Fiscal (In)Solvency, Discretionary Monetary Policy and Multiple Equilibria in a New Keynesian Model*

Andrew P. Blake† Tatiana Kirsanova‡
Bank of England University of Exeter

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Abstract

We study multiple discretionary equilibria in dynamic linear quadratic rational expectations models. We show that these models not only *can* have multiple equilibria, but in some situations *do*. We demonstrate that first order conditions are non-linear in policy parameters, despite the fact that agents' responses are linear; and multiple equilibria, if they exist, are isolated. We rule out multiple stable private sector equilibria and show that policy traps are impossible.

We demonstrate existence of multiple discretionary equilibria by example. In a simple New Keynesian model of optimal monetary policy, but with fiscal solvency constraint, monetary policy can be either 'active' or 'passive' in the sense of Leeper (1991), depending on the strength of fiscal control of debt. There is an intermediate strength of fiscal control when both active and passive policies are possible. Although the policy maker can choose the welfare maximising path, and the private sector will follow, the policy maker must know about the existence of both paths.

Key Words: Time Consistency, Discretion, Multiple Equilibria

JEL References: E31, E52, E58, E61, C61

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†Address: CCBS, Bank of England, Threadneedle Street, London EC2R 8AH; e-mail: andrew.blake@bankofengland.co.uk.

‡Address: School of Business and Economics, University of Exeter, Streatham Court, Rennes Drive, Exeter EX4 4PU; e-mail t.kirsanova@exeter.ac.uk.

1 Introduction

It is well known that if a policy is formulated from today's perspective, such policy can deliver a first-best outcome, but the passage of time can lead to an incentive for the policymaker to renege on the initially chosen plan (Kydland and Prescott (1977), Calvo (1978)). Rational private agents anticipate this and so the initial policy plan is not credible. The appreciation of this problem has motivated the study of policymaking under the additional constraint of time consistency.

One drawback of time consistency, however, is that it can generate multiple equilibria. Under a time consistent policy the policymaker takes into account current and future economic conditions. These conditions are affected by the response of the rational private sector; any response is also based on the forecast of future economic conditions. Consequently, multiple equilibria can arise: Future policy responds to a state that is at least partly determined by forecasts of the future policy.[1]

In this paper we illustrate how interactions between discretionary policy and forward-looking private behaviour can lead to multiple equilibria *in linear-quadratic dynamic models*, those most commonly used in monetary policy analysis. We consider a relatively standard New Keynesian model of monetary and fiscal interactions with a solvency constraint for fiscal policy. We assume that fiscal policy is non-strategic: it operates as a feedback rule designed to keep government debt under control. This rule takes a very simple form, with spending adjusted by some fixed proportion of debt disequilibrium. We examine the implications for monetary policy and policy equilibria of alternative values for this feedback parameter.

We show that depending on the strength of the fiscal control of debt, monetary policy can be either 'active' or 'passive' in the sense of Leeper (1991), i.e. it can either be totally devoted to the control of inflation and output, or it can be forced to take a part in the control debt. In the latter case, 'active' fiscal policy, that pursues other targets, impacts on the type of monetary policy that can be followed. However, there is an intermediate strength of *fiscal control* when both 'active' and 'passive' *monetary policy* are possible.

This situation arises because monetary and fiscal policies can complement each other in the control of domestic debt, but the main task of monetary policy is not debt but macroeconomic stabilisation. If fiscal feedback is present but weak, then the monetary policymaker and the private sector can have the mutually consistent belief that the fiscal feedback is insufficiently strong to prevent a real debt spiral. In this case, the interest rate does not respond strongly to any inflation disequilibrium. Alternatively, the monetary policymaker and the private sector can have a mutually consistent belief that the fiscal feedback *is* sufficient strong to prevent the real debt spiral, so that the interest rate does respond conventionally strongly to an inflation disequilibrium. The existence of multiple equilibria leads to a problem of choosing the best one on welfare grounds.

[1] Multiple equilibria under discretionary policy is a known feature of some types of models. The early literature (Rogoff (1987), Ireland (1997)) discussed the Barro and Gordon (1983) model with trigger strategies that generate a continuum of steady state *levels* of inflation. Recently, in a similar model Albanesi et al. (2003) demonstrated 'policy traps', in which discretionary monetary policy had to accommodate private sector's expectations about a particular type of policy. Another kind of multiple equilibria in a non-linear dynamic (two-period) Barro-Gordon model is discussed in King and Wolman (2004) who find a multiplicity of responses of the private sector to monetary policy actions.

This choice might seem innocuous: discretionary policy as described is a game between the private sector and the policymaker in which the policymaker acts as a Stackelberg leader.[2] In theory the policymaker can always choose the equilibrium with the highest welfare. But a key ingredient to enable a policymaker to choose the best equilibrium is that it must be aware of the existence of both paths. This may not always be the case in practice.

Although well established for non-linear models, the multiplicity of equilibria under discretion has never been shown for dynamic *linear-quadratic* rational expectations models. But because one needs to use an iterative *and* numerical algorithm to solve even moderately sized models, it can be difficult to grasp the analytical complexity of the underlying problem. As a result, the conditions under which multiple discretionary solutions exist for the linear-quadratic problem have never been rigorously determined. This also reflects a belief that discretionary equilibria obtained in this way are probably unique. Indeed, Oudiz and Sachs (1985), whose seminal paper provides one of the most popular iterative approaches, explicitly state that they believe their algorithm generates a unique equilibrium, expressly because of the linear-quadratic nature of the problem. Although the subsequent *theoretical* literature has never ruled out a multiplicity of discretionary solutions, examples of multiple solutions to linear-quadratic models with rational expectations seem never to have been reported.[3] In many applications researchers routinely use the Oudiz and Sachs (1985) or Backus and Driffill (1986) iterative algorithm and perhaps note the linear-quadratic structure of the model.

Of course, *any* iterative procedure, if it converges, delivers only one fixed point. In order to find different solutions we need to search for an equilibrium starting from different initial values for system matrices, but it might be helpful to have some understanding of whether multiplicity is a conceivable option for a particular problem. Moreover, the precise form of the multiplicity can differ. Equilibria exist in which both players have unique well-defined and mutually-consistent responses to each other's actions but where the equilibrium differs along with those mutually consistent beliefs. An alternative is private-sector equilibria, in which the private sector could rationally react in more than one way to the same action of the policymaker.[4] In this paper we provide a formal analysis of the generic properties of discretionary equilibria in general mathematical terms before we discuss our model: this facilitates the interpretation of any multiplicity of equilibria and makes it possible to present analytical results for the model that otherwise would have to be solved numerically.

The paper is organized as follows. In the next section we formally set up a general linear-quadratic optimisation problem and describe the benchmark method of solution proposed by Oudiz and Sachs (1985) and Backus and Driffill (1986). We highlight the fixed point aspect of this approach. We then re-examine the first order conditions to the optimisation system and demonstrate that they constitute a system of well-studied quadratic matrix equations, i.e. they

[2] This also ensures that there are no coordination problems in setting the same beliefs about the future economic conditions.

[3] As we make explicit below this is perhaps surprising given the structure of the underlying problem. Papavassilopoulos and Olsder (1984) noted that LQ Nash dynamic games can admit multiple solutions and such games involve the solution of similar (but crucially not the same) sets of equations as the problem we consider. This was exploited by Lockwood and Philippopoulos (1994), who provide an example of multiple equilibria in a bargaining model. However, this setup is the one without rational expectations, and this is important.

[4] We demonstrate the existence of discretionary equilibria of the first type; King and Wolman (2004) demonstrate the existence of equilibria of the second type in a similar model.

are non-linear in policy parameters and so can produce multiple equilibria. We provide a complete taxonomy of discretionary solutions to controllable linear-quadratic rational expectations models. We demonstrate explicitly that discretionary equilibria, if they exist, are necessarily distinct or isolated equilibria. Further, we show that the multiplicity of stable *private sector equilibria* as well as so-called 'policy traps' can be both ruled out. In Section 4 we present an example of a New Keynesian model that *has* multiple discretionary equilibria. Multiple discretionary equilibria can result depending on the degree of fiscal policy activism. Section 5 contains a summary of the results and their implications.

2 Discretionary Equilibrium as a Fixed Point

2.1 A class of models

We assume a non-singular linear deterministic rational expectations model of the type described by Blanchard and Kahn (1980), augmented by a vector of control instruments. Specifically, the evolution of the economy is explained by the following system:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1}^e \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [u_t] \tag{1}$$

where y_t is an n_1-vector of predetermined variables with initial conditions y_0 given, x_t is n_2-vector of non-predetermined (or jump) variables, and u_t is a $k-$vector of policy instruments of the policymaker. For notational convenience we define the n-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$.

Typically, this system represents the solved out optimisation problem for one of the players in the policy game. This player also has 'instruments', represented by x_t, so the second equation in (1) is essentially its reaction function. Additionally, there is an equation explaining the evolution of predetermined variable y_t. These two equations together describe the 'evolution of the economy' as observed by policymakers. One can draw an example with the behaviour of the private sector in macroeconomic models, presented by the Euler consumption equation and the Phillips curve, see Clarida et al. (1999) among many others.

The policymaker has the following loss function:

$$W_t = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}(g_s' Q g_s) = \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}(z_s'\mathcal{Q}z_s + 2z_s'\mathcal{P}u_s + u_s'\mathcal{R}u_s). \tag{2}$$

The vectors g_s is the goal variables of the policymaker, $g_s = \mathcal{C}(z_s', u_s')'$. The loss function can include instrument costs, but no assumptions of invertibility of $\mathcal{R}$ are made.

It should be noted that none of the results depend on the deterministic setup outlined and the consequent assumption of perfect foresight. We can add an appropriate vector of shocks, but this unneccesarily complicates the analysis to demonstrate the main point.[5]

[5] Shocks can be included into vector y_t, see e.g. Dennis (2007).

2.2 A Dynamic Programming Algorithm

The two fixed-point algorithms to search for discretionary solution that are most commonly used in the literature are those developed by Oudiz and Sachs (1985) and Backus and Driffill (1986). Both algorithms solve for optimal discretionary rules using dynamic programming principles, but Oudiz and Sachs (1985) solve for a stationary feedback rule by taking the limit as $t \to -\infty$ of a finite horizon problem while Backus and Driffill (1986) solve the asymptotic problem directly. Söderlind (1999) provides a popular implementation of this algorithm. We summarize it here in order to have a frame of reference and to introduce notation which is convenient for our purposes.

Under discretion, the policymaker reoptimizes every period by taking the process by which private agents form their expectations as given, but where the expectations are consistent with actual policy. Since the model is linear-quadratic, the solution in any time t gives a value function which is quadratic in the state variables:

$$W_t = \frac{1}{2} y_t' S y_t \tag{3}$$

a linear relation between the the forward-looking variables:

$$x_t = -N y_t \tag{4}$$

and a linear policy reaction function:

$$u_t = -F y_t. \tag{5}$$

The value function of the policymaker in t will then satisfy the Bellman equation

$$y_t' S y_t = \min_{u_t} \left(z_t' \mathcal{Q} z_t + 2 z_t' \mathcal{P} u_t + u_t' \mathcal{R} u_t \right) + \beta \left(y_{t+1}' S y_{t+1} \right) \tag{6}$$

subject to (4), the second equation in (1) and y_t given.

To solve the problem, it is convenient to rewrite the optimisation problem excluding forward-looking variables. Relationship (4) can be taken with one lead forward and the state variable is substituted from the first equation (1). We obtain:

$$\begin{aligned} x_{t+1} &= -N y_{t+1} = -N(A_{11} y_t + A_{12} x_t + B_1 u_t) \\ &= A_{21} y_t + A_{22} x_t + B_2 u_t \end{aligned} \tag{7}$$

from where we can obtain:

$$\begin{aligned} x_t &= -(A_{22} + N A_{12})^{-1} [(A_{21} + N A_{11}) y_t + (B_2 + N B_1) u_t] \\ &= -J y_t - K u_t. \end{aligned} \tag{8}$$

where

$$J = (A_{22} + N A_{12})^{-1} (A_{21} + N A_{11}), \tag{9}$$
$$K = (A_{22} + N A_{12})^{-1} (B_2 + N B_1) \tag{10}$$

Substitute (5) into (8) to obtain

$$x_t = -(J - KF)\, y_t = -N y_t$$

so

$$N = J - KF. \tag{11}$$

We substitute (3) into formula (6) and, using (8) and $y_{t+1} = (A_{11} - A_{12}J)y_t + (B_1 - A_{12}K)u_t$, obtain that along the optimal path:

$$\begin{aligned} y_t' S y_t &= y_t'(Q^* + \beta A^{*\prime} S A^*)y_t + u_t'(P^{*\prime} + \beta B^{*\prime} S A^*)y_t \\ &+ y_t'(P^* + \beta A^{*\prime} S B^*)u_t + u_t'(R^* + \beta B^{*\prime} S B^*)u_t, \end{aligned} \tag{12}$$

where:

$$\begin{aligned} &Q^* = \mathcal{Q}_{11} - \mathcal{Q}_{12}J - J'\mathcal{Q}_{21} + J'\mathcal{Q}_{22}J,\ P^* = J'\mathcal{Q}_{22}K - \mathcal{Q}_{12}K + \mathcal{P}_1 - J'\mathcal{P}_2, \\ &R^* = K'\mathcal{Q}_{22}K + \mathcal{R} - K'\mathcal{P}_2 - \mathcal{P}_2'K,\ A^* = A_{11} - A_{12}J, \text{ and } B^* = B_1 - A_{12}K. \end{aligned}$$

The optimal feedback rule can be determined from (12) by differentiating the loss function with respect to u_t:

$$u_t = -(R^* + \beta B^{*\prime} S B^*)^{-1}(P^{*\prime} + \beta B^{*\prime} S A^*)y_t = -F y_t,$$

from where the policymaker's reaction function is:

$$F = (R^* + \beta B^{*\prime} S B^*)^{-1}(P^{*\prime} + \beta B^{*\prime} S A^*) \tag{13}$$

Now, we substitute the reaction rule (5) and obtain recursive equations for S:

$$S = T_0 + \beta T_1' S T_1 \tag{14}$$

where:

$$T_0 = Q^* + F'R^*F - F'P^{*\prime} - P^*F, \quad T_1 = A^* - B^*F \tag{15}$$

Assuming that $S^{(0)}$ and $N^{(0)}$ are known, at step 1 we find $J^{(0)}, K^{(0)}, F^{(0)}$ using (9), (10), (13): and update $S^{(1)}$ and $N^{(1)}$ using (14) and (11) correspondingly. We then continue and obtain $S^{(2)}$, $N^{(2)}$ *etc.* The fixed point is found if at step i

$$\max\left(\left\|S^{(i)} - S^{(i+1)}\right\|, \left\|N^{(i)} - N^{(i+1)}\right\|\right) < \delta^{DP} \tag{16}$$

where δ^{DP} is a given tolerance. As soon as condition (16) is satisfied for some i, the triplet $\{N^{DP}, S^{DP}, F^{DP}\} = \{N^{(i)}, S^{(i)}, F^{(i)}\}$ is a solution to the discretionary optimisation problem. By construction, triplet $\{N^{DP}, S^{DP}, F^{DP}\}$ is a fixed point on the Bellman operator.

Having presented a version of this algorithm, Oudiz and Sachs (1985, p. 311) comment as follows on the existence of solution:

"We do not know of any general result concerning the convergence of this process. However in our empirical applications we have not run into major problems. Cohen and Michel (1984) show that in a one dimensional case this kind of a recursion does have a fixed point."

Of course, convergence of the algorithm neither guarantees the uniqueness of the solution, nor does it imply any particular properties of the resulting equilibrium. Oudiz and Sachs (1985, p. 288) make the following remarks:

"Although we cannot prove that the resulting function is the unique memoryless, time-consistent equilibrium, we suspect that it is in fact unique, in view of the linear-quadratic structure of the underlying problem."

In what follows, we address these two issues. Although we can say very little about the existence of the fixed point, we discuss what solutions look like if they exist. We also show that, despite the fact that decision rules are linear for both the policymaker and the private sector, conditions for the fixed point solution are non-linear in policy parameters. This non-linearity can generate a multiplicity of solutions. This is not of mere theoretical interest: we show that in some cases there *are* multiple equilibria.

3 Multiplicity of Discretionary Equilibria

3.1 The First Order Conditions Re-examined

It is easy to prove the following proposition:

Proposition 1 *The first-order conditions to the dynamic programming problem (11), (14) and (13) can be rewritten in the following form:*

$$S = Q^* + \beta A^{*\prime} S A^* - \left(P^{*\prime} + \beta B^{*\prime} S A^*\right)' \left(R^* + \beta B^{*\prime} S B^*\right)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^*\right) \tag{DARE}$$

$$F = \left(R^* + \beta B^{*\prime} S B^*\right)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^*\right) \tag{POLICY}$$

$$0 = N C_{11} + C_{21} - N C_{12} N - C_{22} N \tag{NCARE}$$

where matrix C describes the evolution of the system under control F:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} - B_1 F & A_{12} \\ A_{21} - B_2 F & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix}, \tag{17}$$

and matrices Q^*, P^*, R^*, A^*, and B^* are defined as in (12).

Proof. A straightforward substitution of (9), (10) into (11) produces equation (NCARE), a Non-symmetric Continuous Algebraic Riccati Equation for N, conventionally abbreviated as 'NCARE'. Equation (14) where coefficients are determined by relationships (15) is equivalent to (DARE), provided we can use (POLICY), with (DARE) a Discrete Algebraic Riccati Equation, again conventionally 'DARE'. In order to see this, we can do the following manipulations, adding and subtracting additional terms:

$$\begin{aligned}
S &= Q^* + \beta A^{*\prime} S A^* - (\beta B^{*\prime} S A^* + P^{*\prime})'(\beta B^{*\prime} S B^* + R^*)^{-1}(\beta B^{*\prime} S A^* + P^{*\prime}) \\
&= Q^* + \beta A^{*\prime} S A^* - (\beta B^{*\prime} S A^* + P^{*\prime})' F \\
&= Q^* + F' R^* F - F' P^{*\prime} - P^* F + \beta \left(A^* - B^* F\right)' S \left(A^* - B^* F\right) \\
&\quad + F' P^{*\prime} - F' R^* F + \beta F' B^{*\prime} S \left(A^* - B^* F\right) \\
&= T_0 + \beta T' S T + F' \left[\beta B^{*\prime} S \left(A^* - B^* F\right) + P^{*\prime} - R^* F\right] \\
&= T_0 + \beta T' S T.
\end{aligned}$$

as the term in square brackets in the penultimate line is zero because of (POLICY). ■

Thus, in order to find a discretionary solution to an optimisation problem, we need to solve a simultaneous system of the first order conditions (DARE), (POLICY) and (NCARE). It is evident that although the decision rules are linear for both the policymaker and the private sector, the first order conditions are non-linear in parameters: they constitute a non-linear *polynomial* system of $q = (n + k) \times n_1$ equations for q unknown coefficients of matrices S, F and N. Such systems generally have many solution sets of different dimensions: that is, a system could have isolated solution points, curves, surfaces, etc., all simultaneously.

Remark 1 *This characterisation of the reaction function of agents as a Riccati equation is implicit in much of the literature, including Blanchard and Kahn (1980).*[6] *It is explicit in much of the literature on dynamic LQ Nash games, but the results are not transferable. A dynamic LQ Nash game would consist of a system of POLICY-type decision rules for each player dependent on appropriately defined an interrelated DARE-type equations. For the ational expectations problems the stability and existence conditions are necesserily different as we instead have a mixture of DARE- and NCARE-type equations as part of simultaneous system.*[7] *We also have a logical structure for the system which we exploit.*

In what follows we describe the necessary properties of discretionary solutions, if they exist. Additionally, our analysis will provide an insight into the nature of discretionary equilibria. It is convenient to split system (DARE) to (NCARE) into two blocks, the first block consisting of the two matrix equations (DARE) and (POLICY), and the second block consisting of a single matrix equation (NCARE). We can discuss solutions to these blocks separately.

3.2 The Policymaker's Reaction Function and the Value Function

Equation (DARE) is a symmetric discrete algebraic Riccati equation for a square symmetric positive semi-definite *value function* matrix S. It is a quadratic matrix equation which coefficients Q^*, P^*, R^*, A^*, and B^* are determined by structural system matrices, given that the household reaction function is given in the form of linear rule (4). The following two results are standard to the literature on Riccati equations (all references are given in Appendix A):

[6] One of few exceptions is Blake (2004).

[7] NCARE is a *non-symmetric* equation that substantially widens the set of possible solutions.

1. There is a *unique* symmetric solution S to (DARE) if the matrix pair (A^*, B^*) is controllable, i.e. if the controllability matrix $[B^*, A^*B^*, A^{*2}B^*, ..., A^{*n_1-1}B^*]$ has full row rank.[8]

2. The policymaker's reaction function F, which is *uniquely* determined from (POLICY) for given S, is *stabilising*, i.e. all eigenvalues of the matrix $\Omega(F(S))$ defined by:

$$\begin{aligned} y_{t+1} &= A_{11}y_t + A_{12}x_t + B_1u_t = (A_{11} - A_{12}J)\,y_t + (B_1 - A_{12}K)\,u_t \\ &= (A^* - B^*F)\,y_t = \Omega(F(S))\,y_t \end{aligned} \tag{18}$$

   which is the dynamic system under control, are strictly inside the unit circle.

Practically, this solution can be found with some iterative procedure that solves (DARE).

3.3 The Private Sector's Reaction Function and the Equilibrium

Equation (NCARE) is a non-symmetric continuous algebraic Riccati equation for a non-square matrix N that describes the *reaction function of the private sector*. This reaction needs to be *stabilising*, i.e. all eigenvalues of matrix $\Omega(N)$ that defines the evolution of the system:

$$y_{t+1} = C_{11}y_t + C_{12}x_t = (C_{11} - C_{12}N)\,y_t = \Omega(N)\,y_t$$

need to be strictly inside the unit circle.

The coefficients of this quadratic matrix equation depend only on structural system matrices and the policy reaction written in the form of linear rule (5). It can be shown (and we provide references in Appendix B) that *if matrix C can be diagonalised* then, under some rather general conditions,[9] one of three situations is possible:

1. If $m = n_2$ then there is a *unique* stabilising solution N^* to (NCARE). This solution can be found using the Blanchard and Kahn (1980) formula for matrix C.

2. If $m > n_2$ then there are *no* stabilising solutions to (NCARE) and therefore no stabilising solutions to (17).

3. If $m < n_2$ then there is *at most one stable*[10] stabilising solution N^* to (NCARE) and *most* iterative methods of solving (NCARE) will converge to it.

The third of these (Point 3) is the most unfamiliar and requires further discussion. System (NCARE) is a system typically including first order conditions of a utility maximizing private sector, as discussed in the derivation of (7)–(8). It constitutes necessary but not sufficient conditions for a welfare maximizing choice of the private sector. For $m \leq n_2$ the time-consistency

[8] The requirement of stabilisability of (A^*, B^*) is standard for the linear-quadratic optimal control. We use this condition as a sufficient condition. We do not discuss whether this is necessary condition.

[9] We discuss the case when solution does not exist in Appendix B. It is unlikely to happen in most economic applications.

[10] Any solution to (NCARE) N^* is a steady state of a corresponding Differential Riccati Equation (DRE): $dN(s)/ds = N(s)C_{11} + C_{21} - N(s)C_{12}N(s) - C_{22}N(s)$. N^* is a stable solution of (NCARE) if it is an asymptotically stable steady state of the DRE.

requirement then singles out a unique solution for the private sector's response.[11] The private sector knows that the policymaker chooses the best response given the private sector's reaction function. If a rational private sector fails to maximise its own welfare, it will be 'punished' by the policymaker's choice of time-consistent F. Situation 3 suggests that even if the private sector makes a mistake in any initial approximation to N, it must end up with the unique stable N, which can be regarded as a *stable private sector equilibrium.* However, if $m < n_2$ then there are also $\binom{n-m}{n_2-m} - 1$ stabilising but *unstable* solutions. They should be regarded as *unstable private sector equilibria.*[12]

Practically, Point 3 suggests that the standard solution procedure of finding discretionary solution presented in Section 2 — which requires the initialisation of both matrices S and N and then finds the solution by an iterative method simultaneously updating S and N at each step — can be de-coupled into two separate dependant iterative procedures. A further iteration is then needed between them. Using this approach we first initialize $F^{(0)}$ and solve (NCARE) by an iterative method. We will obtain a unique stabilising solution $N^{(0)}$, provided $m \leq n_2$. We then solve (DARE) iteratively for a unique symmetric $S^{(0)}$ and compute an updated $F^{(1)}$. We then continue with $N^{(1)}, F^{(2)}$, etc. The fixed point of the Bellman operator is found if at step i:

$$\max\left(\left\|F^{(i)} - F^{(i+1)}\right\|\right) < \delta \tag{19}$$

where δ is given tolerance.

We should note that *if matrix C cannot be diagonalised* then there is a continuum of solutions to (NCARE), as shown by Freiling (2002). We do not consider this case further.

3.4 Equilibrium outcome

We now summarize our findings about the nature of solutions to the discretionary problem in a linear-quadratic framework. As the system of the first order conditions (DARE) to (NCARE) can possibly be solved in two stages: (i) given N to find $F = F(N)$ and $S = S(N)$ and (ii) given F to find $N = N(F)$, it is convenient to present results that correspond to these steps:

1. For every N there is a unique S, if the pair (A^*, B^*) is controllable. A unique matrix $F = F(S)$ then delivers a *stabilising* solution.

2. For every F there is at most one stable *stabilising* N, if matrix C can be diagonalised.

Note that conditions when we can have multiplicity of either F given N, or continuum of N given F are non-restrictive, i.e. they are unlikely to happen in most economic applications.

The results above rule out the possibility to have a *continuum of private sector* equilibria, as frequently happens if the policymaker operates by some simple (unoptimized) feedback rule.[13] They also rule out *multiple, distinct and stable private sector* equilibria, i.e. multiple responses

[11] The same can be applied to the case $m = n_2$.

[12] For non-linear models there can be several stable private-sector equilibria, see King and Wolman (2004). However, in our LQ model *all* private-sector equilibria *except one* are necessarily unstable.

[13] As a consequence, the possibility of continuum of *discretionary* equilibria (curves or surfaces etc.) is also ruled out.

of the private sector to the same policy, as discussed in King and Wolman (2004). However, multiple, distinct and *unstable private sector* equilibria *are* possible.

The results above do not rule out *distinct multiple discretionary* equilibria, namely a possibility to have different stable triplets $\{N_i, S_i, F_i\}$ and $\{N_j, S_j, F_j\}$ each of which solves the optimisation problem, i.e. satisfies (NCARE), (DARE) and (POLICY). The existence of such multiple equilibria we demonstrate in the next section.

Remark 2 *In order to find multiple equilibria numerically, we need to run a search starting with different initial matrices F, N and S. The results in Sections 3.2 and 3.3 suggest that we can iterate between the two matrices N and F, i.e. in the policy space. In this respect, the algorithm which is discussed in Dennis (2007) is close to our method. However, that algorithm updates the policy reaction matrix (analogous to matrix F) and the private sector reaction matrix (analogous to matrix N) simultaneously and so this algorithm requires the initialisation of both matrices. Point 3 in Section 3.3 suggests that we need only initialise the policy reaction matrix, F, in the search for multiple N. Frequently, economic intuition can be used to facilitate the search with only the policy rule to pre-specify.*

Remark 3 *Note that the algorithm described in Section 2 is only one of possibly many practical algorithms to find a fixed point of the Bellman operator, or, equivalently, to find a solution to system (DARE)–(NCARE). It converges to the same point as the procedure we have just discussed: to the discretionary equilibrium in which the private sector equilibrium is stable. This is because the numerical implementation of the algorithm presented in Section 2 uses an iterative solution of equations that is equivalent to iterative solving (NCARE). This procedure cannot converge to an unstable private sector equilibrium.*

4 Multiple Equilibria in a New Keynesian Model

In the foregoing we have established a number of conditions that hold in the analysis of discretionary equilibria. Nothing so far implies that multiple equilibria will generally be a feature of interesting models. We now turn to a case where we can demonstrate important implications of a fiscal solvency constraint on discretionary monetary policy making in a very standard model, where we do find multiple equilibria.

4.1 The Model

Our model is a version of the Woodford (2003) model with a non-zero-weight government sector. The model leads to a three-equation dynamic system that describes dynamics of out of the steady state economy: a system of an IS curve, a Phillips curve and an intertemporal budget constraint. We assume a simple fiscal rule, that is the fiscal authorities stabilise domestic debt. Specifically, the evolution of the economy is determined by the following equations, written in log-linearised form around the steady state with inflation $\pi = 0$, real output Y, real private consumption $C = \theta Y$, real government spending $G = (1-\theta)Y$, real debt B, and the interest rate $1+R=1/\beta$. The model is:

$$
\begin{aligned}
c_t &= c_{t+1} - \sigma(i_t - \pi_{t+1}) && (20)\\
\pi_t &= \beta\pi_{t+1} + \kappa_c c_t + \kappa_y y_t && (21)\\
y_t &= (1-\theta)g_t + \theta c_t && (22)\\
b_{t+1} &= i_t + \frac{1}{\beta}\left(b_t - \pi_t + \frac{1-\theta}{B}g_t - \frac{\tau}{B}y_t\right) && (23)\\
g_t &= -\lambda b_t && (24)
\end{aligned}
$$

where endogenous variables are aggregate real output $y_t = \ln(Y_t/Y)$, real private consumption $c_t = \ln(C_t/C)$, real government spending $g_t = \ln(G_t/G)$, real debt b_t, nominal interest rate $i_t = \ln((1+R_t)/(1+R))$ and inflation π_t. All structural parameters are defined in Appendix C. This system describes the dynamic behaviour of the economy as observed by a policymaker. Equations (20) and (21) describe the reaction function of the private sector. The private sector chooses consumption and inflation at each period in time, such that their future utility and profits are maximized, given the evolution of state variables and policy. The fiscal authority is not treated as a strategic player in this set-up, as it mechanically reacts to the level of domestic debt. Therefore, the level of assets (domestic debt), government expenditures and output are predetermined state variables; the interest rate is the policy variable. The private sector explicitly treats monetary policy as given when making decisions.

Note that this setup implicitly requires $\lambda < \infty$. As the model is entirely deterministic, the only source of disequilibrium is an initial debt disequilibrium, $b_0 \neq 0$. Under any finite λ this is not implausible, provided b_0 is small so that that economic variables C, G and Y remain positive and close to their steady state values, in the neighborhood of which the linearisation of the system remains valid. $\lambda = \infty$ should correspond to a permanently balanced budget case and $b_t \equiv 0$ for any $t \geq 0$. In the latter case equation (24) is redundant and equation (23) determines balanced budget fiscal policy g_t. Formally, it means that if we are to take limit as λ tends to infinity, we also need to impose that b_t tends to zero for any t, including $t = 0$. This problem is a consequence of using a linear rule for G, which is a 'local' fiscal rule.

The central bank uses the short-term interest rate and *acts under discretion.* We assume that the central bank explicitly maximises the aggregate utility function, and this implies the following loss function, where terms independent of policy and all higher order terms are ignored

$$
\frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\pi_s^2 + a_c c_s^2 + a_g g_s^2 + a_y y_s^2\right). \tag{25}
$$

This quadratic approximation to the social loss is obtained assuming that there is a production subsidy that eliminates the distortion caused by both monopolistic competition and income taxes. As a result, welfare can be written in terms of deviations from the natural rate levels for output, consumption and government spending, and inflation.[14]

[14] This derivation follows Woodford (2003). The alternative way of deriving social welfare of Sutherland (2002) and Benigno and Woodford (2004) is inappropriate, as it assumes some sort of precommitment to a policy which is not the case under discretion. Additionally, elimination of monopolistic and tax distortions allow us to concentrate on the effect of the solvency constraint on monetary policy.

4.2 A canonical representation

We can substitute out all static variables, output and government spending, in order to come to the following reduced form linear-quadratic optimisation problem, written in a matrix form:

$$\min_{\{i_s\}_{s=t}^{\infty}} \frac{1}{2}\sum_{s=t}^{\infty}\beta^{s-t}\left(\begin{bmatrix} b_s \\ \pi_s \\ c_s \end{bmatrix}'\begin{bmatrix} \left(a_g + a_y(1-\theta)^2\right)\lambda^2 & 0 & -a_y(1-\theta)\lambda\theta \\ 0 & 1 & 0 \\ -a_y(1-\theta)\lambda\theta & 0 & \left(a_c + a_y\theta^2\right) \end{bmatrix}\begin{bmatrix} b_s \\ \pi_s \\ c_s \end{bmatrix}\right) \tag{26}$$

subject to the dynamic system:

$$\begin{bmatrix} b_{s+1} \\ \pi_{s+1} \\ c_{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta}\left(1-\frac{(1-\theta)(1-\tau)}{B}\lambda\right) & -\frac{1}{\beta} & -\frac{\theta\tau}{B\beta} \\ \frac{\lambda\kappa_y(1-\theta)}{\beta} & \frac{1}{\beta} & -\frac{(\kappa_c+\theta\kappa_y)}{\beta} \\ -\frac{\sigma(1-\theta)\lambda\kappa_y}{\beta} & -\frac{\sigma}{\beta} & 1+\frac{\sigma(\kappa_c+\theta\kappa_y)}{\beta} \end{bmatrix}\begin{bmatrix} b_s \\ \pi_s \\ c_s \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \sigma \end{bmatrix}[i_s]\,. \tag{27}$$

Variables π_s and c_s are non-predetermined, b_s is a predetermined variable and i_s is the monetary policy instrument. The system matrices needed for computation are:

$$A_{11} = \left[\frac{1}{\beta}\left(1-\frac{(1-\theta)(1-\tau)}{B}\lambda\right)\right], \quad A_{12} = \begin{bmatrix} -\frac{1}{\beta} & -\frac{\theta\tau}{B\beta} \end{bmatrix}, \quad B_1 = [1]\,,$$

$$A_{21} = \begin{bmatrix} \frac{\lambda\kappa_y(1-\theta)}{\beta} \\ -\frac{\sigma(1-\theta)\lambda\kappa_y}{\beta} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} \frac{1}{\beta} & -\frac{(\kappa_c+\theta\kappa_y)}{\beta} \\ -\frac{\sigma}{\beta} & 1+\frac{\sigma(\kappa_c+\theta\kappa_y)}{\beta} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix},$$

$$Q_{11} = \left[\left(a_g + a_y(1-\theta)^2\right)\lambda^2\right], \quad Q_{12} = \begin{bmatrix} 0 & -a_y(1-\theta)\lambda\theta \end{bmatrix}, \quad P_1 = [0]\,,$$

$$Q_{21} = \begin{bmatrix} 0 \\ -a_y(1-\theta)\lambda\theta \end{bmatrix}, \quad Q_{22} = \begin{bmatrix} 1 & 0 \\ 0 & \left(a_c + a_y\theta^2\right) \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $R = [0]$.

4.3 The Policymaker's Reaction Function and the Value Function

In order to find the *policymaker's reaction function* we introduce the following notation:

$$\begin{bmatrix} \pi_s \\ c_s \end{bmatrix} = -\begin{bmatrix} J_\pi \\ J_c \end{bmatrix}[b_s] - \begin{bmatrix} K_\pi \\ K_c \end{bmatrix}[i_s]\,, \tag{28}$$

and compute the following scalars:

$$Q^* = J_\pi^2 + 2\theta\lambda a_y(1-\theta)J_c + J_c^2\left(a_c+\theta^2 a_y\right) + \lambda^2\left(a_g + a_y(1-\theta)^2\right)$$

$$P^* = J_\pi K_\pi + J_c K_c\left(a_c+\theta^2 a_y\right) + \theta\lambda K_c a_y(1-\theta)\,, \quad R^* = K_\pi^2 + K_c^2\left(a_c+\theta^2 a_y\right)$$

$$A^* = \frac{1}{\beta}J_\pi + \frac{\theta\tau}{\beta B}J_c + \frac{1}{\beta}\left(1-\frac{(1-\theta)(1-\tau)}{B}\lambda\right), \quad B^* = 1 + \frac{1}{\beta}K_\pi + \frac{\theta\tau}{\beta B}K_c.$$

Then, (DARE) is a quadratic equation for a scalar variable S and can be written:

$$\beta B^{*2}S^2 + \left(R^* - \beta\left(Q^*B^{*2} - 2P^*B^*A^* + R^*A^{*2}\right)\right)S + \left(P^{*2} - Q^*R^*\right) = 0. \tag{29}$$

This equation has two eigenvalues and the product of these eigenvalues is equal to:

$$P^{*2} - Q^* R^* = -\left(\left(a_c a_g + a_y a_c (1-\theta)^2 + \theta^2 a_g a_y\right) \lambda^2 K_c^2 + a_c (K_\pi J_c - J_\pi K_c)^2 + a_g \lambda^2 K_\pi^2 + a_y (\lambda (1-\theta) K_\pi + \theta (K_\pi J_c - J_\pi K_c))^2\right) < 0.$$

with determinant:

$$D = \left(R^* - \beta \left(Q^* B^{*2} - 2P^* B^* A^* + R^* A^{*2}\right)\right)^2 - 4\beta B^{*2} \left(P^{*2} - Q^* R^*\right) > 0$$

Therefore, the two eigenvalues are always real and of different signs. Since we are looking for a positive value function S, the solution is unique. We can easily find it with conventional methods for solving quadratic equations. Having found S we can uniquely determine optimal discretionary policy as the reaction function:

$$F = \frac{P^* + \beta B^* A^* S}{R^* + \beta B^{*2} S}. \tag{30}$$

Note that in equations (29) and (30) all coefficients depend on J_π, J_c, K_π, and K_c, which are, in their turn, functions of N_π and N_c. Therefore $F = \mathcal{F}(\mathcal{S}(N_\pi, N_c)) = \mathcal{F}(N_\pi, N_c)$ depends on the two coefficients of the reaction function.

4.4 The Private Sector's Reaction Function

Suppose that the policymaker operates with a linear rule:

$$[i_t] = -[F][b_t] \tag{31}$$

then the household reaction function will necessarily have a linear form, given by:

$$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = -\begin{bmatrix} N_\pi \\ N_c \end{bmatrix} [b_t]. \tag{32}$$

The private sector's reaction function solves (NCARE), and can be written as a system of two quadratic equations in N_π and N_c:

$$N_\pi^2 + \frac{\theta\tau}{B} N_\pi N_c - N_\pi \left(\frac{(1-\theta)(1-\tau)}{B}\lambda + \beta F\right) + (\kappa_c + \theta\kappa_y) N_c + \lambda\kappa_y (1-\theta) = 0 \tag{33}$$

$$N_c^2 + \frac{B}{\theta\tau} N_c N_\pi + \frac{B}{\theta\tau}\left(1 - \frac{(1-\theta)(1-\tau)}{B}\lambda - \beta\left(1 + \frac{\sigma(\kappa_c + \theta\kappa_y)}{\beta} + F\right)\right) N_c + \frac{B\sigma}{\theta\tau} N_\pi - \frac{B\sigma}{\theta\tau}((1-\theta)\lambda\kappa_y + \beta F) = 0. \tag{34}$$

This system has three solution pairs $\{N_\pi, N_c\}$. They are shown in Figure 1 for some given values of λ and F.[15] The solution of (33), depicted by a dashed line, intersects solution of (34),

[15] The base line calibration is discussed below and in Appendix C. Here $\lambda = 0.108$, $F = 0.01$.

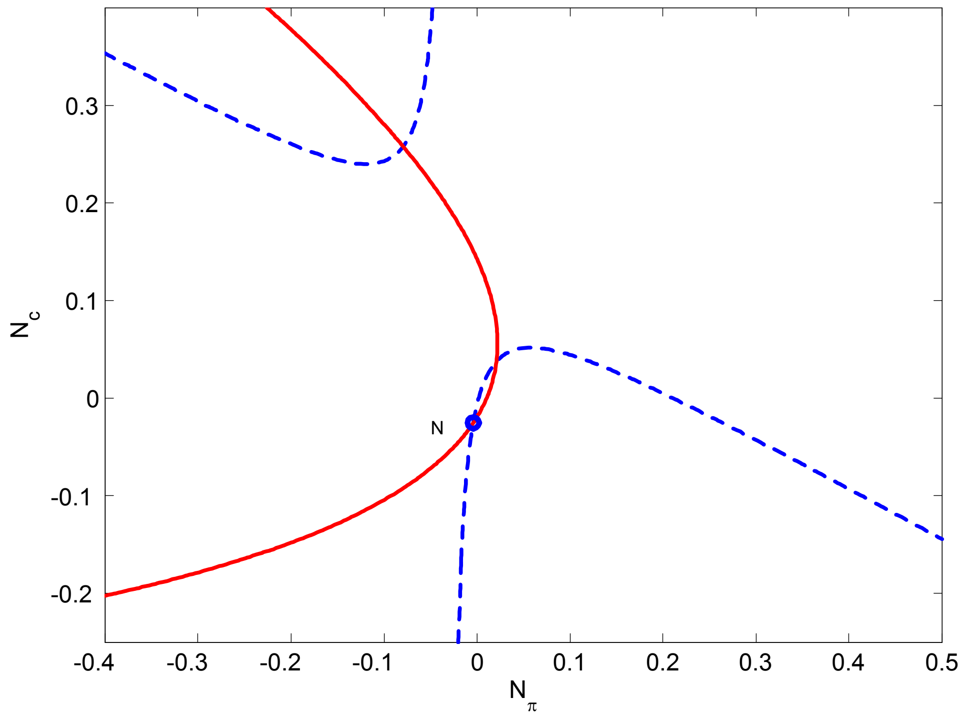

Figure 1: Reaction of the private sector $\{N_\pi, N_c\}$ given the fiscal feedback λ and policy F.

depicted by a solid line, at the three solution pairs. If $\lambda < \infty$ then matrix C is diagonalised and there is at most one stable solution. Remember, *if we solve this system iteratively*, then most iterative routines can only converge to a particular pair $\{N_\pi^*, N_c^*\}$. In Figure 1 this stable solution is labelled as N. Among the remaining two solutions one is neither stabilising nor stable and the other one is stabilising but not stable. The latter solution is the unstable private sector equilibrium. If there is only one eigenvalue of matrix C which is inside the unit circle (as we have only one predetermined variable, debt), then the pair $\{N_\pi^*, N_c^*\}$ can also be obtained via Blanchard and Kahn (1980) decomposition of matrix C. However, this is not the case which is plotted, as our calibration leads to two stable eigenvalues and consequently two stabilising solutions of (33) and (34).

Note that if we vary F but condition on a given fiscal feedback λ we can obtain a curve $\{N_\pi^*(F), N_c^*(F)\}$ of the stable private sector's reactions to policy F. We will use this later.

4.5 Multiplicity of fixed point solutions

4.5.1 Multiple solutions for a given λ

We have derived the system of four equations: (33)-(34), (29) and (30). This system can be interpreted as equation (29) on welfare S given constraints, that are described by equations (33)-

(34) and (30). Alternatively it can be interpreted as the system describing the two reactions: (33)-(34) for N and (30) for F where, in the latter case, there is a constraint on parameter S that is given by equation (29). For our purposes it is convenient to use the second interpretation.

The unknowns are N_π, N_c, S, and F, as J and K depend on N_π, N_c, and F. We can solve this system recursively. We start with initial guess $F^{(0)}$, then solve system (33)–(34) to find $N_\pi^{(0)}$ and $N_c^{(0)}$, then solve the quadratic equation (29) for $S^{(0)}$, finally updating $F^{(1)}$ from (30). For every equilibrium response F^* there is only one stable pair $\{N_\pi^*, N_c^*\}$, but for different F^{*i} and F^{*j} we can have different corresponding $\{N_\pi^{*i}, N_c^{*i}\}$ and $\{N_\pi^{*j}, N_c^{*j}\}$, for $i \neq j$.

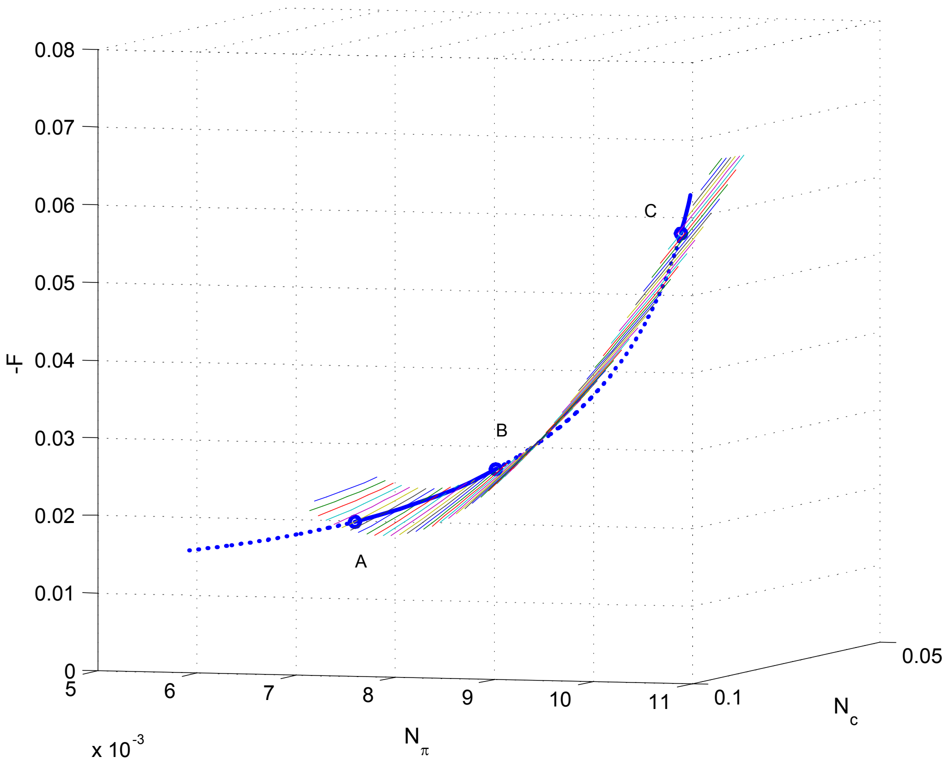

Figure 2: Multiple equilibria for a given λ.

Such multiple equilibria exist in our model and we can demonstrate them in a plot. For every reaction of the private sector $\{N_\pi, N_c\}$ we can plot policy reaction $-F(N_c, N_\pi)$ that is found using formula (30). This defines a two-dimensional surface in the three-dimensional space $\left\{-F(N_c, N_\pi),\ \{N_c, N_\pi\} \in \mathbb{R}^2\right\}$ which is the optimal reaction surface of monetary policy. Similarly, for every policy reaction F there is a private sector's optimal response $\{N_c(F), N_\pi(F)\}$. This response can be presented by a one-dimensional curve in the three-dimensional space

$\{F,\ N_c(F),\ N_\pi(F),\ F \in \mathbb{R}\}$. The points where the curve intersects the surface are the points of discretionary equilibria. We have plotted them in Figure 2 (where we show *minus* F in order to be consistent with all policy reactions presented further in the text). The solid line is the private sector's reaction function and the two-dimensional surface is the reaction surface of monetary policy. It can be seen that the line intersects the surface in three points A, B, and C.[16] We used a solid line to show the reaction curve of the private sector where it goes above the surface, and used a dotted line to show the reaction curve where it goes below the surface. Using the dynamic programming iterative algorithm, or iterating between the policymaker's and the private sector's reaction functions as explained above, we are only able to obtain points A and C. Solution F^B is a separation point of the two areas of attraction: if we initialize $F^{(0)} < F^B$ then the solution algorithm converges to equilibrium C, if $F^{(0)} > F^B$ then we obtain equilibrium A. This is because equilibria A and C are stable while the intermediate equilibrium B is unstable: we can only obtain it numerically by moving along the reaction curve with small increments and checking the equilibrium conditions after each increment.[17] For a higher dimensional case, this method would not work and only stable equilibria could be picked up by an iterative numerical procedure.

4.5.2 Changing fiscal feedback

Picture 2 was obtained for a particular value of λ. We can follow the same procedure as before and check numerically over a wide area for λ large or $\lambda = 0$ to see if there is a unique solution. However, it is difficult to prove uniqueness analytically even for our simple model. One obvious way of doing so could be the following. The solution is unique if the one-dimensional continuous curve that describes the private sector's reaction function passes only *once* through the two-dimensional surface that describes reaction function of the monetary policy maker. It can be formalized as follows.

The private sector's reaction function can be parametrically defined as a vector-function $[N_\pi(F), N_c(F), F]$ with velocity vector $V = \left[\frac{dN_\pi(F)}{dF}, \frac{dN_\pi(F)}{dF}, 1\right]$. The monetary policy reaction function is given by an implicit function (30) $H(J_\pi, J_c, K_\pi, K_c, F, S) = 0$ where $S = S(J_\pi, J_c, K_\pi, K_c)$ because of (29). (Of course, J_π, J_c, K_π and K_c are functions of N_π and N_c and therefore, $H(N_\pi, N_c, F) = 0$ is an implicit function that determines the surface.) Its vector gradient is $G = \left[\frac{\partial H}{\partial N_\pi}, \frac{\partial H}{\partial N_c}, \frac{\partial H}{\partial F}\right]$ and it is orthogonal to the surface.

We can use the following property of a scalar product between two vectors: its value is equal to the product of the two lengths of the vectors, times the cosine of the angle between them. Therefore, if the sign of the scalar product of the two vectors, $(V, G) = \frac{\partial H}{\partial N_\pi}\frac{dN_\pi}{dF} + \frac{\partial H}{\partial N_c}\frac{dN_\pi}{dF} + \frac{\partial H}{\partial F}$, does not change then there can be only one point of intersection.[18] This condition is *sufficient*,

[16] Sufficient conditions in Garcia and Li (1980) suggest that we have no more than $2^2 3^2 = 36$ solutions. Of course, we have different restrictions that reduce the number of them: S must be positive, N must be stable, all solutions must be real; we nevertheless were not able to prove analytically that there are no other points of intersection. We have checked the much bigger area numerically and did not find any more solutions.

[17] Topologically similar cusp catastrophe equilibria were described in Zeeman (1974).

[18] This condition can be generalised for higher dimensional cases; however, given the complexity of even three-dimensional system, it is unlikely that it will be helpful in checking analytically the uniqueness of equilibrium in higher dimensions. Therefore, we do not pursue this line here. For two-dimensional cases it is always straightfor-

and we might hope to show that the sign does not change for the two values of λ, $\lambda = 0$ and $\lambda \gg 1$.

Although this condition is likely to be simple for small models with a unique equilibrium, even for our model it leads to intractable algebraic expressions for the coordinates of V and G, precisely because there can be multiple equilibria. These expressions are so complicated that they do not allow us to determine the sign of the scalar product even for $\lambda = 0$. For large λ they remain intractable; and taking limit as λ tends to infinity makes little economic sense as we need to assume $b_t = 0$ for $t = 0, 1, ..$, as discussed in Section 4.1. Formally, the point $\lambda = \infty$ is singular *for any* F, and matrix C cannot be diagonalised, so there is a continuum of solutions for N_π and N_c. This is also seen from the system of (33) and (34) which collapses into two identical linear equations that relate N_π and N_c.

However, it is obvious that under the balanced budget conditions there is a unique equilibrium: as debt remains at its equilibrium level and there are no shocks, then the system is always in the steady state, i.e. all log-linearised variables are zeros. We study asymptotic behaviour of policy parameters in the next section where we discuss the economics of observed multiplicity of equilibria in this model.

4.6 'Active' and 'Passive' Monetary Policy

Is there any economic intuition behind the two solutions of the model outlined above? Monetary policy has to deal with two problems. First, our model contains nominal rigidities. Second, we have explicitly introduced the government solvency constraint. In dynamic equilibrium, debt is supposed to follow a non-explosive process. As our model is entirely deterministic, the initial state for debt is the only source of disequilibrium. We might expect that if debt is tightly controlled (i.e. λ is reasonably large), then any initial debt disequilibrium is short-lived and monetary policy can control inflation and output in a conventional way as if there are no solvency problems. For example, if the level of debt is higher than in the steady state, this will necessarily result in lower output and negative inflation, so the interest rate should be reduced to stimulate economic activity. If fiscal policy does not control debt at all, then monetary policy must reduce interest rate in response to positive initial debt disequilibrium in order to prevent explosion of debt. However, for the intermediate fiscal regime where debt is only weakly controlled, both these policies can be consistent with the long run equilibrium in which all variables converge to their steady states. The existence of multiple discretionary solutions in our case depends on the possibility of forming varying but consistent beliefs by both players about the future course of monetary stabilisation. The existence of two fiscal regimes and the complementarity of the monetary and fiscal instruments in the stabilisation of debt is the mechanism that provides this possibility in the model.

To explore this further, let us look at the two solutions depicted in Figure 2 by points A and C, and let us describe policy at these points as 'Plan A' and 'Plan C'. These optimal plans can also be written as two reaction functions: policy reaction function and the private sector's reaction function.[19] We present these reaction functions in Table 1.

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ward to see whether there is a unique solution from the form of (DARE), (POLICY) and (NCARE).

[19] We present solutions for the same value of $\lambda = 0.108$.

Table 1: Equilibrium Reactions of the Monetary Policymaker and the Private Sector in the form (31) and (32)

	Monetary Policymaker	Private Sector
Plan A :	$[i_t] = -[0.01][b_t]$	$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = -\begin{bmatrix} -0.005 \\ -0.03 \end{bmatrix} [b_t]$
Plan C:	$[i_t] = -[0.06][b_t]$	$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = -\begin{bmatrix} -0.010 \\ -0.08 \end{bmatrix} [b_t]$

Under Plan A the interest rate reacts weakly to any debt disequilibrium. Under Plan C, the interest rate reacts stronger and controls debt tighter. Figure 3 suggests that only Plan C survives for small λ, in particular for $\lambda = 0$. If λ is sufficiently large, then only Plan A survives. In the latter case, the optimal policy reaction F and household reactions N_π and N_c will grow in line with λ, as monetary policy has to offset the excessive tightness of fiscal control. From the form of the debt accumulation equation it is clearly seen that if the fiscal feedback is large and growing, then at least one of F, N_c and N_π must be close to being linear in λ, otherwise the debt may grow without bounds. Optimally, these feedbacks will follow λ as it helps to stabilise other variables like output: if the fiscal feedback on debt is too strong so the government expenditures are below the equilibrium, then optimal monetary policy will stimulate private consumption to offset the effect on output of lower public spending. Such reaction is associated with positive N_c that is near-linear in λ. But if output is controlled, then under this optimal policy the response of inflation to debt should also be near-linear in λ, otherwise the effect of private consumption on price setting in the Phillips curve cannot be levelled. We can clearly observe this near-linear dependence of F, N_c and N_π on λ if λ becomes large, perhaps unrealistically large.

For realistic values of λ the difference in response to a debt disequilibrium in the two regimes reflects the distinction between 'active' and 'passive' regimes identified in Leeper (1991) and Leith and Wren-Lewis (2000). Under the active regime monetary policy controls debt as weakly as possible so that this task does not prevent it eliminating the resulting inflation and output disequilibrium. Under the passive regime, monetary policy controls debt tightly in order to ensure that it will converge back to equilibrium. In this regime any inflation disequilibrium is accommodated as it actually helps to reduce real debt. If we introduce cost-push shocks (details of such model can be found in the working paper version of this paper, Blake and Kirsanova (2006)) then we can clearly see that the monetary feedback on cost push shocks under Plan C is much smaller than under Plan A. In other words, when debt is not controlled — therefore under 'passive' monetary policy – all cost push shocks are accommodated as they help to reduce debt; with high fiscal feedback so that domestic debt is well controlled by fiscal policy, then conventional – 'active' – monetary policy, which reacts to cost push shocks by raising interest rates, becomes possible.

We therefore have two different monetary policy regimes for the two types of fiscal policy: tough on debt ('passive') and too weak on debt ('active'). Both players are aware of these

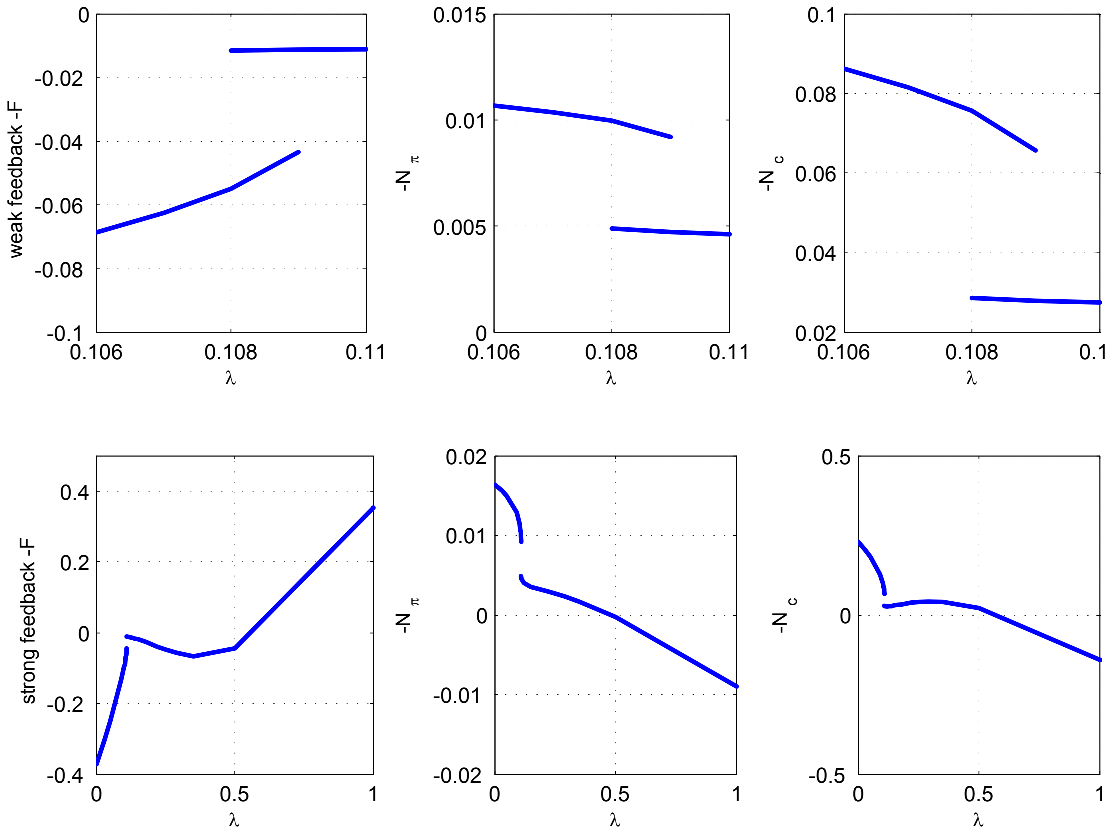

Figure 3: Feedback parameters as a function of fiscal feedback λ.

regimes. If fiscal policy only weakly controls debt, then the policymaker and the private sector *both* perceive both possible regimes and set *consistent* beliefs that one of them prevails. There is no coordination problem because the policymaker always leads in a linear-quadratic model of discretionary policy.[20] The policymaker is able to manipulate the private sector's actions and (rational) beliefs. Given the strength of fiscal feedback and an initial positive debt disequilibrium, if the policymaker decides that fiscal policy is active then a fall in the interest rate will be validated by the belief by the private sector that fiscal policy is active. King (2006) discuss equilibrium selection in a similar setting.

Finally, Table 2 reports the welfare loss for selected values of fiscal feedback. Note that although the model is deterministic any solution is certainty equivalent for additive Gaussian shocks, i.e. we have found a policy which will deliver the best result 'on average' (see Currie and Levine (1985) for discussion). The cost along any particular path can be computed using

[20] See Cohen and Michel (1988), sections 4 and 5.

Table 2: Equilibrium Monetary Policy Reaction Function

Fiscal	feedback:	λ	0.00	0.100	0.106	$0.108^{\dagger}$	$0.109^{\ddagger}$	0.110	0.500	4.000
Passive	feedback:	$-F$	-0.37	-0.10	-0.07	-0.06	-0.04	–	–	–
Monetary Policy	loss:	$L_{b_0}^{\Diamond}$	1.027	0.464	0.398	0.361	0.326	–	–	–
Active	feedback:	$-F$	–	–	–	-0.01	-0.01	-0.01	-0.04	3.58
Monetary Policy	loss:	$L_{b_0}^{\Diamond}$	–	–	–	0.171	0.167	0.165	0.878	101.6

Notes

† – approximate lower bound of multiplicity area;

‡ – approximate lower bound of multiplicity area;

$^{\Diamond}$ – all numbers should be multiplied by 10^{-5}.

the standard formula (Currie and Levine (1993) or Backus and Driffill (1986)) which requires knowledge of initial conditions for state variables, b_0. We compute the loss L_{b_0} assuming $b_0 = 0.01$ and present it as percent reduction in steady-state consumption under the benchmark policy that makes households as well off as under our optimal policy. Under this benchmark policy the economy is permanently in the steady state at the steady state level of consumption of $(1+\Omega)C$. We determine the percentage change in steady state consumption Ω such that we have the same level of welfare under both policies and report $L_{b_0} = 100 \times \Omega$. Although the loss appears to be small, one should remember that this is a deterministic model, where we only have an *initial* disequilibrium and the cost of stochastic volatility, which can be substantial, is ignored.

The results suggest that if there is a belief common to all agents that debt is not sufficiently stabilised by fiscal policy and so monetary policy has to intervene, then such a policy severely harms welfare. However, if there are multiple equilibria then the monetary policymaker can always choose the welfare-maximising equilibrium. This is because a dynamic linear-quadratic model of discretionary policy describes a dynamic game in which the policymaker leads. The policymaker, therefore, faces no conflict in the choice of equilibrium: he can always choose the best outcome and cannot be trapped by the private sector as in Albanesi et al. (2003). Of course, it remains crucial to recognize the possibility of multiple solutions. A false belief by a policymaker that there is a unique solution might well lead to the adoption of a welfare-inferior policy.

The result on multiplicity does not depend on whether debt (or some part of it) is indexed. Indexing debt changes the rate of return on financial wealth, but it does not eliminate potential insolvency problems. When debt is fully indexed then the *ex post* real interest rates enjoyed by holders of financial assets is equivalent to the *ex ante* real interest rate. When debt is instead nominal then surprise inflation due to initial disequilibrium can erode the real value of debt by decreasing the *ex post* real interest rate relative to the *ex ante* one. Although it is possible for *ex ante* real rates to differ from *ex post* rates at the initial moment, at other periods the economy operates under perfect foresight, so the overall quantitative effect is relatively small.[21]

[21] See Leith and Wren-Lewis (2007) who examine the issue in details.

5 Summary and Implications

The main contribution of this paper is to show that linear quadratic RE models of discretionary policy not only *can* have multiple equilibria, but in some situations *do*. This result means that papers applying these methods may not have discovered all of the equilibria to their problem and that researchers should check that they have indeed isolated either the unique or the welfare maximizing equilibrium for their problem. We demonstrated one way this can be done for simple models.

We have examined the properties of discretionary equilibria in general mathematical terms. We have shown that the theory does not rule out *isolated* (or distinct) multiple discretionary equilibria. The theory, however, rules out a continuum of discretionary equilibria, similar to those that are frequently mentioned in the literature on simple rules. It also rules out multiple *stable private sector* equilibria similar to those discussed in King and Wolman (2004). In our New Keynesian example we do indeed find a multiplicity of solutions, and illustrate the welfare consequences of adopting an inferior equilibrium.

In effect, a discretionary equilibrium is a leadership game in which the policymaker leads. Although the private sector is atomistic, and not assumed to set expectations strategically, in any resulting equilibrium the collective actions of private agents have a strategic effect on the policymaker's choice of policy. This result allows us to conclude that in linear-quadratic models of discretionary policy, the policymaker can always choose the equilibrium with highest welfare. It cannot be trapped by the private sector, unlike in discretionary games with a different leadership structure, see Albanesi et al. (2003).

The example we present highlights one reason why multiple equilibria might exist. The crucial features of our model are (i) the intertemporal policy is time consistent, (ii) the private sector has rational expectations, (iii) the monetary policymaker (potentially) has to deal with two tasks and (iv) monetary and fiscal policies complement each other in dealing with one of these tasks. The first two features are generic for any LQ model of discretionary policy, and they *generate the possibility* of multiple discretionary equilibria. The last two features are model-specific, and they *lead* to multiple equilibria. For certain parameterizations there are two solution paths, each of which is associated with a consistent belief about whether one of these tasks is adequately dealt with by fiscal policy or not. Along each of these solution paths the policymaker focuses on one of these tasks at the expense of the other, and along each of them it is commonly known that the policymaker is predominantly occupied with this task. Although the policymaker can choose the welfare maximizing path (and the private sector will necessarily follow) it is imperative that the policymaker is aware of the existence of both.

A Solutions of (DARE)

This appendix states conditions under which a solution to (DARE) exists and is unique.

We obtained (DARE) from the first order conditions to the optimisation problem, assuming that we know the response of the private sector N. By substituting out the rational response of the private sector, we reduce our problem to a well-known engineering one and simply apply that analysis. Explicitly, solving the following *standard* optimal control problem is equivalent to solving (DARE) and (POLICY). The problem is to stabilise the following linear system, given the initial conditions:

$$w_{t+1} = \beta^{\frac{1}{2}} \left(A^* w_t - B^* v_t \right), \quad w_0 = \bar{w} \tag{35}$$

and given the positive semi-definite cost functional:

$$J\left(w_0, v\right) = \sum_{t=0}^{\infty} \left[\begin{array}{cc} w_t' & v_t' \end{array} \right] \left[\begin{array}{cc} Q^* & P^* \\ P^{*\prime} & R^* \end{array} \right] \left[\begin{array}{c} w_t \\ v_t \end{array} \right]. \tag{36}$$

It is well known (Kwakernaak and Sivan (1972, Ch. 6)) that if the pair matrices $(\beta^{\frac{1}{2}} A^*, \beta^{\frac{1}{2}} B^*)$ in (35) is controllable (i.e. if the $k \times n_1$ controllability matrix $[\beta^{\frac{1}{2}} B^*, \beta^{\frac{2}{2}} A^* B^*,\ \beta^{\frac{3}{2}} A^{*2} B^*,\ ...,\ \beta^{\frac{n_1}{2}} A^{*n_1 - 1} B^*]$ has rank n_1 or full row rank) then there exists a unique stabilising optimal control v_t that minimizes (36). The system under this control evolves following:

$$w_{t+1} = \beta^{\frac{1}{2}} \left(A^* - B^* (R^* + \beta B^{*\prime} S B^*)^{-1} \left(P^{*\prime} + \beta B^{*\prime} S A^* \right) \right) w_t = \beta^{\frac{1}{2}} \Omega w_t$$

and all eigenvalues of matrix Ω are strictly inside the unit circle. Obviously, the controllability of $(\beta^{\frac{1}{2}} A^*, \beta^{\frac{1}{2}} B^*)$ is equivalent to the controllability of (A^*, B^*). It follows, therefore, that *the solution pair $\{S, F\}$ to (DARE) and (POLICY) exists and unique if (A^*, B^*) is controllable.*

B Solutions of (NCARE)

This appendix discuss existence and uniqueness of a solution to (NCARE).

Existence of solution of (NCARE). How does the general solution to (NCARE) look like? Suppose matrix C in equation (17) can be diagonalised as[22] $C = M^{-1} \Lambda M$. Matrix M is the matrix of left eigenvectors which correspond to eigenvalues Λ. Arrange the eigenvalues so that Λ_u is a diagonal matrix of size n_2 and Λ_s a diagonal matrix of size $n_1 = n - n_2$. Rearrange similarly M and partition it to give

$$\Lambda = \left[\begin{array}{cc} \Lambda_s & 0 \\ 0 & \Lambda_u \end{array} \right] \text{ and } M = \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right].$$

Now, construct:

$$N = M_{22}^{-1} M_{21}. \tag{37}$$

Matrix M_{22} needs to be invertible and this is the condition mentioned in footnote 9 in Section 3.3. It has been shown in the literature (see e.g. Medanic (1982) Theorem 1.) that *any* solution of (NCARE) can be represented in this way for some adequate Jordan basis of C. *If all eigenvalues of C are simple then there are at most $\binom{n}{n_2}$ of different solutions N.* Note that we did not make any assumptions about matrices Λ_s and Λ_u apart from assuming that they are of particular size.

[22] In what follows we always assume that matrix C is *simple*, i.e. all its eigenvalues are of geometric multiplicity one, and the column rank of M is equal to n. This case is of practical interest; but Freiling (2002) discusses implications of higher geometric multiplicity.

Uniqueness of a stable solution to (NCARE). Can we choose a particular matrix N? Suppose Λ_u collects all eigenvalues that are greater than one in modulus and suppose there are $m \leq n_2$ of them. Clearly, if $x_t = -M_{22}^{-1}M_{21}y_t$ then system (17) collapses to:

$$y_{t+1} = \left(C_{11} - C_{12}M_{22}^{-1}M_{21}\right) y_t \tag{38}$$

and it is easy to show (see e.g. Freiling (2002)) that eigenvalues of $C_{11} - C_{12}M_{22}^{-1}M_{21}$ are those of Λ_s. By construction, they all are not greater than unity and the system is non-explosive. Given y_0 we completely determine the path $y_t, t > 0$. It is clear that if $m = n_2$ then Λ_u is uniquely determined and (37) is a unique solution (this is the Blanchard and Kahn (1980) solution). If $m < n_2$ we can construct Λ_u in several ways, and so there are several solutions to (17).

Definition 1 *We call N stable if it an asymptotically stable steady state of the following differential Riccati equation (DRE): $dN(s)/ds = N(s)C_{11} + C_{21} - N(s)C_{12}N(s) - C_{22}N(s)$.*

It was proved in Medanic (1982), Theorem 5, that, under certain conditions,[23] *there is only one stable solution among solutions to (NCARE).* This is the solution to which most *iterative* methods of solution of (NCARE) will converge, see Fital and Guo (2006). (Most iterative methods, including the Oudiz and Sachs (1985) method, exploit asymptotic stability of the unique stable solution to DRE.) When $m = n_2$ then the solution that can be obtained by the Blanchard and Kahn (1980) decomposition of matrix C can also be obtained by an iterative method. If $m > n_2$ then Λ_s will always contain at least one explosive eigenvalue. Equation (38) will describe an explosive path of state variables in this case.

C Calibration of the Model

One period is taken as equal to one quarter of a year. All behavioural parameters below are taken from Rotemberg and Woodford (1997). A more detailed derivation is given in Blake and Kirsanova (2006).

Assumed parameters			
$\beta = 0.99$	Household discount rate	$\psi = 2.0$	Labour elasticity
$\sigma = 0.5$	Intertemporal elasticity of substitution	$\frac{B}{Y} = 0.1$	Steady state level of debt and assets
$\varepsilon = 5.0$	Elasticity of substitution between domestic goods	$\theta = \frac{C}{Y} = 0.75$	Steady state consumption to output ratio
$\gamma = 0.75$	Probability of that price remains unchanged		
Derived parameters			
$\kappa_c = (1-\gamma\beta)(1-\gamma)\psi/\left(\gamma\sigma\left(\psi+\epsilon\right)\right)$		$a_c = \psi(1-\gamma\beta)\left(1-\gamma\right)\theta/\left(\left(\epsilon+\psi\right)\gamma\epsilon\sigma\right)$	
$\kappa_y = (1-\gamma\beta)(1-\gamma)/\left(\gamma\left(\psi+\epsilon\right)\right)$		$a_g = \psi(1-\gamma\beta)\left(1-\gamma\right)\left(1-\theta\right)/\left(\left(\epsilon+\psi\right)\gamma\epsilon\sigma\right)$	
		$a_y = \psi(1-\gamma\beta)\left(1-\gamma\right)/\left(\left(\epsilon+\psi\right)\gamma\epsilon\psi\right)$	

[23] The condition is that C is *dichotomically separable*, i.e. there exist n_2 eigenvalues of C such that $Re(\lambda_i) > Re(\lambda_j)$, $i = 1, ..., n_2$, $i = n_2 + 1, ..., n_2 + n_1$. For example, if $n_2 = 3$ and the four biggest eigenvalues form two complex pairs then C is not dichotomically separable.

The tax rate, τ, is found from the equilibrium condition: $\frac{B}{Y} = \frac{1}{\beta}\left(\frac{B}{Y} + (1-\theta) - \tau\right)$. When computing losses we assume that in the initial moment debt is 1% higher than it is in the steady state.

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