Appendum to A Smoothed Least Squares Estimator For Threshold Regression Models: Simulations*

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A Monte Carlo Simulation

The purpose of these experiments is to investigate the performance of the estimator and the accuracy of the distributional approximation in the context of the design of Hansen (2000). This allows us to make comparisons with his procedures in a context of his choosing. The estimators we consider converge at a slower rate than the unsmoothed least squares estimator and so should lead to asymptotically longer confidence intervals than those of Hansen (2000) and Gozalo and Wolf (2005) [although neither of these authors investigate the length of their confidence intervals in finite samples]. On the other hand, the coverage rate of the respective confidence intervals is asymptotically the same and may be better or worse in finite samples - the first order asymptotic theory has nothing to say on this issue.

The design is

$$y_t = \beta^\top x_t + \delta^\top x_t 1(q_t \leq \psi) + \varepsilon_t,$$

where $x_t = (1, x_{2t})$, $q_t \sim N(2,1)$, $\varepsilon_t \sim N(0,1)$, $\delta = (\delta_1, \delta_2)^\top$, $\beta = 0$, $\delta_1 = 0$, and $\psi = 2$. In case I, $x_{2t} = q_t$ and in case II, $x_{2t} \sim N(0,1)$. We compute ψ_n, ψ_n^+ using the kernel $\mathcal{K}(x) = \Phi(x) + x\phi(x)$,

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where Φ and ϕ are the standard Gaussian c.d.f. and density functions respectively. The estimators are computed by grid search over the sample of observed threshold values. We consider parameter values $\delta_2 \in \{0.25, 0.5, 1.0, 1.5, 2.0\}$ and sample sizes $n \in \{50, 100, 250, 500, 1000\}$ and do $ns = 1000$ replications for each experiment. In other work we have examined larger sample sizes, and we comment on these results.

A.1 Performance of the Estimator

In this section we investigate the performance of the unsmoothed and smoothed threshold estimators in comparison with the predicted performance from the asymptotic theory. We take bandwidth parameter $\sigma_n = (\log n)n^{-1/2}$. In Table 1a we report results for the estimates of ψ, while in Table 1b we present the results for the estimates of δ_2. We present the interquartile range divided by 1.35, which is a robust estimate of the standard deviation of the estimates. The (median) biases are very small in all cases and are not reported. We also present the asymptotic standard deviation as predicted by the asymptotic theory. There are several main results:

1. Results improve with sample size and as δ_2 increases.
2. The small sample variability of all estimates is much higher than predicted by the asymptotic theory, but this overprediction reduces considerably with sample size and as δ_2 increases. This overprediction is also implicitly true for the unsmoothed least squares estimator.
3. The estimator ψ_n^+ is nearly always better than ψ_n

We have also examined the case with very large sample sizes and find that with $n = 12,800$ the mean squared errors are within 5% of the asymptotic predictions. Also in this case q-q plots in Figure 1 reveal that normality is a good approximation for the largest sample size.

A.2 Performance of the Confidence Intervals

We next compare our confidence intervals based on estimating the asymptotic variance and on the bootstrap method with those of Hansen (2000). This implicitly evaluates the quality of our distributional approximations and in a context of practical interest.

We compute the estimators by the two different smoothing methods and we investigated three different t-statistic confidence intervals: those based on estimates of the asymptotic variance, those based on the percentile bootstrap, and those based on the pivotal bootstrap using the asymptotic standard errors to studentize. Hansen (2000) used the likelihood ratio, which can be expected to work particularly well in this design as it assumes normality and homoskedasticity. We report results for the parameters ψ and δ_2 for the fifty different combinations of sample sizes ($n \in \{50, 100, 250, 500, 1000\}$) and parameter values ($\delta_2 \in \{0.25, 0.5, 1.0, 1.5, 2.0\}$) for cases I and II. We implemented the two methods as in the previous section. We report results for the 90% intervals throughout.

We first report the results for confidence intervals around ψ. In Table 2 we give the coverage rate for the intervals based on ψ_n and the percentile bootstrap, pivotal bootstrap, and the asymptotic methods. In Table 3 we give the same for intervals based on ψ_n^+. These tables correspond to Table II of Hansen (2000). The main results are:

1. Apart from the smallest value of δ_2, the bootstrap coverage rates are close to the nominal rate and because of the small number of replications are generally within 2 standard errors of the target value (0.02) except for the small δ_2 case.

2. The coverage rates of the asymptotic intervals are less satisfactory for smaller samples sizes, but improve steadily with sample size and are competitive for $n = 1000$.

3. There does not seem to be much difference between the intervals based on ψ_n and the intervals based on ψ_n^+.

We next report the results for confidence intervals around δ_2. In Tables 4 and 5 we give the bootstrap intervals for δ_2 based on the two estimators. These correspond to Table III of Hansen (2000).[1] The main results are:

1. The coverage rates of the bootstrap intervals are close to the nominal throughout.

2. There is not much difference between the two bootstrap methods.

In conclusion, the bootstrap seems to work well for our smoothed estimators in agreement with the asymptotic theory, at least for moderate sized thresholds and sample sizes near 1000. On the other hand the bootstrap for the unsmoothed estimator does not appear to work well in simulations. The coverage rates were very low (and are not reported here) even in the largest sample sizes. We take this as evidence of inconsistency, which would agree with results of Abrevaya and Huang (2005) for the unsmoothed maximum score procedure.

The results do suggest that the small threshold case, $\delta_2 = 0.25$, is problematic. Indeed the asymptotic intervals are considerably undercovered for this case, although the bootstrap intervals are overcovered. It should be pointed out that the threshold effect is indeed very small in this case (Figure 2 shows a typical sample from this process), which is why Hansen's method works well. Note also that the bandwidth parameter $\sigma_n = (\log n)n^{-1/2}$ is used here, and there is reason to suspect that this very small bandwidth while good for the rate of convergence of the estimator is not likely to be optimal for the size properties of the test. In the working paper version of this paper we report some results for bandwidths $\sigma_n = cn^{-1/5}$ in the special case $\delta_2 = 0.25$ and found slightly better size

[1] In Table III of Hansen, the critical level of the table is 95%. And the confidence interval is constructed as union of confidence intervals based on a given set of threshold values that is a confidence interval for the threshold estimate.

properties.

B Figures and Tables

B.1 Monte Carlo

		$z = q$					z is $N(0,1)$				
		0.25	0.50	1.0	1.5	2.0	0.25	0.50	1.0	1.5	2.0
$\widehat{\psi}_{LS}$	n=50	1.2514	0.8047	0.2335	0.1076	0.0793	1.0039	0.3917	0.0614	0.0441	0.0410
	n=100	1.1387	0.4255	0.1020	0.0539	0.0364	0.9337	0.1230	0.0296	0.0217	0.0205
	n=250	0.7991	0.1400	0.0371	0.0201	0.0155	0.2845	0.0300	0.0103	0.0083	0.0087
	n=500	0.2960	0.0590	0.0167	0.0099	0.0077	0.0785	0.0139	0.0054	0.0046	0.0046
	n=1000	0.1516	0.0272	0.0086	0.0051	0.0034	0.0278	0.0068	0.0026	0.0021	0.0021
$\widehat{\psi}^{+}$	n=50	1.4526	0.8655	0.2274	0.1241	0.1061	1.3251	0.5302	0.1012	0.0704	0.0646
	n=100	1.2934	0.3735	0.1159	0.0765	0.0607	1.1106	0.1498	0.0597	0.0402	0.0364
	n=250	0.8721	0.1287	0.0527	0.0380	0.0317	0.4307	0.0558	0.0271	0.0227	0.0185
	n=500	0.2895	0.0673	0.0307	0.0223	0.0198	0.0934	0.0306	0.0175	0.0119	0.0113
	n=1000	0.1257	0.0379	0.0191	0.0151	0.0131	0.0402	0.0176	0.0103	0.0074	0.0064
$\widehat{\psi}$	n=50	1.5129	1.1006	0.2645	0.1461	0.1423	1.4388	1.2621	0.1870	0.1282	0.1075
	n=100	1.5580	0.5378	0.1359	0.0934	0.0839	1.8743	0.8384	0.0892	0.0724	0.0580
	n=250	1.3445	0.1480	0.0633	0.0521	0.0444	1.9138	0.0766	0.0379	0.0309	0.0278
	n=500	0.3638	0.0742	0.0384	0.0291	0.0273	0.1721	0.0361	0.0228	0.0185	0.0183
	n=1000	0.1366	0.0372	0.0243	0.0196	0.0186	0.0475	0.0196	0.0126	0.0120	0.0105
as$\widehat{\psi}^{+}$	n=50	0.4984	0.2492	0.1246	0.0831	0.0623	0.2492	0.1246	0.0623	0.0415	0.0312
	n=100	0.3216	0.1608	0.0804	0.0536	0.0402	0.1608	0.0804	0.0402	0.0268	0.0201
	n=250	0.1771	0.0885	0.0443	0.0295	0.0221	0.0885	0.0443	0.0221	0.0148	0.0111
	n=500	0.1117	0.0559	0.0279	0.0186	0.0140	0.0559	0.0279	0.0140	0.0093	0.0070
	n=1000	0.0700	0.0350	0.0175	0.0117	0.0088	0.0350	0.0175	0.0088	0.0058	0.0044
as$\widehat{\psi}$	n=50	0.4958	0.2640	0.1603	0.1324	0.1212	0.2498	0.1355	0.0857	0.0728	0.0677
	n=100	0.3199	0.1703	0.1034	0.0854	0.0782	0.1611	0.0874	0.0553	0.0470	0.0437
	n=250	0.1762	0.0938	0.0569	0.0470	0.0430	0.0887	0.0481	0.0304	0.0259	0.0241
	n=500	0.1111	0.0592	0.0359	0.0297	0.0272	0.0560	0.0304	0.0192	0.0163	0.0152
	n=1000	0.0697	0.0371	0.0225	0.0186	0.0170	0.0351	0.0190	0.0120	0.0102	0.0095

Table 1a: Standard deviation of estimates of ψ along with asymptotic predictions (as)

		$z = q$					z is $N(0,1)$				
		0.25	0.50	1.0	1.5	2.0	0.25	0.50	1.0	1.5	2.0
	n=50	0.8073	0.4849	0.2930	0.3083	0.2978	1.3119	0.8594	0.6130	0.5061	0.5341
	n=100	0.4971	0.2362	0.2020	0.1862	0.2072	0.8628	0.4635	0.3507	0.3775	0.3469
$\widehat{\delta}_{2LS}$	n=250	0.2012	0.1163	0.1256	0.1281	0.1292	0.4133	0.2280	0.2154	0.2195	0.1984
	n=500	0.0959	0.0852	0.0876	0.0911	0.1017	0.1957	0.1587	0.1493	0.1472	0.1426
	n=1000	0.0616	0.0649	0.0630	0.0616	0.0677	0.1223	0.1073	0.1028	0.1138	0.1057
	n=50	0.9516	0.5244	0.3132	0.2946	0.2986	1.9424	1.0380	0.6577	0.6658	0.7245
	n=100	0.5319	0.2433	0.2063	0.1883	0.2088	1.0608	0.5103	0.3957	0.4332	0.4330
$\widehat{\delta}_2^+$	n=250	0.2024	0.1133	0.1231	0.1277	0.1327	0.4253	0.2335	0.2403	0.2447	0.2460
	n=500	0.0954	0.0866	0.0879	0.0879	0.1002	0.2113	0.1635	0.1619	0.1624	0.1583
	n=1000	0.0607	0.0642	0.0645	0.0607	0.0662	0.1178	0.1143	0.1037	0.1178	0.1163
	n=50	1.1238	0.6689	0.3491	0.3505	0.3341	7.7863	6.1208	1.3128	1.2141	1.2464
	n=100	0.7613	0.2886	0.2303	0.2127	0.2235	8.2338	1.4964	0.6105	0.5978	0.6238
$\widehat{\delta}_2$	n=250	0.2972	0.1292	0.1343	0.1338	0.1392	1.9394	0.3259	0.2766	0.2929	0.2934
	n=500	0.1091	0.0925	0.0881	0.0942	0.1014	0.3161	0.1806	0.1734	0.1797	0.1827
	n=1000	0.0671	0.0661	0.0639	0.0631	0.0722	0.1302	0.1131	0.1121	0.1228	0.1306
	n=50	0.2000	0.2000	0.2000	0.2000	0.2000	0.4692	0.4692	0.4692	0.4692	0.4692
	n=100	0.1414	0.1414	0.1414	0.1414	0.1414	0.3318	0.3318	0.3318	0.3318	0.3318
as	n=250	0.0894	0.0894	0.0894	0.0894	0.0894	0.2098	0.2098	0.2098	0.2098	0.2098
	n=500	0.0632	0.0632	0.0632	0.0632	0.0632	0.1484	0.1484	0.1484	0.1484	0.1484
	n=1000	0.0447	0.0447	0.0447	0.0447	0.0447	0.1049	0.1049	0.1049	0.1049	0.1049

Table 1b: Standard deviation of estimates of δ_2 along with asymptotic predictions (as)

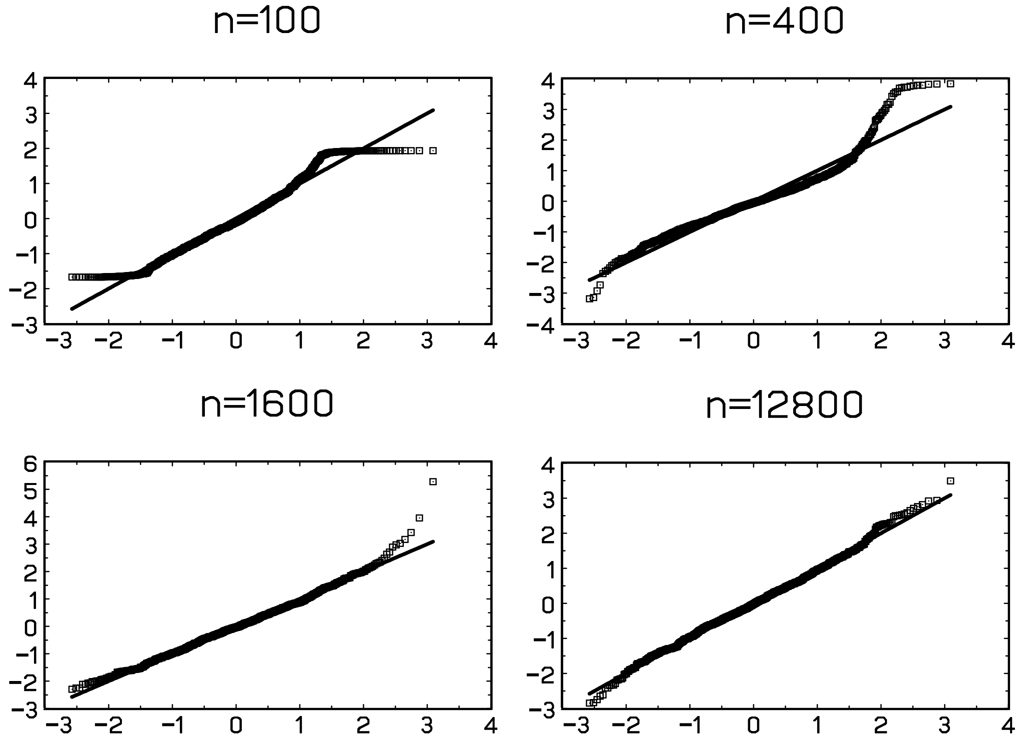

Figure 1. Q-Q plots of estimator ψ_n for sample sizes $n = 100, 400, 1600$, and 12,800.

	$z = q$					z is $N(0,1)$				
δ_2	0.25	0.50	1.0	1.5	2.0	0.25	0.50	1.0	1.5	2.0
(a)										
n=50	0.9713	0.9952	0.9665	0.9426	0.9282	0.9856	0.9713	0.9426	0.9330	0.9282
n=100	0.9904	0.9904	0.9282	0.9187	0.9234	0.9856	0.9809	0.9426	0.9234	0.9282
n=250	0.9952	0.9617	0.9091	0.9139	0.8660	0.9904	0.9139	0.9139	0.9043	0.9474
n=500	0.9809	0.9569	0.9043	0.8469	0.8947	0.9713	0.9234	0.9139	0.9234	0.9378
n=1000	0.9617	0.9330	0.8947	0.8804	0.9091	0.9234	0.8852	0.9187	0.8995	0.8947
(b)										
n=50	0.9282	0.9426	0.9569	0.9665	0.9187	0.9378	0.9713	0.9617	0.9474	0.9187
n=100	0.9330	0.9522	0.9426	0.9282	0.9234	0.9617	0.9713	0.9713	0.9522	0.9234
n=250	0.9569	0.9713	0.9282	0.8995	0.9091	0.9474	0.9522	0.9474	0.9330	0.9665
n=500	0.9713	0.9569	0.9330	0.8612	0.8900	0.9665	0.9474	0.9234	0.9569	0.9474
n=1000	0.9617	0.9474	0.9234	0.9043	0.9330	0.9234	0.8995	0.9187	0.9139	0.9091
(c)										
n=50	0.1483	0.3158	0.5167	0.6794	0.6746	0.1914	0.4163	0.6411	0.6986	0.7321
n=100	0.1340	0.3780	0.6603	0.7464	0.7608	0.2727	0.5933	0.7799	0.7703	0.7560
n=250	0.2727	0.6411	0.7943	0.8230	0.7560	0.4258	0.8182	0.8469	0.7847	0.8804
n=500	0.4019	0.7512	0.8852	0.8182	0.8373	0.6651	0.8900	0.8708	0.9139	0.8900
n=1000	0.5694	0.8612	0.8660	0.8660	0.8995	0.7943	0.8995	0.8756	0.8804	0.8708

Table 2. Coverage of 90% intervals for ψ. (a) Percentile Bootstrap based on ψ_n; (b) Pivotal bootstrap based on ψ_n; (c) Asymptotic interval based on ψ_n.

δ_2	$z = q$ 0.25	0.50	1.0	1.5	2.0	z is $N(0,1)$ 0.25	0.50	1.0	1.5	2.0
(a)										
n=50	0.9904	0.9904	0.9952	0.9474	0.9187	1.0000	0.9952	0.9904	0.9856	0.9904
n=100	1.0000	0.9904	0.9569	0.9378	0.9282	0.9952	0.9952	0.9761	0.9713	0.9856
n=250	0.9952	0.9761	0.9139	0.9234	0.8852	1.0000	0.9809	0.9522	0.9426	0.9139
n=500	0.9809	0.9713	0.9187	0.8660	0.8612	1.0000	0.9617	0.9091	0.9282	0.8947
n=1000	0.9761	0.9569	0.9091	0.8660	0.8947	0.9569	0.8852	0.9043	0.8900	0.8708
(b)										
n=50	0.9856	0.9809	0.9856	0.9426	0.9187	0.9904	1.0000	0.9952	0.9713	0.9330
n=100	0.9809	0.9952	0.9761	0.8995	0.9043	0.9952	1.0000	0.9761	0.9474	0.9282
n=250	0.9856	0.9809	0.9043	0.8995	0.8565	1.0000	0.9809	0.9139	0.9043	0.8995
n=500	0.9904	0.9904	0.8995	0.8660	0.8612	1.0000	0.9378	0.8995	0.9378	0.9043
n=1000	0.9809	0.9569	0.8947	0.8708	0.9234	0.9665	0.8947	0.9043	0.8756	0.8708
(c)										
n=50	0.1340	0.2823	0.4354	0.5694	0.6124	0.1148	0.3014	0.5167	0.5502	0.6459
n=100	0.0718	0.2679	0.6124	0.6746	0.7177	0.1770	0.4163	0.6746	0.6842	0.7033
n=250	0.1818	0.5598	0.7847	0.8038	0.7943	0.3158	0.6507	0.7416	0.7656	0.8086
n=500	0.3062	0.7081	0.8278	0.7990	0.7895	0.5120	0.8565	0.8182	0.8469	0.8373
n=1000	0.4641	0.8325	0.8373	0.8421	0.8612	0.7033	0.8373	0.8469	0.8182	0.8325

Table 3. Coverage of 90% intervals for ψ. (a) Percentile Bootstrap based on ψ_n^+; (b) Pivotal bootstrap based on ψ_n^+; (c) Asymptotic interval based on ψ_n^+.

	$z = q$					z is $N(0,1)$				
δ_2	0.25	0.50	1.0	1.5	2.0	0.25	0.50	1.0	1.5	2.0
(a)										
n=50	0.9665	0.9426	0.9665	0.9378	0.9378	1.0000	0.9952	0.9665	0.9617	0.9426
n=100	0.9474	0.9617	0.9282	0.8995	0.8804	1.0000	0.9809	0.9426	0.9187	0.9282
n=250	0.9330	0.9617	0.9187	0.8947	0.8469	0.9952	0.9617	0.8756	0.9378	0.8995
n=500	0.9282	0.9234	0.8852	0.8565	0.8852	0.9952	0.8947	0.8565	0.9139	0.9139
n=1000	0.8756	0.9426	0.8995	0.8804	0.8565	0.9522	0.8995	0.9139	0.8947	0.9091
(b)										
n=50	0.9665	0.9330	0.9665	0.9378	0.9187	0.9952	0.9856	0.9569	0.9569	0.9426
n=100	0.9378	0.9713	0.9330	0.8947	0.8947	1.0000	0.9665	0.9282	0.9234	0.9043
n=250	0.9330	0.9665	0.9234	0.8995	0.8421	0.9904	0.9426	0.8900	0.9474	0.9043
n=500	0.9378	0.9187	0.8804	0.8708	0.8852	0.9713	0.9043	0.8660	0.9234	0.9139
n=1000	0.8804	0.9474	0.8995	0.8756	0.8612	0.9426	0.8995	0.9139	0.9043	0.8947

Table 4. Coverage of 90% intervals for δ_2. (a) Percentile bootstrap based on δ_{2n}; (b) Pivotal bootstrap based on δ_{2n}.

	$z = q$					z is $N(0,1)$				
δ_2	0.25	0.50	1.0	1.5	2.0	0.25	0.50	1.0	1.5	2.0
(a)										
n=50	0.9617	0.9569	0.9522	0.9282	0.8804	1.0000	0.9952	0.9856	0.9617	0.9856
n=100	0.9809	0.9617	0.9282	0.9091	0.9043	1.0000	1.0000	0.9904	0.9809	0.9426
n=250	0.9569	0.9330	0.9282	0.8900	0.8995	1.0000	0.9952	0.9330	0.9569	0.9282
n=500	0.9617	0.9378	0.9139	0.8852	0.9139	1.0000	0.9569	0.8756	0.9187	0.9043
n=1000	0.8852	0.9617	0.9426	0.9187	0.9091	0.9809	0.9139	0.9139	0.8804	0.8995
(b)										
n=50	0.8900	0.8565	0.9234	0.8900	0.9043	0.9091	0.8900	0.8804	0.8900	0.9282
n=100	0.8804	0.8852	0.9043	0.8804	0.8660	0.9522	0.8804	0.9330	0.9043	0.9234
n=250	0.9043	0.8756	0.8995	0.8947	0.8900	0.9569	0.8947	0.8469	0.9234	0.8995
n=500	0.8900	0.8900	0.8708	0.8852	0.9043	0.9426	0.8804	0.8421	0.9139	0.8995
n=1000	0.8469	0.9474	0.9426	0.9187	0.9091	0.9282	0.8804	0.9139	0.8660	0.8995

Table 5. Coverage of 90% intervals for δ_2. (a) Percentile bootstrap based on δ_{2n}^{+}; (b) Pivotal bootstrap based on δ_{2n}^{+}

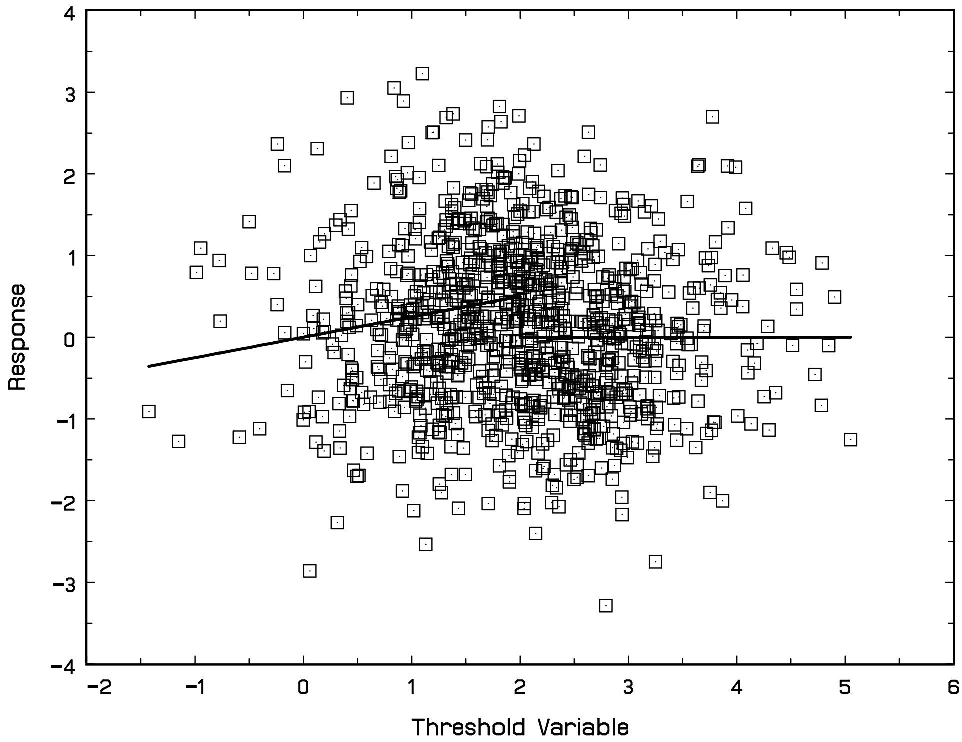

Figure 2. Typical sample from small threshold case $\delta_2 = 0.25$.