Fiscal (In)Solvency, Discretionary Monetary Policy and Multiple Equilibria in a New Keynesian Model

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Introduction

- Discretionary policy can generate multiple equilibria: Future policy responds to a state that is at least partly determined by forecasts of the future policy.
- Multiplicity is known for some types of models: trigger strategies (Rogoff 1987), policy traps (Albanesi et al 2003), private sector equilibria (King and Wolman 2004)
- It has never been shown in LQ models, that are frequently used in monetary policy.
- Most commonly algorithm to find the solution: Oudiz & Sachs (1985). But *any* iterative algorithm delivers only one solution.
- Theoretical literature does not rule out multiple equilibria, but no formal discussion of the multiplicity.

In this paper we

- Demostrate that in *some* cases there *are* multiple equilibria
- Discuss general properties of discretionary equilibria

Evolution of Economy

... is explained by the following system:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [u_t] + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

where y_t is an n_1-vector of predetermined variables with initial conditions y_0 given, x_t is n_2-vector of non-predetermined (or jump) variables, u_t is a $k-$vector of policy instruments of the policymaker, and ε_{t+1} is vector of innovations to predetermined variables with covariance matrix Σ. For notational convenience we define the n-vector $z_t = (y_t', x_t')'$ where $n = n_1 + n_2$.

Policymaker's Objective

$$W_t = \frac{1}{2}\mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t}(g_s' Q g_s) = \frac{1}{2}\mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t}(z_s' \mathcal{Q} z_s + 2 z_s' \mathcal{P} u_s + u_s' \mathcal{R} u_s).$$

The vectors g_s is the goal variables of the policymaker, $g_s = \mathcal{C}(z_s', u_s')'$. The loss function can include instrument costs, but no assumptions of invertibility of $\mathcal{R}$ are made.

Evolution of Economy under Control

Suppose we know that solution is a pair of linear rules

$$\begin{aligned} u_t &= -Fy_t \\ x_t &= -Ny_t. \end{aligned}$$

Then, the evolution of the economy under control can be summarised by the equation

$$\begin{aligned} \begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} &= \begin{bmatrix} A_{11} - B_1 F & A_{12} \\ A_{21} - B_2 F & A_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}. \end{aligned}$$

Bellman Equation

A value function is quadratic in the state variables:

$$W_t = \frac{1}{2} y_t' S y_t,$$

And it satisfies the Bellman Equation:

$$y_t' S y_t = \min_{u_t} \left(z_t' \mathcal{Q} z_t + 2 z_t' \mathcal{P} u_t + u_t' \mathcal{R} u_t \right) + \beta \mathcal{E}_t \left(y_{t+1}' S y_{t+1} \right).$$

Iterative Algorithm

The fixed point operator can be written as:

$$S^{(i+1)} = T_0\left(F^{(i)}, N^{(i)}\right) + \beta T_1\left(F^{(i)}, N^{(i)}\right)' S^{(i)} T_1\left(F^{(i)}, N^{(i)}\right)$$

$$N^{(i+1)} = (A_{22} + N^{(i)} A_{12})^{-1}\left(A_{21} + N^{(i)} A_{11} - (B_2 + N^{(i)} B_1) F^{(i)}\right)$$

where

$$\begin{aligned} F^{(i)} &= \left(R^*\left(N^{(i)}\right) + \beta B_K\left(N^{(i)}\right)' S^{(i)} B_K\left(N^{(i)}\right)\right)^{-1}\left(P^*\left(N^{(i)}\right)'\right. \\ &\quad \left. + \beta B_K\left(N^{(i)}\right)' S^{(i)} A_J\left(N^{(i)}\right)\right) = F^{(i)}\left(S^{(i)}, N^{(i)}\right). \end{aligned}$$

Sentiments

Oudiz & Sachs (1985), p. 311:

> *"We do not know of any general result concerning the convergence of this process. However in our empirical applications we have not run into major problems. Cohen & Michel(1984) show that in a one dimensional case this kind of a recursion does have a fixed point."*

Of course, convergence of the algorithm neither guarantees a uniqueness of the solution, nor implies any particular properties of the resulting equilibrium. Oudiz & Sachs (1985), p. 288:

> *"Although we cannot prove that the resulting function is the unique memoryless, time-consistent equilibrium, we suspect that it is in fact unique, in view of the linear-quadratic structure of the underlying problem."*

First Order Conditions Revisited

The FOCs constitute the system of three matrix equations. Two of them describe the reaction function of the policymaker:

$$
\begin{aligned}
S &= Q^* + \beta A_J' S A_J - \left(P^{*\prime} + \beta B_K' S A_J\right)' \left(R^* + \beta B_K' S B_K\right)^{-1} \\
&\quad \times \left(P^{*\prime} + \beta B_K' S A_J\right) && \text{(DARE)} \\
F &= \left(R^* + \beta B_K' S B_K\right)^{-1} \left(P^{*\prime} + \beta B_K' S A_J\right) && \text{(POLICY)}
\end{aligned}
$$

and one describes the response of the private sector:

$$
N C_{11} + C_{21} - N C_{12} N - C_{22} N = 0. \qquad \text{(NCARE)}
$$

Obviously, the system is highly non-linear and can have many solutions of different types.

Policymaker's Reaction Function: existence and uniqueness

For given N, if (A^*, B^*) is controllable then

- Symmetric S is unique
- F is stabilising, i.e. all eigenvalues of $A^* - B^* F$ are strictly inside the unit circle

As we 'substituted out' the rational response of the private sector, the problem is essentially an engineering one: find optimal control of a dynamic system with initial conditions. Nothing new to discuss.

Household's Reaction Function: existence and uniqueness

We need to find rational expectations reaction subject to time-consistency requirement If C can be diagonalised then one of the three situations is possible

- If the number of jump variables = the number of explosive eigenvalues then there is a unique solution that can be found using Blanchard and Kahn formula. This solution N is stabilising i.e. all eigenvalues of $C_{11} - C_{12}F$ are strictly inside the unit circle.
- If the number of jump variables $<$ the number of explosive eigenvalues then there are no stabilising solutions.
- If the number of jump variables $>$ the number of explosive eigenvalues then there is unique *stable* solution to CARE, this solution is stabilising. *Most* iterative procedures of solving CARE will converge to this solution.

In the third case there *are* multiple private sector equilibria, but only one of them is stable. *This* solution is picked up by the Oudiz and Sachs procedure.

Iterations

Start with $F^{(i)}$. Solve NCARE for unique stable $N^{(i)}$, solve DARE for $S^{(i)}$, update $F^{(i+1)}$. Do until $\max\left(\left\|F^{(i)} - F^{(i+1)}\right\|\right) < \delta$ where δ is given tolerance. This locates a fixed point.

This method is different from the Oudiz and Sachs algorithm in that

- we iterate in policy space (N, F) instead of (N, S)
- we need to initialise F only. Intuition might help to initialise F.

Log-Linearised System

Evolution of the economy is described by:

$$
\begin{aligned}
c_t &= c_{t+1} - \sigma(i_t - \pi_{t+1}), \\
\pi_t &= \beta\pi_{t+1} + \frac{(1-\gamma\beta)(1-\gamma)\psi}{\gamma(\psi+\epsilon)}\left(\frac{1}{\sigma}c_t + \frac{1}{\psi}y_t\right) + \eta_t, \\
y_t &= (1-\theta)g_t + \theta c_t, \\
b_{t+1} &= i_t + \frac{1}{\beta}\left(b_t - \pi_t + \left(\frac{\theta}{B} - (1-\beta)\right)y_t - \frac{\theta}{B}c_t\right), \\
g_t &= -\lambda b_t,
\end{aligned}
$$

It is described by Phillips curve, IS curve, national income identity, and budget constraint. The role of fiscal policy is limited to control of domestic debt: fiscal authority is not a strategic player.

Monetary Policy's Objective

...is standard: choose interest rate to minimise loss

$$\frac{1}{2}\mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{\psi(1-\gamma\beta)(1-\gamma)}{\epsilon(\epsilon+\psi)\gamma} \left(\frac{\theta}{\sigma} c_t^2 + \frac{(1-\theta)}{\sigma} g_t^2 + \frac{1}{\psi} y_t^2 \right) + \pi_t^2 \right)$$

It can be shown that this form reflects the private sector's preferences.

Linear Form of Solution

Suppose that the policymaker operates with a linear rule

$$[i_t] = -[F][b_t]$$

then the household reaction function will necesserily have a linear form of

$$\begin{bmatrix} \pi_t \\ c_t \end{bmatrix} = -\begin{bmatrix} N_\pi \\ N_c \end{bmatrix} [b_t]$$

Policymaker's reaction Function

DARE is a quadratic equation for a scalar variable S and can be written as

$$\beta B_K^2 S^2 + \left(R^* - \beta\left(Q^* B_K^2 - 2P^* B_K A_J + R^* A_J^2\right)\right) S + \left(P^{*2} - Q^* R^*\right) = 0.$$

This equation has two eigenvalues and the product of these eigenvalues is equal to:

$$\begin{aligned} P^{*2} - Q^* R^* \;=\; & -\Big(\left(a_c a_g + a_y a_c (1-\theta)^2 + \theta^2 a_g a_y\right) \lambda^2 K_c^2 \\ & + a_c a_\pi (K_\pi J_c - J_\pi K_c)^2 + a_g a_\pi \lambda^2 K_\pi^2 \\ & + a_y a_\pi \left(\lambda (1-\theta) K_\pi + \theta (K_\pi J_c - J_\pi K_c)\right)^2\Big) < 0. \end{aligned}$$

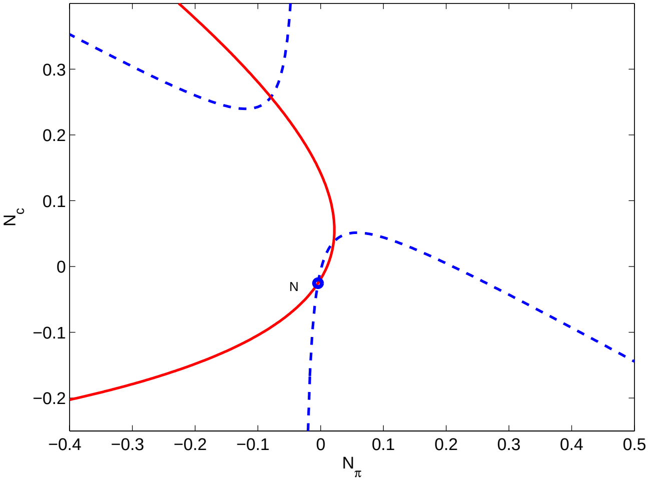
Positive solution for value function is unique.

Household's Reaction Function

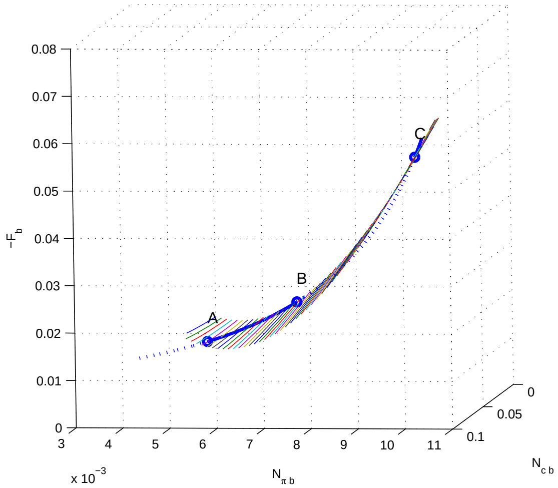
It solves NCARE, which can be written as a system of two quadratic equations in N_π and N_c:

$$
\begin{aligned}
&N_\pi^2 + \frac{\theta\tau}{B} N_\pi N_c - N_\pi \left(\frac{(1-\theta)(1-\tau)}{B}\lambda + \beta F \right) \\
&\quad + (\kappa_c + \theta\kappa_y) N_c + \lambda\kappa_y (1-\theta) = 0, \\
&N_c^2 + \frac{B}{\theta\tau} \left(1 - \frac{(1-\theta)(1-\tau)}{B}\lambda - \beta \left(1 + \frac{\sigma(\kappa_c + \theta\kappa_y)}{\beta} + F \right) \right) N_c \\
&\quad + \frac{B}{\theta\tau} N_c N_\pi + \frac{B\sigma}{\theta\tau} N_\pi - \frac{B\sigma}{\theta\tau} ((1-\theta)\lambda\kappa_y + \beta F) = 0.
\end{aligned}
$$

Household's Reaction Function



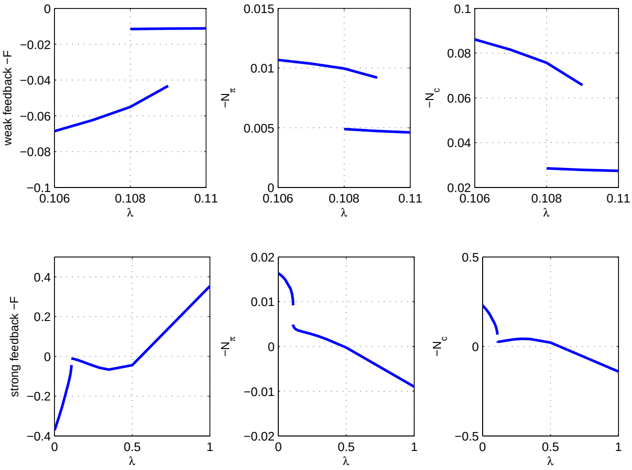
Policy Reactions



Active and Passive Monetary Policy

	Monetary Policymaker	Private Sector
Plan A :	$i_t = -0.01 b_t$	$\pi_t = 0.005 b_t$ $c_t = 0.03 b_t$
Plan C:	$i_t = -0.06 b_t$	$\pi_t = 0.010 b_t$ $c_t = 0.08 b_t$

Full range of fiscal feedback



Welfare

λ			0.00	0.100	0.106	0.108†
Passive	feedback	$-F$	-0.37	-0.10	-0.07	-0.06
	loss	L_{b_0}	1.027	0.464	0.398	0.361
Active	feedback	$-F$	–	–	–	-0.01
	loss	L_{b_0}	–	–	–	0.171
λ			0.109‡	0.110	0.500	4.000
Passive	feedback	$-F$	-0.04	–	–	–
	loss	L_{b_0}	0.326	–	–	–
Active	feedback	$-F$	-0.01	-0.01	-0.04	3.58
	loss	L_{b_0}	0.167	0.165	0.878	101.6

Conclusions

- Linear-Quadratic models *can* generate multiple equilibria under discretionary policy and in some cases *do*. Researchers need to make sure that they found all of them or that they found the welfare-maximising.
- In the model with fiscal policy and debt we demonstrate the existence of 'active' and 'passive' monetary policy for a weak fiscal control of debt.
- Discussed properties of multiple equilibria in general mathematical terms
    - Equilibria are distinct. Ruled out 'indeterminacy'.
    - Demonstrated slightly different way of computing discretionary equilibria.
    - Leadership structure ensures that the policymaker cannot be trapped by the private sector. The policymaker can always choose the best equilibrium. *But the policymaker needs to know that different equilibria can exist.*