Fiscal (In)Solvency, Discretionary Monetary Policy and Multiple Equilibria in a New Keynesian Model

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Introduction

Discretionary policy can generate multiple equilibria: Future policy responds to a state that is at least partly determined by forecasts of the future policy.

Multiplicity is known for some types of models: trigger strategies (Rogoff 1987), policy traps (Albanesi et al 2003), private sector equilibria (King and Wolman 2004)

It has never been shown in LQ models, that are frequently used in monetary policy.

Most commonly algorithm to find the solution: Oudiz & Sachs (1985). But any iterative algorithm delivers only one solution.

Theoretical literature does not rule out multiple equilibria, but no formal discussion of the multiplicity.

In this paper we

- Demonstrate that in some cases there are multiple equilibria
- Discuss general properties of discretionary equilibria
Evolution of Economy

... is explained by the following system:

\[
\begin{bmatrix}
y_{t+1} \\
x_{t+1}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
y_t \\
x_t
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}
\begin{bmatrix}
u_t
\end{bmatrix}
+ \begin{bmatrix}
\varepsilon_{t+1} \\
0
\end{bmatrix}
\]

where \( y_t \) is an \( n_1 \)-vector of predetermined variables with initial conditions \( y_0 \) given, \( x_t \) is \( n_2 \)-vector of non-predetermined (or jump) variables, \( u_t \) is a \( k \)-vector of policy instruments of the policymaker, and \( \varepsilon_{t+1} \) is vector of innovations to predetermined variables with covariance matrix \( \Sigma \). For notational convenience we define the \( n \)-vector \( z_t = (y_t', x_t')' \) where \( n = n_1 + n_2 \).
Policymaker’s Objective

\[ W_t = \frac{1}{2} \mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t} (g_s' Q g_s) = \frac{1}{2} \mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t} (z_s' Q z_s + 2z_s' \mathcal{P} u_s + u_s' \mathcal{R} u_s). \]

The vectors \( g_s \) is the goal variables of the policymaker, \( g_s = C(z_s', u_s')' \). The loss function can include instrument costs, but no assumptions of invertibility of \( \mathcal{R} \) are made.
Evolution of Economy under Control

Suppose we know that solution is a pair of linear rules

\[
\begin{align*}
u_t &= -Fy_t \\
x_t &= -Ny_t.
\end{align*}
\]

Then, the evolution of the economy under control can be summarised by the equation

\[
\begin{bmatrix}
y_{t+1} \\
x_{t+1}
\end{bmatrix} = \begin{bmatrix}
A_{11} - B_1 F & A_{12} \\
A_{21} - B_2 F & A_{22}
\end{bmatrix} \begin{bmatrix}
y_t \\
x_t
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{t+1} \\
0
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix} \begin{bmatrix}
y_t \\
x_t
\end{bmatrix} + \begin{bmatrix}
\varepsilon_{t+1} \\
0
\end{bmatrix}.
\]
Bellman Equation

A value function is quadratic in the state variables:

\[ W_t = \frac{1}{2} y_t' S y_t, \]

And it satisfies the Bellman Equation:

\[ y_t' S y_t = \min_{u_t} (z_t' Q z_t + 2 z_t' P u_t + u_t' R u_t) + \beta E_t (y_{t+1}' S y_{t+1}). \]
Iterative Algorithm

The fixed point operator can be written as:

\[ S^{(i+1)} = T_0 \left( F^{(i)}, N^{(i)} \right) + \beta T_1 \left( F^{(i)}, N^{(i)} \right)' S^{(i)} T_1 \left( F^{(i)}, N^{(i)} \right) \]

\[ N^{(i+1)} = (A_{22} + N^{(i)} A_{12})^{-1} \left( A_{21} + N^{(i)} A_{11} - (B_2 + N^{(i)} B_1) F^{(i)} \right) \]

where

\[ F^{(i)} = \left( R^* \left( N^{(i)} \right) + \beta B_K \left( N^{(i)} \right)' S^{(i)} B_K \left( N^{(i)} \right) \right)^{-1} \left( P^* \left( N^{(i)} \right) \right)' + \beta B_K \left( N^{(i)} \right)' S^{(i)} A_J \left( N^{(i)} \right) \right) = F^{(i)} \left( S^{(i)}, N^{(i)} \right). \]
**Sentiments**

Oudiz & Sachs (1985), p. 311:

“We do not know of any general result concerning the convergence of this process. However in our empirical applications we have not run into major problems. Cohen & Michel (1984) show that in a one dimensional case this kind of a recursion does have a fixed point.”

Of course, convergence of the algorithm neither guarantees a uniqueness of the solution, nor implies any particular properties of the resulting equilibrium. Oudiz & Sachs (1985), p. 288:

“Although we cannot prove that the resulting function is the unique memoryless, time-consistent equilibrium, we suspect that it is in fact unique, in view of the linear-quadratic structure of the underlying problem.”
First Order Conditions Revisited

The FOCs constitute the system of three matrix equations. Two of them describe the reaction function of the policymaker:

\[ S = Q^* + \beta A_J S A_J - \left( P^{*'} + \beta B'_K S A_J \right)' \left( R^* + \beta B'_K S B_K \right)^{-1} \times \left( P^{*'} + \beta B'_K S A_J \right) \]  \hspace{1cm} \text{(DARE)}

\[ F = \left( R^* + \beta B'_K S B_K \right)^{-1} \left( P^{*'} + \beta B'_K S A_J \right) \]  \hspace{1cm} \text{(POLICY)}

and one describes the response of the private sector:

\[ NC_{11} + C_{21} - NC_{12} N - C_{22} N = 0. \]  \hspace{1cm} \text{(NCARE)}

Obviously, the system is highly non-linear and can have many solutions of different types.
For given $N$, if $(A^*, B^*)$ is controllable then

- Symmetric $S$ is unique
- $F$ is stabilising, i.e. all eigenvalues of $A^* - B^*F$ are strictly inside the unit circle

As we ‘substituted out’ the rational response of the private sector, the problem is essentially an engineering one: find optimal control of a dynamic system with initial conditions. Nothing new to discuss.
Household’s Reaction Function: existence and uniqueness

We need to find rational expectations reaction subject to time-consistency requirement. If $C$ can be diagonalised then one of the three situations is possible:

- If the number of jump variables $= \text{the number of explosive eigenvalues}$ then there is a unique solution that can be found using Blanchard and Kahn formula. This solution $N$ is stabilising i.e. all eigenvalues of $C_{11} - C_{12}F$ are strictly inside the unit circle.
- If the number of jump variables $< \text{the number of explosive eigenvalues}$ then there are no stabilising solutions.
- If the number of jump variables $> \text{the number of explosive eigenvalues}$ then there is unique stable solution to CARE, this solution is stabilising. Most iterative procedures of solving CARE will converge to this solution.

In the third case there are multiple private sector equilibria, but only one of them is stable. This solution is picked up by the Oudiz and Sachs procedure.
Iterations

Start with $F^{(i)}$. Solve NCARE for unique stable $N^{(i)}$, solve DARE for $S^{(i)}$, update $F^{(i+1)}$. Do until $\max(\| F^{(i)} - F^{(i+1)} \|) < \delta$ where $\delta$ is given tolerance. This locates a fixed point.

This method is different from the Oudiz and Sachs algorithm in that

- we iterate in policy space $(N, F)$ instead of $(N, S)$
- we need to initialise $F$ only. Intuition might help to initialise $F$. 
Log-Linearised System

Evolution of the economy is described by:

\[ c_t = c_{t+1} - \sigma (i_t - \pi_{t+1}), \]
\[ \pi_t = \beta \pi_{t+1} + \frac{(1 - \gamma \beta)(1 - \gamma) \psi}{\gamma (\psi + \epsilon)} \left( \frac{1}{\sigma} c_t + \frac{1}{\psi} y_t \right) + \eta_t, \]
\[ y_t = (1 - \theta) g_t + \theta c_t, \]
\[ b_{t+1} = i_t + \frac{1}{\beta} \left( b_t - \pi_t + \left( \frac{\theta}{B} - (1 - \beta) \right) y_t - \frac{\theta}{B} c_t \right), \]
\[ g_t = -\lambda b_t, \]

It is described by Phillips curve, IS curve, national income identity, and budget constraint. The role of fiscal policy is limited to control of domestic debt: fiscal authority is not a strategic player.
Monetary Policy’s Objective

...is standard: choose interest rate to minimise loss

$$\frac{1}{2} \mathcal{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left( \frac{\psi(1-\gamma\beta)(1-\gamma)}{\epsilon(\epsilon + \psi)\gamma} \left( \frac{\theta}{\sigma} c^2_t + \frac{(1-\theta)}{\sigma} g^2_t + \frac{1}{\psi} y^2_t \right) + \pi^2_t \right)$$

It can be shown that this form reflects the private sector’s preferences.
Linear Form of Solution

Suppose that the policymaker operates with a linear rule

$$[i_t] = - [F] [b_t]$$

then the household reaction function will necessarily have a linear form of

$$\begin{bmatrix}
\pi_t \\
c_t
\end{bmatrix} = - \begin{bmatrix}
N\pi \\
Nc
\end{bmatrix}[b_t]$$
Policymaker’s reaction Function

DARE is a quadratic equation for a scalar variable $S$ and can be written as

$$\beta B_K^2 S^2 + (R^* - \beta (Q^* B_K^2 - 2P^* B_K A_J + R^* A_J^2)) S + (P^*^2 - Q^* R^*) = 0.$$  

This equation has two eigenvalues and the product of these eigenvalues is equal to:

$$P^*^2 - Q^* R^* = \left(-\left(\left(a_c a_g + a_y a_c (1 - \theta)^2 + \theta^2 a_g a_y \right) \lambda^2 K_c^2 + a_c a_\pi (K_\pi J_c - J_\pi K_c)^2 + a_g a_\pi \lambda^2 K_\pi^2 + a_y a_\pi \left(\lambda (1 - \theta) K_\pi + \theta (K_\pi J_c - J_\pi K_c)\right)^2\right)\right) < 0.$$  

Positive solution for value function is unique.
Household’s Reaction Function

It solves NCARE, which can be written as a system of two quadratic equations in $N_\pi$ and $N_c$:

$$N_\pi^2 + \frac{\theta \tau}{B} N_\pi N_c - N_\pi \left( \frac{(1 - \theta)(1 - \tau)}{B} \lambda + \beta F \right) + (\kappa_c + \theta \kappa_y) N_c + \lambda \kappa_y (1 - \theta) = 0,$$

$$N_c^2 + \frac{B}{\theta \tau} \left( 1 - \frac{(1 - \theta)(1 - \tau)}{B} \lambda - \beta \left( 1 + \frac{\sigma (\kappa_c + \theta \kappa_y)}{\beta} + F \right) \right) N_c$$

$$+ \frac{B}{\theta \tau} N_c N_\pi + \frac{B \sigma}{\theta \tau} N_\pi - \frac{B \sigma}{\theta \tau} ((1 - \theta) \lambda \kappa_y + \beta F) = 0.$$
Household’s Reaction Function
Policy Reactions
### Active and Passive Monetary Policy

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<th>Monetary Policymaker</th>
<th>Private Sector</th>
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<tr>
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<td>$\pi_t = 0.005b_t$</td>
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<td>$c_t = 0.03b_t$</td>
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<tr>
<td><strong>Plan C:</strong> $i_t = -0.06b_t$</td>
<td>$\pi_t = 0.010b_t$</td>
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<td>$c_t = 0.08b_t$</td>
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**Full range of fiscal feedback**
### Welfare

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Conclusions

- Linear-Quadratic models *can* generate multiple equilibria under discretionary policy and in some cases *do*. Researchers need to make sure that they found all of them or that they found the welfare-maximising.

- In the model with fiscal policy and debt we demonstrate the existence of ‘active’ and ‘passive’ monetary policy for a weak fiscal control of debt.

- Discussed properties of multiple equilibria in general mathematical terms
  - Equilibria are distinct. Ruled out ‘indeterminacy’.
  - Demonstrated slightly different way of computing discretionary equilibria.
  - Leadership structure ensures that the policymaker cannot be trapped by the private sector. The policymaker can always choose the best equilibrium. *But the policymaker needs to know that different equilibria can exist.*