}$. Hence there are cases in which the seller's payoff is unambiguously higher in the private version. Whether or not this holds in general is an open question.

## 5.1 Conjectures and Numerical Evaluations

We have been unable to explicitly solve for the equilibria of the game, except in special cases discussed below. As shown above, any equilibrium can be partitioned into two "phases". Provided that the first player's payoff from submitting an offer accepted with probability one is negative, there must be a first buyer (say, player $N > 1$) who must be precisely indifferent between submitting a losing offer or submitting an offer accepted with probability one. From this point on, all buyers randomize over those kinds of offers, and as long as the latter offer is not submitted with probability one by some buyer, the conditional beliefs of the seller do not change any more. The first phase (up to player $N - 1$) is more complicated. In particular, all buyers but the first must use a mixed strategy, with strictly positive probability assigned to at least two offers. While it is possible to rule out many configurations with simple considerations, a wide range of possibilities remain; in fact, simple examples of multiple equilibria can be constructed.

There are many properties of equilibria that one may conjecture. One is (i) to look for an equilibrium in which every buyer submits a losing offer with positive probability; a second is (ii) to look for an equilibrium in which the expected offer is increasing over time; a third is (iii) to look for an equilibrium in which every buyer randomizes over at most two equilibrium offers.

It is simple to show that it follows from either (i) and (ii) or from (i) and (iii) that the equilibrium should be as follows: each buyer randomizes over two offers. The lower one is a losing offer (for all buyers but possibly for the first buyer) while the higher one is accepted by all types up to $x_i$, where $x_i$ is strictly increasing in $i$ for $i < N$. (This randomization must be strict up to buyer $N - 1$).

From numerical simulations, it appears that such equilibria exists for all $\alpha$ and $\delta$, but they require that $\underline{x}$ be sufficiently high. See Figure 1. [Horizontal dotted lines are support of distributions, solid curves are payoffs as the function of the marginal type accepting an offer; all Figures omit buyers beyond $N_0$]

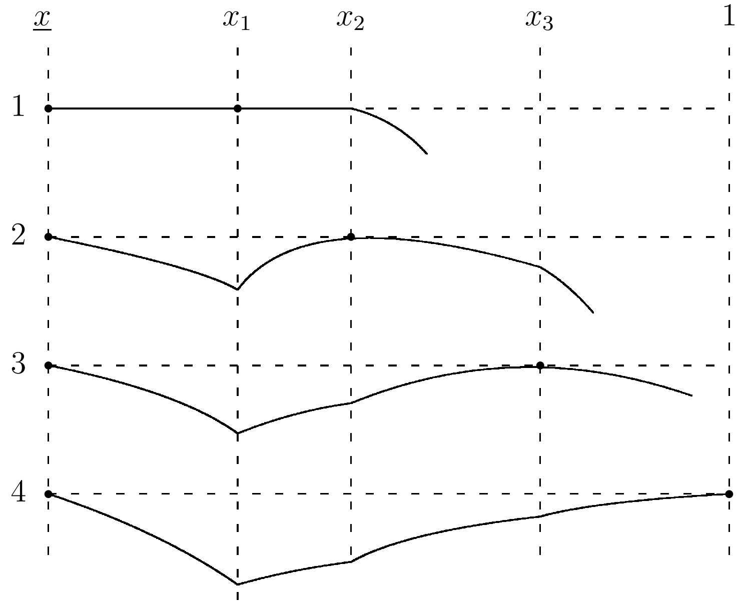


Figure 1 ( not to scale)

For lower values of $\underline{x}$, this does not work, because buyer 2 strictly gains from submitting an offer accepted with small but positive probability. This problem can be remedied by assuming instead that buyer 2's lower offer is serious as well, so that only the low offer of buyers $n \geq 3$ is a losing offer (in this revised conjecture, buyer 2's payoff is still zero). Such equilibria exist, and indeed, they can be constructed for lower values of $\underline{x}$ than is consistent with the first conjecture. However, it is again necessary that $\underline{x}$ be sufficiently high, for otherwise the same problem arises with buyer 4.

It seems therefore natural to amend the conjecture further, by considering the case in which, for a subsequence of the buyers in the first phase, the low offer is serious, while it is losing for the others. (It is easy to see that no two consecutive buyers can belong to that subsequence). Unfortunately, the resulting systems of equations is quite untractable. In the special case in which $\alpha(1-\delta) > 1/2 > \alpha\left(1-\delta^2\right)$, we could construct an equilibrium as depicted in Figure 2, which is consistent with a more general conjecture along the lines of Figure 3.

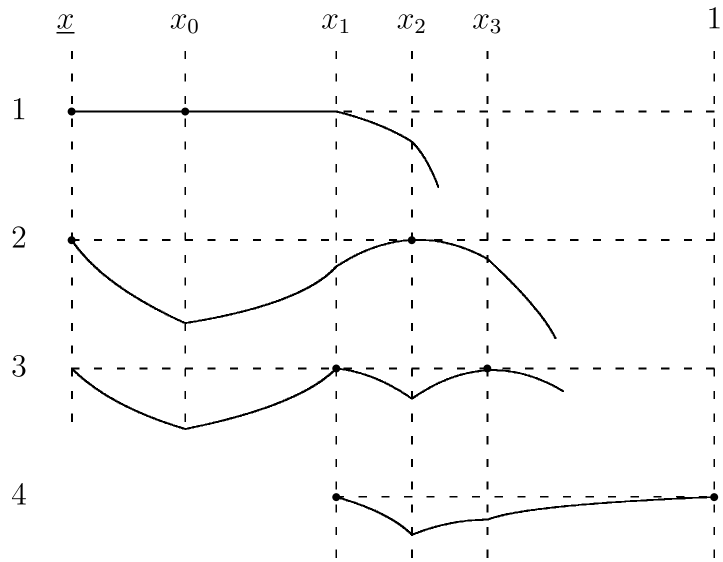


Figure 2 ( not to scale)

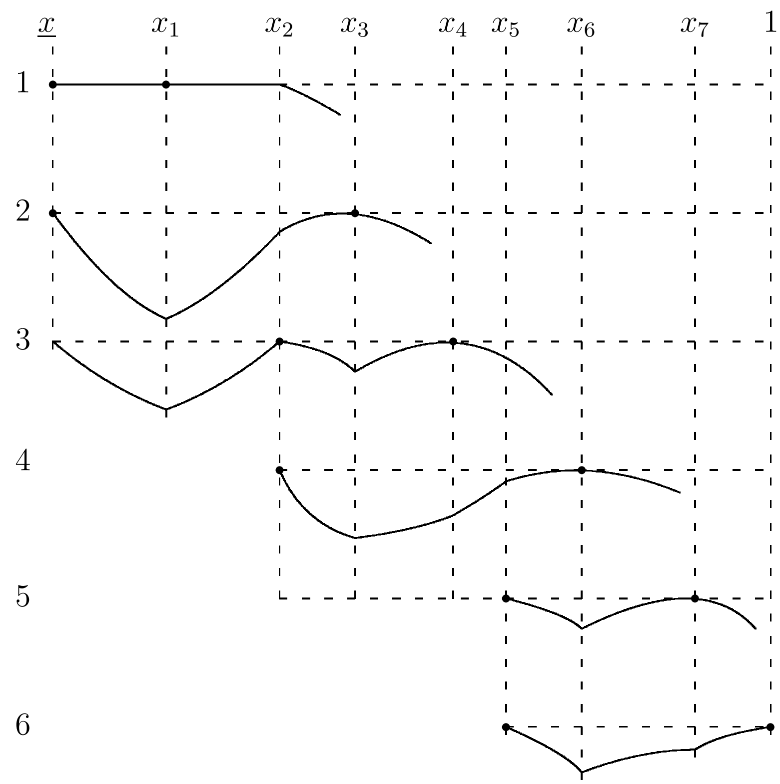


Figure 3 : Conjecture ( not to scale )
$\alpha(1-\delta) > \frac{1}{2} > \alpha(1-\delta^2)$

# 6 Proofs

## 6.1 Proof of Proposition 5.2

The next lemma is repeatedly used in the sequel.

**Lemma 6.1** *If $\underline{x}_{n+2} > \underline{x}_{n+1}$, buyer $n$ submits no offer in the interval $(\underline{x}_n, \underline{x}_{n+1})$. In particular, $\underline{x}_{n+1} = \underline{x}_n$, and buyer $n$'s equilibrium payoff is zero.*

**Proof**: //**in progress**// For $x > \underline{x}_n$, let $p_n(x)$ denote the offer that buyer $n$ must submit for type $x$ to be the marginal type. For $x < \underline{x}_{n+2}$, the hypothesis of the lemma implies that $p_n(x) = (1-\delta)\,c(x) + C$ for some constant $C$. The (unconditional) payoff of buyer $n$ from submitting an offer for which the marginal type is $x$ is given by:

$$\int_{\underline{x}}^{x} (v(t) - p_n(x))\, dF_n(t) .$$

Clearly, if $v(x) = p(x)$ for some $x > \underline{x}_n$, offering $p(x)$ results in a strict loss, so in order to find the maximizers of this payoff, we can restrict attention to $x$ such that $v(x) > p(x)$. Since $c'$ is bounded and $v$ is strongly increasing, this implies that we can restrict attention to $x$ above some threshold $\bar{x}$, provided $\delta$ is sufficiently close to one. Since $F_n$ is convex (given the cream-skimming property), it is piecewise continuously differentiable on $(\underline{x}_n, \min S_{n+1})$, with derivative, on each subinterval, given by:

$$\frac{dF_n(x)}{dt}(v(x) - p(x)) - (1-\delta)\, c'(x)\, F_n(x) ,$$

which is strictly increasing for all $\delta$ sufficiently large, since $v$ is strongly increasing, while $F_n$ is convex. Therefore, the payoff is strictly convex on each of those subintervals, a result that holds more generally over $(\bar{x}, \min S_{n+1})$ since at all points this interval, $D_{-}F_n(x) < D_{+}F_n(x)$. Therefore, this payoff is first strictly negative and then strictly convex in $x$, so that it admits no maximum in this interval. ■

**Proof of Proposition 5.2**: From Proposition 5.1, we know that $\lim_n F_n(1) = 0$. Fix some $N$ such that $F_N(1) < \varepsilon/\bar{c}'$, where $\bar{c}' > 0$ is an upper bound on the derivative of $c$ over $[\underline{x}, 1]$. Let:

$$\tilde{V}_n(x) := \int_{\underline{x}}^{x} \{v(t) - c(x)\}\, dF_n(t) .$$

Observe that $\tilde{V}_n(x)$ is an upper bound to the (unconditional) payoff buyer $n$ obtains when submitting an offer for which $x$ is the marginal type, with equality if and only if either it is a losing offer ($F_n(x) = 0$), or a winning offer ($x = 1$).

We first prove that $1 \in T_n$, for some $n \in \mathbf{N}$. Let's argue by contradiction. In particular, the highest offer $\bar{s}_N := \max_{n<N} \max S_n$ is lower than 1, and the function $F_N(x)$ is differentiable on the interval $[\bar{s}_N, 1]$, with derivative equal to one. Hence, on that interval, $\tilde{V}_N$ is differentiable and its derivative equals:

$$\tilde{V}_N'(x) = v(x) - c(x) - F_N(x)\, c'(x) ,$$

which is strictly positive. Therefore $\tilde{V}_N$ is strictly increasing on $(\bar{s}_N, 1)$, so that buyer $n$'s payoff cannot be maximized on this open interval. Hence, any such buyer can only submit either a winning offer –which is ruled out by assumption– or offers for which the marginal type is less than $T_N$, implying that such all marginal types are bounded away from one, contradicting Proposition 3.

Consider now the second statement. For the sake of contradiction, suppose that some buyer $n \geq N_0$ submits w.p.p. an offer in $\left(c(\underline{x}_{N_0}), \bar{c}\right)$. Then, conditional on submitting a winning offer

$\bar{c}$, the expected value of the unit is higher for buyer $n+1$ than for buyer $n$. Thus, buyer $n+1$'s payoff is positive. By Lemma 6.1, it cannot be the case that buyer $n+1$ assigns probability one to $\bar{c}$. Hence, buyer $n+2$'s equilibrium payoff is also positive – a contradiction to Lemma 6.1. ■

## 6.2 Proof or Proposition 5.3

Denote $\tau$ the (random) first buyer who offers $\bar{c}$, and define $n_* \in \mathbf{N}$ by $\delta^{n_*+1} \leq \mathbf{E}[\delta^\tau] < \delta^{n_*}$. Consider the strategy profile $\tilde{\sigma}$ in which (i) $\tilde{\sigma}_B^n = \sigma_B^n$ for all $n < N_0$, (ii) all buyers $n = N_0, \ldots, n_* - 1$ submit a losing offer, (iii) buyer $n_*$ offers a winning offer $\bar{c}$ with probability $\dfrac{\mathbf{E}[\delta^\tau] - \delta^{n_*+1}}{\delta^{n_*}}$, and a losing offer otherwise, (iv) all buyers $n \geq n_* + 1$ offer $\bar{c}$ with probability one.

By construction, for all buyers $n < N_0$, and after any history $h^{n-1} \in H^{n-1}$, the offer $p_n(s)$ with marginal type $s$ is the same under the two profiles $\sigma$ and $\tilde{\sigma}$. Hence, for any such $n$, $\sigma_B^n$ is a best-reply to $\tilde{\sigma}$. It remains to check that no buyer $n = N_0, \ldots, n_*$ finds it profitable to submit a serious offer in the open $(c(\underline{x}_{N_0}), \bar{c})$. For any such $n$, the expected payoff that type $x$ obtains from rejecting the offer is $(\bar{c} - c(x))\mathbf{E}\left[\delta^{\tau-n} | \tau > n\right]$. This continuation payoff is higher when $\mathbf{E}\left[\delta^{\tau-n} | \tau > n\right]$ is computed using $\tilde{\sigma}$ then when using $\sigma$. Hence the marginal type for an offer in $(c(\underline{x}_N), \bar{c})$ is lower under $\tilde{\sigma}$ than it is under $\sigma$. Such a deviation cannot be profitable under $\tilde{\sigma}$, for it would also be profitable under $\sigma$.

We turn to the proof of the second statement. We assume that the equilibrium is such that $T_n = \{1\}$ for some $n$, for otherwise the conclusion holds trivially. As above, we adjust the probability of submitting $\bar{c}$, while keeping the optimal decision rule of the seller in stages $n = 1, \ldots, N_0 - 1$ unchanged.

Set $N := \inf\{n \geq N_0 : T_n = \{1\}\}$, and denote by $\pi$ the probability assigned to $\bar{c}$ by buyer $N-1$. We first check that this definition is meaningful when $N = N_0$, and consider buyer $N-1$. By Lemma 6.1, $T_{N_0-1} \subset \{\underline{x}_{N_0-1}, 1\}$. By definition of $N_0$, it must be that buyer $N_0 - 1$ assigns probability one to a losing offer. Hence, $\pi = 0$ in that case.

Consider the strategy profile $\tilde{\sigma}$ in which (i) $\tilde{\sigma}_B^n = \sigma_B^n$ for all $n < N - 1$, (ii) buyer $N - 1$ assigns probability $\tilde{\pi}$ to $\bar{c}$, and submits a losing offer otherwise, (iii) all buyers $n \geq N$ submit $\bar{c}$ with probability $\alpha$, and a losing offer otherwise.

The parameters $\tilde{\pi}$ and $\alpha$ are chosen so that (i) the expected discounted time $\mathbf{E}\left[\delta^\tau\right]$ is the same under both profiles $\sigma$ and $\tilde{\sigma}$, and (ii) no buyer $n \geq N-1$ would find it profitable to submit a serious offer in $(c(\underline{x}_N), \bar{c})$.

Condition (i) writes

$$\pi + (1-\pi)\delta = \tilde{\pi} + (1-\tilde{\pi})\frac{\alpha\delta}{1-\delta(1-\alpha)}. \tag{3}$$

Condition (ii) holds as soon as the payoff $\displaystyle\int_{\underline{x}_N}^{s} \{v(t) - p_n(s)\}\, dF_{N-1}(t)$ is convex in $s$. A sufficient condition is

$$m - 2M_{c'}\left(1 - \frac{\alpha\delta}{1-\delta(1-\alpha)}\right) > 0.$$

Provided $\alpha$ is close enough to one, the condition is satisfied and the solution $\tilde{\pi}$ to (3) is in $(0,1)$. $\blacksquare$

## 6.3 Proof of Proposition 5.4

We start with the second assertion. Assume that, for some $n \leq \underline{N}_0$, buyer $n$ submits a losing offer with probability 1. Then, $n < \underline{N}_0$ by the choice of $\underline{N}_0$, and buyer $n$'s equilibrium payoff is zero. Denote $n^* > n$ the first buyer after $n$ who submits a serious offer with positive probability. Denote $p_{n^*}(s) = \max P_{n^*}$ the highest offer of buyer $n^*$, with marginal type $s$. Observe that $s < 1$, hence $p_{n^*}(s) < \bar{c}$. As a consequence, the price $p_n(s)$ that buyer $n$ needs to submit so that $s$ is indifferent between accepting and declining is strictly smaller than $p_{n^*}(s)$. Observe also that the expected value of the unit, conditional on an offer being accepted by all seller's types less than $s$ is the same for buyer $n$ and $n^*$. Therefore, buyer $n$ gets a strictly positive payoff from submitting such an offer, a contradiction.

We now deal with the second assertion. Assume that, for some $n \leq \underline{N}_0$, buyer $n$'s strategy is pure. By the previous paragraph, buyer $n$'s unique offer must be serious. Denote $x < 1$ the marginal type for this offer. By Lemma 6.1, buyer $n-1$ cannot submit any offer with marginal type in $(\underline{x}_n, x)$. This implies that the expected value conditional on the marginal type being equal to $x$ is the same both for buyer $n$ and buyer $n+1$. Since buyer $n$'s offer is serious, buyer $n-1$ must thus submit a losing offer with positive probability, and therefore have a zero equilibrium payoff. However, the offer he must submit such that the marginal type equals $x$ is strictly less than the unique offer submitted by buyer $n$ (because of discounting), a contradiction. $\blacksquare$

## 6.4 Proof of Proposition 5.5

We proceed by induction over the index $n$ of the buyers. The proof of the initial step is similar to the proof of the induction step, hence we provide only the latter.

Consider any buyer $n \geq 1$. By the induction hypothesis, the probability $G_{n-1}(t)$ that type $t$ has rejected all previous offers, assumes only finitely many values: there exists $\underline{x} = x_0 < x_1 < \cdots < x_K = 1$, and $\lambda_0 < \cdots < \lambda_{K-1} = 1$, such that $G_n(t) = \lambda_k$ over the interval $[x_k, x_{k+1})$.

Denote by $\pi_n^* \geq 0$ the equilibrium payoff of buyer $n$. For each type $x$, the offer $p_n(x)$ with marginal type $x$ yields a payoff of

$$\pi_n(x) = F_n(x)\left(\mathbf{E}_n[v(t)|t \leq x] - p_n(x)\right) :$$

in this expression, $\mathbf{E}_n$ stands for the expectation under the distribution of types faced by buyer $n$.

Therefore, one has

$$\mathbf{E}_n[v(t)|t \le x] \le \frac{\pi_n^*}{F_n(x)} + p_n(x), \tag{4}$$

with equality for each $x$ in the support of $\sigma_n$. Recall that $p_n(\cdot)$ is convex. It is immediate to check that $1/F_n(\cdot)$ is also convex. We prove below that $x \mapsto \mathbf{E}_n[v(t)|t \le x]$ is strictly concave on each interval $[x_k, x_{k+1}]$. Therefore, equality in (4) holds for at most one value in $[x_k, x_{k+1}]$, and this will conclude the proof of the induction step.

**Lemma 6.2** *The map $R : x \mapsto \mathbf{E}_n[v(t)|t \le x]$ is strictly concave on the interval $[x_k, x_{k+1}]$, for each $k < K$.*

**Proof.** Fix $k$, and observe that, for $x \in (x_k, x_{k+1}]$

$$\begin{aligned} R(x) &= \frac{1}{F_n(x)} \left\{ \sum_{i=0}^{k-1} \lambda_i \int_{x_i}^{x_{i+1}} v(t)dt + \lambda_k \int_{x_k}^{x} v(t)dt \right\} \\ &= \frac{N(x)}{F_n(x)} \end{aligned}$$

The second derivative of $R$ is

$$R^{''}(x) = \frac{F_n(x)^2 N^{''}(x) - 2F_n'(x)(N'(x)F_n(x) - N(x)F_n'(x))}{F_n^3(x)}.$$

It is of the same sign as the expression

$$E(x) := v'(x)F_n^2(x) - 2\lambda_k v(x)F_n(x) + 2\lambda_k N(x).$$

Note that $E^{'''}(x)F_n(x)^2 < 0$, hence $E(x) < E(x_k)$, that is:

$$E(x) < v'(x_k)F_n(x_k)^2 - 2\lambda_k v(x_k)F_n(x_k) + 2\lambda_k \left\{ \sum_{i=1}^{k-1} \lambda_i \int_{x_i}^{x_{i+1}} v(t)dt \right\}. \tag{5}$$

For fixed $x_0, \ldots, x_{k-1}, \lambda_0, \ldots, \lambda_k$, we check that the right-hand side in (5) is non-increasing in $x_k$. The derivative of this right-hand side w.r.t. $x_k$ is $v^{''}(x_k)F_n(x_k)^2 + 2(\lambda_{k-1} - \lambda_k)v'(x_k)F_n(x_k) < 0$, hence the right-hand side is maximal when $x_k$ gets close to $x_{k-1}$. At the limit, we obtain

$$E(x) < v'(x_{k-1})F_n(x_{k-1})^2 - 2\lambda_k v(x_{k-1})F_n(x_{k-1}) + 2\lambda_k \left\{ \sum_{i=0}^{k-2} \lambda_i \int_{x_i}^{x_{i+1}} v(t)dt \right\}.$$

Repeating this proof inductively yields, for each $j \ge 0$:

$$E(x) < v'(x_j)F_n(x_j)^2 - 2\lambda_k v(x_j)F_n(x_j) + 2\lambda_k \left\{ \sum_{i=0}^{j-1} \lambda_i \int_{x_i}^{x_{i+1}} v(t)dt \right\}.$$

For $j = 0$, this reads $E(x) < 0$. ■

## 6.5 Proof of Proposition 5.6

For $n \in \mathbf{N}$, and $x > \underline{x}_n$, we denote $\bar{q}_n(s)$ the expected value of the unit for buyer $n$, conditional on an offer with marginal type $x$ being accepted:

$$\bar{q}_n(x) = \frac{1}{F_n(x)} \int_{\underline{x}_n}^{x} v(t) dF_n(t).$$

Plainly, $\underline{x}_n \leq \bar{q}_n(x) \leq x$. Besides, $\bar{q}_n(x) \leq \bar{q}_{n+1}(x)$, with equality if and only if buyer $n$ submits no serious offer with marginal type in $(\underline{x}_n, x)$.

**Proof of Proposition 5.6**: We let here $n$ be any buyer with positive profit. Note that $n \leq \underline{N}_0$.

Since buyer $n$'s payoff is positive, $\underline{x}_{n+1} > \underline{x}_n$. consider buyer $n$'s lowest offer, $p_n(\underline{x}_{n+1})$, with marginal type $\underline{x}_{n+1}$. For notational concision, we abbreviate $\underline{x}_{n+1}$ to $x$. The probability that this lowest offer is accepted is $F_n(x)$. Conditional on it being accepted, buyer $n$'s surplus is $\bar{q}(x) - p_n(x)$. Hence, buyer $n$'s equilibrium payoff is $F_n(x)\left(\bar{q}_n(x) - p_n(x)\right)$. We first bound the conditional surplus.

Since buyer $n$'s payoff is positive, buyer $n+1$'s payoff must be zero, hence $x \leq p_{n+1}(x)$, for otherwise buyer $n+1$ would obtain a positive payoff when submitting a low, serious offer. Thus,

$$\delta \bar{q}_n(x) \leq \delta x \leq \delta p_{n+1}(x) \leq p_n(x) - (1-\delta)c(x).$$

This yields

$$\bar{q}_n(x) - p_n(x) \leq (1-\delta)\left(v(x) - c(x)\right). \tag{6}$$

We next bound the probability $F_n(x)$ that the lowest offer is accepted. We will rely on the inequality

$$\bar{q}_n(x) \geq \delta v(x) + (1-\delta)c(x).$$

It is convenient to observe that $F_n(x) = \displaystyle\int_{\underline{x}_n}^{x} f_n(t)dt$, where $f_n(t)$ is the probability that type $t$ rejects the offers from all buyers $k < n$.

Introduce the optimization problem $\mathcal{P}$:

$$\sup \int_{\underline{x}_n}^{x} f(t)dt,$$

where the supremum is taken over all non-decreasing, $[0,1]$-valued functions $f$, such that

$$\int_{\underline{x}_n}^{x} v(t)f(t)dt \geq (\delta v(x) + (1-\delta)c(x)) \int_{\underline{x}_n}^{x} f(t)dt.$$

We will prove that the value $v^*$ of $\mathcal{P}$ satisfies $v^* \leq 2/m(1-\delta)(v(x)-c(x))$. Together with (6), and since $F_n(x) \leq v^*$, this will conclude the proof.

We analyze $\mathcal{P}$ by introducing an auxiliary problem $\mathcal{P}_\omega$:

$$\sup \frac{\int_{\underline{x}_n}^{x} v(t)f(t)dt}{\int_{\underline{x}_n}^{x} f(t)dt},$$

where the supremum is taken over all non-decreasing, $[0,1]$-valued functions $f$ that satisfy $\displaystyle\int_{\underline{x}_n}^{x} f(t)dt \geq \omega$ ($\omega \leq x - \underline{x}_n$). The problem $\mathcal{P}_\omega$ is some kind of dual to $\mathcal{P}$. Existence of an optimal solution to $\mathcal{P}_\omega$, and continuity of the value function $V(\omega)$ follow from standard arguments.

Plainly, one has $v^* \geq \omega$ as soon as $V(\omega) \geq \delta v(x) + (1-\delta)c(x)$ – indeed, any optimal solution to $\mathcal{P}_\omega$ is then a feasible point for $\mathcal{P}$. Hence,

$$v^* = \sup\{\omega : V(\omega) \geq \delta v(x) + (1-\delta)c(x)\}. \tag{7}$$

Any solution $f^*$ to $\mathcal{P}_{v^*}$ must satisfy the feasibility constraint $\displaystyle\int_{\underline{x}_n}^{x} f(t)dt \geq v^*$ with equality, for otherwise $V(\omega')$ would be equal to $V(v^*)$ for $\omega'^*$, close enough to $v^*$. Hence, $f^*$ is also a solution to the problem $\sup \displaystyle\int_{\underline{x}_n}^{x} v(t)f(t)dt$, subject to the constraint $\displaystyle\int_{\underline{x}_n}^{x} f(t)dt = v^*$. Since $v$ is strictly increasing, $f^*$ is a step function, with $f^*(t) = 0$ for $t < x^*$, and $f^*(t) = 1$ for $t \geq x^*$. The value of $x^*$ is determined by the constraint $\displaystyle\int_{\underline{x}_n}^{x} f(t)dt = v^*$: $v^* = x - x^*$, and

$$V(v^*) = \frac{1}{x - x^*}\int_{x^*}^{x} v(t)dt.$$

Note that $v(t) \leq v(x) - (x-t)m$, hence $V(v^*) \leq v(x) - \frac{1}{2}(x - x^*) = v(x) - \frac{m}{2}$. on the other hand, by (7), $V(v^*) = \delta v(x) + (1-\delta)c(x)$, hence

$$v^* \leq \frac{2}{m}(1-\delta)(v(x) - c(x)),$$

as claimed. ■

## 6.6 Proof of Proposition 5.7

The proof of Proposition 5.7 relies on the following inequality.

**Lemma 6.3** *For every $a < b$ in $[\underline{x}, 1]$, and every non-decreasing function $h : [\underline{x}, 1] \to [0, 1]$, with $\int_a^b h(t)dt > 0$, one has*

$$\frac{h(b)}{\int_a^b h(t)dt} \times \int_a^b v(t)h(t)dt + \frac{m}{2}\int_a^b h(t)dt \leq v(b)h(b). \tag{8}$$

**Proof.** The inequality is homogenous in $h$. We may thus assume that $h(b) = 1$.

We prove the statement for functions $H$ that take finitely many values. The result will then follow using a density argument.

The proof goes by induction over the number of values in the range of $H$. To start the induction, assume that $h(t) = 1$ for each $t \in [a, b]$. Since $v(t) \leq v(b) + m(t - b)$ for each $t \in [a, b]$, one has

$$\int_a^b v(t)dt \leq v(b)(b-a) - \frac{m(b-a)^2}{2}.$$

Hence, the left-hand side of (8) is at most $v(b) - \frac{m}{2}(b-a) + \frac{m}{2}(b-a) = v(b)$.

Assume that (8) holds for every $a < b$, provided the range of $h$ contains at most $n$ points. Let $h = \sum_{k=0}^{n} x_k 1_{[y_k, y_{k+1})}(\cdot)$, where $0 \leq x_0 \leq \cdots \leq x_n = 1$ and $a = y_0 < y_1 < \cdots < y_n = b$.

For given $x_1, \ldots, x_n, y_0, \ldots, y_n$, the left-hand side if (8) is a function of $x_0$, which we denote $\psi(x_0)$. We prove below that $\psi$ is convex. This will imply that $\psi(x_0) \leq (1 - \frac{x_0}{x_1})\psi(0) + \frac{x_0}{x_1}\psi(x_1)$. Since, for both $x = 0$ and $x = x_1$, the range of $h$ contains at most $n$ points, one has $\psi(0) \leq v(b)$ and $\psi(x_1) \leq v(b)$, hence the result follows.

The second term in $\psi(x_0)$ is linear, hence we only prove the convexity of

$$\phi(x_0) := \frac{1}{\int_a^b h(t)dt} \times \int_a^b v(t)h(t)dt = \frac{\sum_{i=0}^{n} x_i \int_{y_i}^{y_{i+1}} v(t)dt}{\sum_{i=0}^{n} x_i(y_{i+1} - y_i)}.$$

Observe that

$$\phi'(x_0) = \frac{1}{D^2(x_0)}\left\{\sum_{i=0}^{n} x_i \left((y_{i+1} - y_i)\int_{y_0}^{y_1} v(t)dt - (y_1 - y_0)\int_{y_i}^{y_{i+1}} v(t)dt\right)\right\},$$

where $D(x_0) = \sum_{i=0}^{n} x_i(y_{i+1} - y_i)$.

Since, $v$ is increasing, one has

$$\frac{1}{y_1 - y_0}\int_{y_0}^{y_1} v(t)dt < \frac{1}{y_{i+1} - y_i}\int_{y_i}^{y_{i+1}} v(t)dt, \text{ for } i \geq 1,$$

hence $\phi'(x_0)$ is increasing in $x_0$. ■

**Lemma 6.4** *Let $n \in \mathbf{N}$, and $x > \underline{x}_n$ in $T_n$ be given. Then $D^+ p_n(x) \geq \frac{m}{2}$.*

**Proof.** Write $F_n(x) = \displaystyle\int_{\underline{x}_n}^{x} f_n(t)dt$. If he submits an offer $p_n(s)$ with marginal type $s$, buyer $n$ obtains a payoff equal to

$$\int_{\underline{x}_n}^{s} f_n(t)(v(t) - p_n(s))dt.$$

Since $f_n$ is right-continuous, and $p_n$ is convex, the integral has a right-derivative at $x$, equal to

$$f_n(x)(v(x) - p(x)) - D^+ p(x)) \int_{\underline{x}_n}^{x} f_n(t)dt.$$

Since $x \in T_n$ is a serious offer, the right-hand side is non-positive and $v(x) \geq p_n(x) \geq \bar{q}_n(x)$. Hence,

$$f_n(x)\left(v\left(x\right) - \frac{\int_{\underline{x}_n}^{x} v(t)f - n(t)dt}{\int_{\underline{x}_n}^{x} f(t)dt}\right) - D^+ p_n(x) \int_{\underline{x}_n}^{x} f_n(t)dt \leq 0.$$

The conclusion follows, using Lemma 6.3. ■

It is easily checked that $D^+ p_n(x) = c'(x)\left(1 - \mathbf{E}\left[\delta^{\tau_n(x)-n}\right]\right)$, where $\tau_n(x)$ is the first buyer following $n$, who submits an offer with marginal type above $x$. Hence, by Lemma 6.4, one has

$$\mathbf{E}\left[\delta^{\tau_n(x)-n}\right] \leq 1 - \frac{m}{2M_{c'}}. \tag{9}$$

We now turn to the proof of Proposition 5.7. Apply (9) with $n = n_1$, and $x = \underline{x}_{n_1+1}$. Plainly, $\tau_n(x) \leq n_2$. We obtain $\delta^{n-2-n_1} \leq 1 - \dfrac{m}{2M_{c'}}$, as desired.

## 6.7 Proof of Proposition 5.8

Finding an upper bound in $\mathbf{E}\left[\delta^{\tau}\right]$ is straightforward. Let $x \in T_1$ be (the marginal type of) any serious equilibrium offer of buyer 1. By Lemma 6.3, one has $D^+ p_1(x) \geq \frac{m}{2}$. Observe that, for each $y$:

$$p_1(y) - c(y) = \sum_{\tilde{\tau}} \mathbf{E}[\delta^{\tilde{\tau}-1}(p_{\tilde{\tau}} - c(y))]. \tag{10}$$

For a given $y$, denote $\tilde{\tau}_y$ an optimal stopping time in (10), and set $\tau_x = \lim_{y \searrow x} \tilde{\tau}_y$. Then

$$D^+ p_1(x) = c'(x)\left(1 - \mathbf{E}\left[\delta^{\tau_x - 1}\right]\right).$$

Since $\tau_x \leq \tau$ and $D^+ p_1(x) \geq \frac{m}{2}$, one obtains

$$\mathbf{E}\left[\delta^{\tau}\right] \leq 1 - \frac{m}{2M_{c'}}.$$

The existence of the lower bound relies on the following insight. Consider any buyer $n$ with zero equilibrium payoff, and $x > \underline{x}_n$, close to $\underline{x}_n$. The offer $p_n(x)$ with marginal type $x$ must exceed $c(x$ by a significant amount, for otherwise buyer $n$'s payoff would be positive when submitting the offer $p_n(x)$. Hence, it is likely that type $x$ will receive a serious offer from some future buyer, which he will accept. To be specific, the probability (discounted back to stage $n$) that type $x$ will accept some later offer is bounded away from zero.

In the equilibrium first phase – prior to stage $N_1$ – this allows to get an estimate of the probability of trade, and we prove that $F_{N_1}(1) \leq \dfrac{\nu}{M_{c'}}$, where $N_1 \leq \beta/(1-\delta)$, for some constant $\beta$.

From the proof of Proposition 5.1, we know that between stages $N_1$ and $N_0$, no equilibrium offer has a marginal type higher than $\bar{x}_{N_1} = \max_{n<N_1} \max T_n$, and the expected value $\bar{q}_n(1)$ of the unit steadily increases with time. The insight above allows us to prove that it takes at most $\beta_2/(1-\delta)$ additional stages for $\bar{q}_n(1$ to reach the level $\bar{c}$.

From this stage on, submitting the winning offer $\bar{c}$ yields a nonnegative payoff, hence the average value $\bar{q}_n(1)$ no longer increases: the only serious offer equilibrium offer is $\bar{c}$, and $N_0 \leq (\beta_1+\beta_2)/(1-\delta)$. From that stage on, and relying once more on the insight, it will take no more than $\beta_3/(1-\delta)$ stages on average, to receive a winning offer.

**Lemma 6.5** *Let $a, b \in (0,1)$, and a stopping time be given, such that $\mathbf{E}\left[\delta^{\tau-n}|\tau > n\right] \geq a$, for each $n \in \mathbf{N}$. Then $\mathbf{P}(\tau > N) \leq b$, where $N = \dfrac{1-a}{ab} \times \dfrac{1}{1-\delta} + 2$.*

**Proof.** Consider the optimization problem $\sup \mathbf{P}(\tau > N)$, where the supremum is taken over all $\tau$'s such that $\mathbf{E}[\delta^{\tau-n}|\tau > n] \geq a$, for all $n \leq N$. Existence of optimal solution $\tau^*$ is standard, and we set $p_n^* = \mathbf{P}(\tau^* = n|\tau^* \geq n)$. Plainly, $\tau^* > 1$ and $\tau^* \leq N+1$. Besides, the constraint $\mathbf{E}[\delta^{\tau^*-(n-1)}|\tau^* \geq n] \geq a$ is binding for each $n \leq$ such that $p_n^* > 0$.[8]

Thus, $\tau^*$ is uniquely defined by:

1. $p_n^* = 0$ as soon as $\delta^{N+1-(n-1)} \geq a$ or, equivalently, $n \geq \underline{N} := [N+2-\frac{\ln a}{\ln \delta}]$;
2. $p_{\underline{N}}^* \in [0,1]$ is set so that $\mathbf{E}\left[\delta^{\tau^*-(n-1)}|\tau^* \geq n\right] = a$;
3. for each $n < \underline{N}$, one has $\mathbf{E}\left[\delta^{\tau^*-(n-1)}|\tau^* \geq n\right] = a$, hence $p_n^* = \dfrac{a(1-\delta)}{(1-a)\delta}$.

Thus,

$$\mathbf{P}(\tau^* > N) \geq \left(1 - \frac{a(1-\delta)}{(1-a)\delta}\right)^{\underline{N}-2}.$$

[8]For otherwise, lowering $p_n^*$ by a small amount $\eta$, while increasing $p_n^*$ by the amount $\eta\delta$ would improve upon $\tau^*$.

The result follows, using the inequalities $\frac{h}{1+h} \leq \ln(1+h) \leq h$. ∎

**Lemma 6.6** *Let $s^* < 1$, and $n \in \mathbf{N}$ be given. Assume no buyer $k \leq n$ ever submits an offer with marginal type in $[s^*, 1]$. Then*

$$\bar{q}_{n+1}(1) - \bar{q}_n(1) \geq \frac{\pi_n(1-s^*)m(1-\underline{x})}{2F_{n+1}(1)},$$

*where $\pi_n := (F_n(1) - F_{n+1}(1))/F_n(1)$ is the probability that the seller accepts buyer $n$'s offer, conditional on having rejected all previous offers.*

**Proof.** For $k \in \mathbf{N}$, the expected value of the unit, as seen by buyer $k$, is

$$\bar{q}_k(1) = \frac{1}{F_k(1)} \int_{\underline{x}}^1 v(t) dF_k(t),$$

with $F_k(x) = \int_{\underline{x}}^x f_k(t)dt$, where $f_k(t)$ is the probability that type $t$ rejected the offers of all buyers prior to $k$. Note also that $f_{n+1}(t) = f_n(t)g(t)$, where $g(t)$ is the probability that the marginal type of buyer $n$'s offer does not exceed $t$. Thus,

$$\bar{q}_{n+1}(1) - \bar{q}_n(1) = \frac{1}{F_{n+1}(1)} \int_{\underline{x}}^1 v(t)g(t)dF_n(t) - \frac{1}{F_n(1)} \int_{\underline{x}}^1 v(t)dF_n(t). \tag{11}$$

For fixed $F_n$ and $\pi_n$, one has $F_{n+1}(1) = (1-\pi_n)F_n(1)$, hence the difference in (11) is minimal when $\int_{\underline{x}}^1 v(t)g(t)dF_n(t)$ is minimal. Thus, consider the auxiliary optimization problem

$$\inf \int_{\underline{x}}^1 v(t)g(t)dF_n(t),$$

where the infimum is taken over all non-decreasing and $[0,1]$-valued functions $g$, such that $g(t) = 1$ for $t \geq s^*$, and $\int_{\underline{x}}^1 g(t)dF_n(t) = (1-\pi_n)F_n(1)$. Since $v$ is strictly increasing, the solution $g^*$ to this optimization problem must satisfy $g^*(t) = \omega$ for $t < s^*$ (and $g^*(t) = 1$ for $t \geq \omega$). The value of $\omega$ is determined by the constraint $\int_{\underline{x}}^1 g(t)dF_n(t) = (1-\pi_n)F_n(1)$, which boils down to $\omega F_n(s^*) = \pi_n F_n(1)$.

It then follows that

$$\begin{aligned} \bar{q}_{n+1}(1) - \bar{q}_n(1) \quad &\geq \quad \frac{1}{F_n(1) - \omega \int_{\underline{x}}^{s^*} dF_n(t)} \left\{ \int_{\underline{x}}^1 v(t)dF_n(t) - \omega \int_{\underline{x}}^{s^*} v(t)dF_n(t) \right\} \\ &\quad - \frac{1}{F_n(1)} \int_{\underline{x}}^1 v(t)dF_n(t). \end{aligned}$$

Using the identity $\dfrac{a-a'}{b-b'} - \dfrac{a}{b} = \dfrac{b'}{b-b'}\left(\dfrac{a}{b} - \dfrac{a'}{b'}\right)$, this also yields

$$\bar{q}_{n+1}(1) - \bar{q}_n(1) \geq \omega \frac{F_n(s^*)}{F_{n+1}(1)} \left\{ \frac{1}{F_n(1)} \int_{\underline{x}}^1 v(t) dF_n(t) - \frac{1}{F_n(s^*)} \int_{\underline{x}}^{s^*} v(t) dF_n(t) \right\} \tag{12}$$

It remains to bound the right-hand side RHS in (12). Rewrite first

$$RHS = \frac{\pi_n}{F_{n+1}(1)F_n(s^*)} \left( F_n(s^*) \int_{\underline{x}}^1 v(t) dF_n(t) - F_n(1) \int_{\underline{x}}^{s^*} v(t) dF_n(t) \right),$$

and recall that $F_n(1) = F_n(s^*) + (1 - s^*)$, hence

$$RHS = \frac{\pi_n}{F_{n+1}(1)F_n(s^*)} \left( F_n(s^*) \int_{s^*}^1 v(t) dF_n(t) - (1 - s^*) \int_{\underline{x}}^{s^*} v(t) dF_n(t) \right).$$

Use now the inequality $v(t) \geq v(s^*) = m(t - s^*)$ $(t \in [s^*, 1])$ to obtain

$$\begin{aligned} RHS &\geq \frac{\pi_n}{F_{n+1}(1)F_n(s^*)} \left( mF_n(s^*) \int_{s^*}^1 (t - s^*) dt + F_n(s^*) v(s^*)(1 - s^*) - (1 - s^*) \int_{\underline{x}}^{s^*} v(t) dF_n(t) \right) \\ &\geq \frac{\pi_n (1 - s^*)}{F_{n+1}(1)F_n(s^*)} \left( mF_n(s^*) \frac{1 - s^*}{2} + \int_{\underline{x}}^{s^*} \left( v(s^*) - v(t) \right) dF_n(t) \right). \end{aligned}$$

Use now $v(s^*) - v(t) \geq m(s^* - t)$ $(t \in [\underline{x}, s^*])$:

$$HS \geq \frac{\pi_n (1 - s^*) m}{F_{n+1}(1)F_n(s^*)} \left( F_n(s^*) \frac{1 - s^*}{2} + \int_{\underline{x}}^{s^*} (s^* - t) dF_n(t) \right). \tag{13}$$

This lower bound involves the integral $\displaystyle\int_{\underline{x}}^{s^*} (s^* - t) f_n(t) dt$. For a given value of $\displaystyle\int_{\underline{x}}^{s^*} f_n(t) dt$, and since $s^* - t$ is decreasing, while $f_n$ is non-decreasing, this integral is minimal when $f_n$ is constant over the interval $[\underline{x}, s^*]$. It is then equal to

$$\frac{F_n(s^*)}{s^* - \underline{x}} \int_{\underline{x}}^{s^*} (s^* - t) dt = \frac{1}{2} (s^* - \underline{x}) F_n(s^*).$$

Substituting into (13), one obtains

$$\begin{aligned} RHS &\geq \frac{\pi_n (1 - s^*) m}{F_{n+1}(1)F_n(s^*)} \left\{ F_n(s^*) \frac{1 - \underline{x}}{2} \right\} \\ &= \frac{\pi_n (1 - s^*) m (1 - \underline{x})}{2F_{n+1}(1)}. \end{aligned}$$

■

We will find a constant $c_1 > 0$ such that $\mathbf{E}[\delta^\tau] \geq c_1$ for each equilibrium. We fix $\alpha, \eta$ such that $0 < 2v(1)\eta < \alpha < \eta$. The constant $c_1$ will depend on the choice of $\alpha$ and $\eta$, but we will make no attempt at optimizing these values.

We first define $x_0 = \underline{x} < x_1 < \cdots < x_K = 1$ recursively: given $x_k$, we set $x_{k+1} = 1$ and $K = k+1$ if $\mathbf{E}[v(t)|t \geq x_k] \geq \bar{c} + \alpha$; otherwise, we choose $x_{k+1}$ such that

$$\mathbf{E}[v(t)|t \geq x_k] = c(x_{k+1}) + \alpha \tag{14}$$

**Lemma 6.7** *One has* $K \leq \frac{M_{v'}(1-\underline{x}}{2(\nu-\alpha)}$.

**Proof.** Any solution $x_{k+1}$ to (14) satisfies $x_{k+1} > x_k$ and

$$c(x_{k+1}) + \alpha = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} v(t)dt \geq v(x_{k+1}) - \frac{M_{v'}}{2}(x_{k+1} - x_k),$$

where the inequality holds since $v(x_{k+1}) - v(t) \leq M_{v'}(x_{k+1} - t)$. Since $v(x_{k+1}) - c(x_{k+1}) \geq \nu$, this yields $x_{k+1} - x_k \geq \dfrac{2(\nu - \alpha)}{M_{v'}}$, and the result follows. ■

Next, consider an arbitrary buyer $n \in \mathbf{N}$, and denote $k_n$ the smallest $k$ such that $F_n(x_k) \geq \eta^{K-k} F_n(1)$. If $k_n > 0$, one must have $F_n(x_{k_n - 1}) < \eta F_n(x_{k_n})$, hence

$$|\mathbf{E}\left[v(t)|t \leq x_{k_n}\right] - \mathbf{E}\left[v(t)|t \in [x_{k_n-1}, x_{k_n}]\right]| \leq 2\eta v(x_{k_n}), \tag{15}$$

since $|\mathbf{E}(X) - \mathbf{E}(X|A)| \leq 2\mathbf{P}(\bar{A}) \sup X$, for each bounded random variable $X$.

By (14), the inequality (15) also yields

$$\bar{q}_n(x_{k_n}) \geq c(x_{k_n}) + \alpha - 2\eta v(1). \tag{16}$$

Since the expected value $\bar{q}_m(x)$ is non-decreasing in $m$ for fixed $x$, one thus has, for $x := x_{k_n}$,

$$\bar{q}_m(x) - c(x) \geq \alpha - 2\eta v(1), \text{ for each } m \geq n. \tag{17}$$

Denote $\tau_x$ the first buyer who submits an offer which type $x$ accepts. At stage $m$, the seller is indifferent between accepting or declining an offer with marginal type at least $x$, hence

$$\begin{aligned} p_m(x) - c(x) &= \mathbf{E}\left[\delta^{\tau_x - m}(p_{\tau_x} - c(x))|\tau_x > m\right] \\ &\leq (\bar{c} - c(x))\mathbf{E}\left[\delta^{\tau_x - m}|\tau_x > m\right], \end{aligned}$$

where $p_{\tau_x}$ is the (random) offer of buyer $\tau_x$.

By (17), and with $a := \bar{\delta}\frac{\alpha - 2\eta v(1)}{\bar{c} - c(x)}$, one has

$$\mathbf{E}\left[\delta^{\tau_x - m} | \tau_x > m\right] \geq \frac{a}{\bar{\delta}},$$

for every buyer $m \geq n$ with zero equilibrium payoff. Since there are no two consecutive buyers with positive payoff, one actually has

$$\mathbf{E}\left[\delta^{\tau_x - m} | \tau_x > m\right] \geq a, \text{ for each } m \geq n. \tag{18}$$

By Lemma, with $N_a := \frac{1-a}{a}\frac{2}{1-\delta} + 4$, one has

$$\mathbf{P}(\tau_x > n + N_a | \tau_x > n) \leq 1/2.$$

Thus, assuming type $x$ has rejected all offers up to $n$, there is a probability of at least 1/2 that he will accept an offer from some buyer $m \leq n + N_a$. Hence,

$$F_n(1) - F_{n+N_a}(1) \geq \frac{1}{2} F_n(1),$$

which, by the choice of $x$, also yields $F_n(x) \geq \eta^K F_n(1)$.

We thus have proven that, for each buyer $n$,

$$F_{n+N_a}(1) \leq (1 - \frac{1}{2}\eta^K) F_n(1). \tag{19}$$

Applying repeatedly (19), one obtains $F_n(1) \leq (1 - \frac{1}{2}\eta^K)^i$ for every $i \in \mathbf{N}$ and $n \geq iN_a$. In particular, with $N_1 := \dfrac{4}{\eta^K} \ln \dfrac{M_{c'}}{\nu} N_a$, one has $F_{N_1}(1) \leq \frac{\nu}{M_{c'}}$.

Set $s^* = \max_{n < N_1} \max T_n$. From Proposition, we know that no buyer $n \geq N_1$ ever submits a serious offer with marginal type in $(s^*, 1)$. Besides, observe that $\bar{q}_{N_1}(s^*) \geq c(s^*)$, for otherwise no buyer $n < N_1$ would have been willing to submit an offer with marginal type $s^*$.

Let $m \geq N_1$ be arbitrary. By (19) again, the probability that the seller accepts the offer of some buyer $m \in \{n, \ldots, n + N_a - 1\}$, conditional on having rejected al previous offers, is at least $\pi := (1 - \frac{1}{2}\eta^K)$. By Lemma, and assuming $F_{n+N_a}(1) > 0$, one has

$$\bar{q}_{n+N_a}(1) - c(s^*) \geq \bar{q}_{n+N_a}(1) - \bar{q}_n(1) \geq \frac{\pi}{2}(1 - s^*) m (1 - \underline{x}).$$

Note next that $\bar{c} - c(s^*) \leq M_{c'}(1 - s^*)$. Hence, with $N_2 := \dfrac{2M_{c'}}{(1 - \underline{x}) m (1 - \frac{1}{2}\eta^K)} \times N_a$, one obtains $\bar{q}_{N_1 + N_2}(1) = \bar{c}$: from stage $N_1 + N_2$ on, all equilibrium offers are either winning ones or losing ones: $N_0 \leq N_1 + N_2$.

Finally, consider buyer $N_0$, and apply (15) with $x = x_{k_{N_0}}$:

$$\mathbf{E}\left[\delta^{\tau_x - N_0} | \tau_x > N_0\right] \geq a.$$

Now, if $\tau_x > N_0$, one must have $\tau_x = \tau$, hence $\mathbf{E}[\delta^\tau] \geq a\delta^{N_0}$.

Observe that $N_1 + N_2$ is equal to a constant times $\frac{1}{1-\delta}$, for some constant. Since $\delta^{1/(1-\delta)} \geq e^{-\frac{1}{\delta}}$, the result follows.

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