Coordination-Free Equilibria in Cheap Talk Games*

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Abstract

In Crawford and Sobel's (1982) one-sender cheap talk model, information transmission is partial: in equilibrium, the state space is partitioned into a finite number of intervals, and the receiver learns the interval where the state lies. Existing studies of the analogous model with multiple senders, however, offer strikingly different predictions: with a large state space, equilibria with full information transmission exist, as do convoluted almost fully revealing equilibria robust to certain classes of small noise in the senders' information. This paper shows that such equilibria are ruled out when robustness to a broad class of perturbations is required. The surviving equilibria confer a less dramatic informational advantage to the receiver from consulting multiple senders. They feature an intuitive interval structure, and are coordination-free in the sense that boundaries between intervals reported by different players do not coincide. The coordination-free property implies many similarities between these profiles and one-sender equilibria, and is further justified by appealing to an alternative set of assumptions.

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‡This version is identical to the November 12, 2010 version, save for one added sentence at the bottom of page 16.

1 Introduction

The transmission of information is an integral part of many economic models, whether implicitly or explicitly. In certain settings, such transmission is strategic: the side sending the information may choose the extent of the disclosure in order to achieve a higher payoff. At the same time, the party receiving the information may be unable to offer incentives that significantly improve disclosure.

The seminal work of Crawford and Sobel (1982, henceforth CS) examines such a setting. A sender observes the state of the world in one dimension, sends a message to the receiver, who then takes an action. Both the sender and the receiver desire a higher action when the state is higher, but the optimal action for the sender differs from the optimal action for the receiver. Talk is cheap in the sense that neither player's utility depends on the sender's message. CS show that equilibria in this setting feature the sender revealing an interval of the state space. Moreover, there is a finite upper bound on the number of intervals that can be revealed in equilibrium, and this bound increases as the sender's bias relative to the receiver becomes small.

This paper examines a model very similar to CS's, but with multiple senders simultaneously sending their messages. For example, consumers may approach several merchants for advice about a product (without revealing the messages of merchants previously visited), or a policymaker may seek the opinion of multiple experts. In multi-sender cheap talk games, because the actions that a given sender can induce depend on what messages other senders use, there exists a large multiplicity of equilibria. Some of these profiles lead to full revelation of the state, even with two senders biased in the same direction, as long as the state space is much larger than the biases. Thus, there are large qualitative differences between equilibria in single- and multi-sender cheap talk, both in terms of their predictive power and in terms of their properties.

This paper proposes a strong selection criterion called *disagreement-robustness*, which requires equilibria to survive the possibility of small noise in the senders' observation of the state, even when the senders disagree about the noise structure. Formally, an equilibrium is disagreement-robust if, whenever there is common knowledge that noise is small, even if players may have different priors about the exact form of the noise, there exists a profile near the equilibrium where every player's strategy is almost optimal. Optimality is in an interim sense: each sender's message must be nearly optimal given her observed signal, and the receiver's action must be nearly optimal given the senders' messages. The motivation for allowing small disagreement is that when the possibility of noise exists, common knowledge of the exact noise structure is a strong assumption: it is quite plausible that the players have different beliefs on the fine details of noise.

Generic disagreement-robust equilibria belong to a class called *coordination-free* and retain many properties of CS equilibria. Coordination-free equilibria, like the ones in CS, have interval structure, in that each message vector reveals an interval of states. Moreover, at each boundary between two intervals, only one sender changes her message. That sender must therefore be indifferent at the boundary between inducing the action corresponding to the left interval and the action corresponding to the right interval, just like in CS equilibria. Several desirable properties follow, such as an upper bound on the number of intervals, and, fixing the leftmost inducible action and the list of senders changing their message at each boundary, a procedure for determining the existence of such an equilibrium and for computing it. In terms of the best coordination-free equilibrium for the receiver, consulting two senders instead of one generally improves the receiver's payoff the most when the second sender is less biased than the first and/or biased in the opposite direction. The benefit of having the extra sender is much less dramatic than in the best overall equilibrium, which features full revelation whenever the two senders' biases are sufficiently small.

Call an equilibrium *complete* if all combinations of equilibrium messages by each sender occurs on path. Theorem 1 shows that complete coordination-free equilibria are disagreement-robust, while generic equilibria that are not coordination-free and non-complete equilibria fail the criterion. The key to this result is that in coordination-free profiles, when small noise is added, even when a sender faces multiple possible message vectors by opponents, she knows that the message prescribed by the profile is nearly optimal for all likely such vectors. She therefore has little incentive to modify her course of action in order to coordinate with other senders given her beliefs about the noise. By contrast, in generic non-coordination-free equilibria, a sender's message at a state can be substantially suboptimal in response to other senders' messages at nearby states. Differing beliefs about the noise can then lead to the unraveling of such profiles. Completeness is required because the receiver's beliefs following an out-of-equilibrium message vector can be drastically affected by arbitrarily small noise.

Although completeness implies the desirable property that the receiver's action after any combination of equilibrium messages is determined by play and not arbitrary, it is a characteristic that some intuitive equilibria lack. *Local robustness* is a weakening of disagreement-robustness that admits such equilibria by only considering "local" noise, where the support of the senders' signals conditional on a state is a small interval around the state. Theorem 2 shows that subject to assumptions concerning specific pathological cases, the set of generic equilibria thus selected is the set of coordination-free equilibria.

One appealing characteristic of complete and coordination-free equilibria is that at almost all states, in the induced normal-form game played by senders given a state and a receiver strategy, the Nash equilibrium is unique. At the same time, equilibria that are not coordination-free generally do not have this property, and are therefore vulnerable to a change in the induced-game Nash equilibrium on which senders coordinate.

The extension of the CS model with multiple senders has been studied by several papers focusing on the existence of fully revealing equilibria and their robustness. Krishna and Morgan (2001) note that if the senders' biases are sufficiently small relative to the state space, full revelation is achievable, as the receiver may threaten an action unappealing to all parties if the senders' messages diverge. However, Battaglini (2002) shows that fully revealing equilibria rely on implausible out-of-equilibrium beliefs: in the above example, were the receiver to face slightly divergent messages, she should probably take an action corresponding to a close-by state rather than a "crazy" action. Ambrus and Lu (2010) exhibit equilibria where every vector of messages from senders occurs, so out-of-equilibrium considerations do not arise, and where each vector identifies an interval of states. These equilibria approach full revelation in the sense that as the state space gets large relative to the senders' biases, the intervals get small relative to the biases. However, they are not particularly intuitive, and one may doubt whether such equilibria ever arise in actual communication.

To exclude fully revealing equilibria, Battaglini (2002) defines a particular class of noise, and requires that for some sequence of vanishing noise within that class, after any out-of-equilibrium message vector, the receiver's optimal action converges to her action as specified in the equilibrium profile, holding the senders' strategies fixed. One might hope to obtain a tractable class of equilibria through strengthening Battaglini's criterion, by applying it to all message vectors (rather than just out-of-equilibrium ones) and requiring robustness to all sequences of vanishing noise (rather than just one sequence). Appendix B determines the set of equilibria surviving such a criterion; many unappealing equilibria, such as the ones proposed by Ambrus and Lu (2010), are part of this set.[1]

There are many other strands of the literature related to CS.[2] One that this paper may

[1] This is not surprising given that Ambrus and Lu show that given any small noise within certain classes, their equilibria have a nearby profile that constitute an equilibrium with noise.

[2] Battaglini (2002), Battaglini (2004) and Ambrus and Takahashi (2008) study a multidimensional setting, while Krishna and Morgan (2001) examines sequential cheap talk. For cheap talk in a finite state space, see Green and Stokey (2007), and for finite state space with nontrivial noise, see Austen-Smith (1990b), Austen-Smith (1993), Wolinski (2002) and Gerardi, Maclean and Postlewaite (2009). Blume, Board and Kawamura

contribute to is the study of delegation: is it better for the receiver to retain the decision right and play a cheap talk game, or to delegate the action to the sender(s)? Melumad and Shibano (1991), Dessein (2002) and Alonso and Matoushek (2008) study this question in a single-sender setting. Since fully revealing equilibria and robust almost fully revealing equilibria exist when there are multiple senders (and when the state space is large relative to the biases), there has been little scope for studying this question in a multi-sender setting. However, if one expects a coordination-free equilibrium to arise from multi-sender cheap talk, then the delegation issue becomes nontrivial since coordination-free equilibria cannot approach full revelation.

2 Model

There are $n+1$ players: a set N of senders $1, 2, ..., n$ and a receiver R. Each sender observes a common state of the world $\theta \in \Theta = [0, 1]$, which is drawn from a probability distribution $F(.)$ with a continuous and strictly positive prior density $f(.) \in [d, D]$.

Each sender's pure strategy $m_i : \Theta \rightarrow M_i$ assigns a message from a Borel set M_i to each state. When referring to a specific pure-strategy profile Γ, it is convenient to denote sender i's message at θ by $m_i^\Gamma(\theta)$, and to let $M_i^\Gamma = \{m_i : m_i^\Gamma(\theta) = m_i \text{ for some } \theta \in \Theta\}$. Let $m^\Gamma(\theta) = (m_1^\Gamma(\theta), m_2^\Gamma(\theta), ..., m_n^\Gamma(\theta))$ be the message vector sent at state θ, and let $\theta^\Gamma(m)$ denote the set of states where message vector m is sent.

Upon observing message vector m, the receiver takes action $a(m)$. Her pure strategies thus take the form $a : \times_{i=1}^n M_i \rightarrow \Theta$. When referring to a specific profile Γ, denote the receiver's action given m by $a^\Gamma(m)$.

The equilibrium concept is Bayesian equilibrium. The senders' strategies are required to be measurable, and for each m, $\theta^\Gamma(m)$ is either empty or allows the receiver's optimum to be well defined.

(2007) examine nontrivial noise with one sender in the CS setting. For a dynamic model, see Eső and Fong (2008). Baliga and Morris (2002) and Aumann and Hart (2003) study cheap talk with multiple decision makers. Matthews, Okuno-Funiwara and Postlewaite (1991) and Chen, Kartik and Sobel (2008) propose refinements in the one-sender case. For models where the sender's reputation is at stake, see Olszewski (2004) and Ottaviani and Sørensen (2006). Mylovanov and Zapechelnyuk (2010) study the impact of commitment power. Finally, Gilligan and Krehbiel (1989), Austen-Smith (1990a) and Krehbiel (2001) apply the model to a political science setting.

For simplicity, assume throughout the paper that in any equilibrium Γ, if $a^\Gamma(m_i, m_{-i}) = a^\Gamma(m_i', m_{-i})$ for all $m_{-i} \in \times_{j \neq i} M_j^\Gamma$, then $m_i = m_i'$. Note that this simplification does not impose any restrictions on equilibrium because m_i and m_i' are assumed equal only if they lead to the same action for *all* $m_{-i} \in \times_{j \neq i} M_j^\Gamma$, whether on-path or out-of-equilibrium.[3] Also for convenience, let $\lambda(.)$ denote the Lebesgue measure.

All players' utilities depend only on the state θ and an action $a \in \Theta$ taken by the receiver. Let $u_i(a, \theta)$ denote player i's utility when the action is a and state is θ, for $i = 1, 2, ..., n, R$. The following standard assumptions are maintained throughout the paper:

1. all utility functions are Lipschitz continuous;
2. given θ, the receiver's utility is strictly concave with a maximum at $a = \theta$;
3. given θ, sender i's utility has a unique maximum at $a = \theta + b_i(\theta)$, is strictly increasing to the left, and strictly decreasing to the right;
4. $\exists \eta > 0 : |b_i(\theta)| > \eta$ for all $i \in N$ and $\theta \in \Theta$; and
5. for all $i \in N$, if $a < a'$, $\theta < \theta'$ and $u_i(a', \theta) \geq u_i(a, \theta)$, then $u_i(a', \theta') > u_i(a, \theta')$.

Note that Assumption 4 and continuity imply that for each i, either $b_i(\theta) > \eta$ everywhere or $b_i(\theta) < -\eta$ everywhere. Assumption 5 is the commonly encountered single-crossing condition.

The receiver's preferences are described by her utility function. Because of Assumption 2, in equilibrium, the receiver's optimal action after any message vector that occurs on path is unique. I assume, in the spirit of perfection, that the receiver's strategy after out-of-equilibrium message vectors is pure.[4]

I also assume that in an equilibrium Γ, each sender i has a strict preference ranking $\succ_{i,\Gamma}$ over messages,[5] and that her overall preferences are lexicographic: among messages yielding

[3]For example, suppose message vector $a^\Gamma(1,1,1) = a^\Gamma(1,1,2) = a^\Gamma(1,2,1) = a^\Gamma(2,1,1) = a$, while $a^\Gamma(2,2,2) = a^\Gamma(2,2,1) = a^\Gamma(2,1,2) = a^\Gamma(1,2,2) = a' \neq a$. Also assume that of these eight message vectors, only $(1,1,1)$ and $(2,2,2)$ are sent in equilibrium. Then even though no deviation leads to a different action, 1 and 2 remain distinct messages for all senders: for example, if sender 1 sends 1 and sender 2 sends 2, then the actions induced by sender 3 through sending 1 and sending 2 are not the same.

[4]Perfect Bayesian equilibrium is not explicitly assumed due to the lack of a widely accepted definition with a continuous action space.

[5]Because messages are assumed equal when they lead to the same action for all $m_{-i} \in \times_{j \neq i} M_j^\Gamma$, senders need only have a strict preference over messages that are truly distinct.

the highest expected utility, she picks her preferred one under $\succ_{i,\Gamma}$. For example, given the choice between two messages leading to the same outcome, the sender may pick the simpler one. Note that this ranking is endogenous to Γ, since in a cheap talk setting, messages acquire their meaning endogenously. This assumption plays two roles. First, attention in this paper is restricted to pure-strategy equilibria, as is standard in the simultaneous multi-sender cheap talk literature. Second, if two message vectors m and m' both occur on the path of an equilibrium Γ and induce the same action, then m and m' cannot differ in exactly one component. The results in this paper do not change with standard sender preferences if these two assumptions are made instead.

3 Examples

This section presents examples of unintuitive equilibria that can arise in multi-sender cheap talk. In each of these examples, there are two senders, and the parameters follow the popular "uniform-quadratic" specification: $\theta \sim U[0,1]$ and $u_i(a,\theta) = -(a-(\theta+b_i))^2$ (with of course $b_R = 0$), so for all θ, sender i's utility is maximized at $\theta + b_i$. In all examples, every message vector in $\times_{i=1}^{n} M_i^{\Gamma}$ occurs in equilibrium,[6] so any selection criterion that only places restrictions on out-of-equilibrium beliefs would not rule out any of the following equilibria. For later reference, call such equilibria *complete*.

A feature common to all of these examples is that at some states θ, sender i's optimal message changes if sender $-i$ deviates to a message $m_{-i}^{\Gamma}(\theta') \neq m_{-i}^{\Gamma}(\theta)$ for some θ' close to θ. Therefore, these equilibria require precise coordination among the senders.

Example 1: No Full Interval Structure

Suppose $b_1 = b_2 = 0.04$. Each sender has two equilibrium messages, x and y. The prescribed message vector $m^{\Gamma}(\theta)$ is:

- (x,x) if $\theta \in [0,0.01]\backslash\mathbb{Q}$;
- (y,y) if $\theta \in ([0,0.01]\cap\mathbb{Q})\cup(0.01,0.18]$;
- (x,y) if $\theta \in (0.18,0.51]\cup((0.51,1]\cap\mathbb{Q})$;
- (y,x) if $\theta \in (0.51,1]\backslash\mathbb{Q}$.

It is easy to check that the receiver's optimal actions are $a^{\Gamma}(x,x) = 0.005$, $a^{\Gamma}(y,y) = 0.095$, $a^{\Gamma}(x,y) = 0.345$ and $a^{\Gamma}(y,x) = 0.755$, and that this profile is indeed an equilibrium. As a result, within $[0,0.01]$, the action following a rational state is 0.095, while the action

[6] *I.e.* every combination of equilibrium messages from each sender is sent in equilibrium.

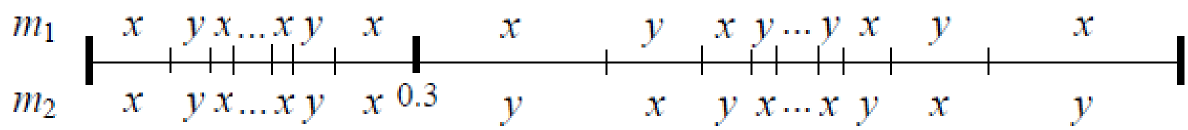

Figure 1: Example 2

following an irrational state is 0.005; a similar situation arises within $(0.51, 1]$. Thus, outside of $[0.01, 0.51]$, there is no nontrivial interval of states following which the same messages are sent.

Example 2: Infinitely Many Intervals

Even when almost all states belong to an nontrivial interval leading to the message vector, there can be infinitely many such intervals.

Suppose $b_1 = b_2 = 0.1$. Define the following sets:

$A = \left[\cup_{k=0}^{\infty}[0.15(1-(\frac{1}{4})^k), 0.15(1-\frac{1}{2}(\frac{1}{4})^k))\right] \cup \left[\cup_{k=0}^{\infty}(0.15(1+\frac{1}{2}(\frac{1}{4})^k), 0.15(1+(\frac{1}{4})^k)]\right]$, and $B = \{z \in (0.3, 1] : \frac{3}{7}(z - 0.3) \in A\}$.

Each sender has two equilibrium messages, x and y. The prescribed message vector $m^{\Gamma}(\theta)$ is (see figure 1):

- (x, x) if $\theta \in A$;
- (y, y) if $\theta \in [0, 0.3] \backslash A$;
- (x, y) if $\theta \in B$;
- (y, x) if $\theta \in (0.3, 1] \backslash B$.

Because A and B are essentially symmetric around 0.15 and 0.65, the midpoints of $[0, 0.3]$ and $(0.3, 1]$, the receiver's optimal actions are $a^{\Gamma}(x, x) = a^{\Gamma}(y, y) = 0.15$ and $a^{\Gamma}(x, y) = a^{\Gamma}(y, x) = 0.65$. Both senders are indifferent between 0.15 and 0.65 when the state is 0.3, so this profile is an equilibrium.

It is also possible to build an equilibrium with interval structure where the number of equilibrium actions by the receiver is infinite.

Example 3: Convoluted Finite Interval Structure

This example is borrowed from Ambrus and Lu (2010). Even when the entire state space is divided into intervals that are revealed, the equilibrium profile may still be counterintuitive.

Suppose b_1 and b_2 are small, and divide Θ into q equally sized *blocks*, each of which is divided into q equally sized *cells*. Both sender's messages are labeled $1, 2, ..., q$ and are used

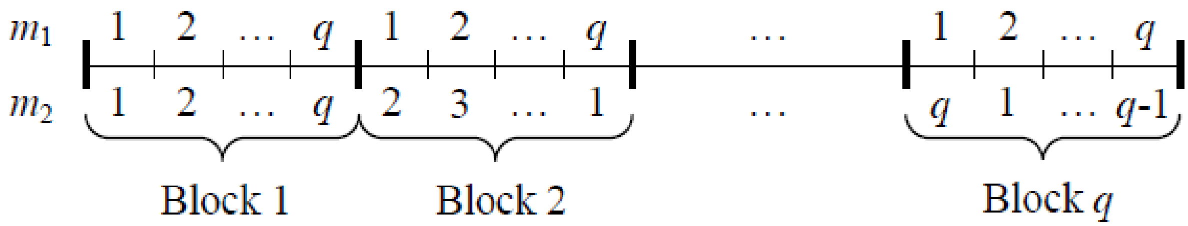

Figure 2: Example 3

as follows (see Figure 2):

- sender 1 sends message k in the k^{th} cell of each block;

- sender 2 sends message $k+l-1(\text{mod}\, q)$ in the k^{th} cell of the l^{th} block (when the formula gives 0, message q is sent).

This profile guarantees that every message vector in $\{1, ..., q\} \times \{1, ..., q\}$ occurs in exactly one cell, and that any deviation leads to an action at least almost a block away when q is large. Thus, if the biases b_1 and b_2 are small with respect to Θ, then the number of blocks can be large, and as the biases shrink, the size of cells shrinks faster and becomes much smaller than the biases.

Therefore, with small biases, this construction can yield a very informative equilibrium. Yet, it appears to be asking too much of the senders, who must change their messages very frequently at arbitrarily set boundaries between cells.

4 Coordination-Free Equilibria

This section defines coordination-free equilibria and examines some of its basic properties.

Definition: Given a pure-strategy profile Γ, a *cell in Γ* is a maximal[7] positive-measure interval of states throughout which m^{Γ} remains constant.

Note that in some equilibria, such as Example 1, cells do not cover Θ because m^{Γ} does not remain constant over any nontrivial interval in certain regions. These are equilibria that I wish to exclude. The definition of coordination-free equilibria is then straightforward.

Definition: An equilibrium Γ is *coordination-free* if:

[7] That is, m^{Γ} does not remain constant in any strict superset of a cell.

1. $\Theta = \cup_{I \text{ is a cell in } \Gamma} I$, the message vectors sent in any two adjacent cells in Γ differ in exactly one component, and the induced action differs in any two adjacent cells; or

2. $|\{i \in N : |M_i^\Gamma| \geq 2\}| = 1$.[8]

The forward induction procedure of Crawford and Sobel (1982) can be used to determine candidate coordination-free equilibria given the identity of the sender whose message changes at each boundary between two cells[9]. Given the receiver's leftmost action, the location of the leftmost boundary θ_1 is determined by the prior and u_R. The indifference condition for the sender whose message changes at θ_1 then pins down the receiver's second leftmost action, and so on. *Candidate play*[10] is achieved if some boundary falls on 1, the right endpoint of Θ. Clearly, on-path play in any coordination-free equilibrium can be computed through such a procedure.

However, unlike in the one-sender case, not all candidate play corresponds to play in an equilibrium. The reason is that there may be nowhere to place the receiver's action after an out-of-equilibrium vector in $\times_{i=1}^n M_i^\Gamma$ such that no sender desires a deviation. The remainder of this section examines when this may be the case.

Consider candidate play with two senders and three cells where the equilibrium message vectors are, from left to right, (L, L), (H, L) and (H, H) (so that $a^\Gamma(L, H)$ must be placed).[11] It is easy to think of two cases where any location of $a^\Gamma(L, H)$ induces a deviation:

a) Sender 1 has a small leftward bias while sender 2 has a large rightward bias, such that the middle interval (H, L) is very small, and the rightmost interval (H, H) is very big (see Figure 3).[12] Then choosing $a^\Gamma(L, H) < a^\Gamma(H, H)$ induces sender 1 to deviate from H to L at the left end of the (H, H) interval, choosing $a^\Gamma(L, H) > a^\Gamma(H, H)$ induces sender 1 to deviate at the right end of the (H, H) interval, while choosing $a^\Gamma(L, H) = a^\Gamma(H, H)$ induces sender 2 to deviate from L to H at the right end of the (L, L) interval.

b) Preferences are very asymmetric, so that at the boundary θ_1 between (L, L) and (H, L), sender 2 prefers any action greater than θ_1 to $a^\Gamma(L, L)$, and at the boundary θ_2

[8] The latter ensures that CS-type one-sender equilibria where an endpoint of Θ (*i.e.* 0 or 1) has its own message are considered coordination-free. They otherwise would not since $\cup_{I \text{ is a cell in } \Gamma} I$ misses the endpoint.

[9] trivially always the only sender in the one-sender case

[10] This is not a full strategy profile: neither the senders' messages nor the receiver's actions after out-of-equilibrium message vectors are specified.

[11] Obviously, given such simple candidate play, there is no other way to assign messages.

[12] For example, $\theta \sim U[0, 1]$, $u_R(a, \theta) = -(a - \theta)^2$, $u_1(a, \theta) = -(a - (\theta - 0.05 + \varepsilon))^2$, $u_2(a, \theta) = -(a - (\theta + 0.2 - \varepsilon))^2$ for small ε yield a small middle interval around $a^\Gamma(H, L) = 0.2$.

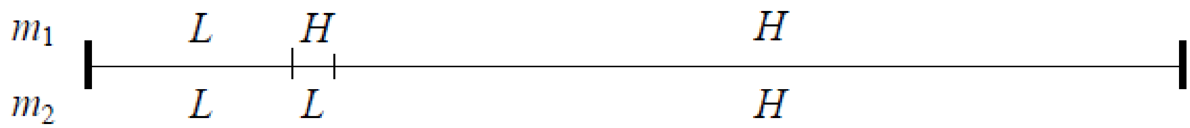

Figure 3: Case a

between (H, L) and (H, H), sender 1 prefers any action smaller than θ_2 to $a^\Gamma(H, H)$. Since $a^\Gamma(H, H) > a^\Gamma(L, L)$, it is impossible for $a^\Gamma(L, H)$ to be simultaneously less than $a^\Gamma(L, L)$ and greater than $a^\Gamma(H, H)$, so once again a deviation is always desired.

Proposition 1 below shows that the above two scenarios, *i.e.* a large cell or highly asymmetric preferences, essentially depict the only reasons why candidate play may not correspond to equilibrium play.

Given candidate play, call an *i-block* a maximal interval of states where all senders other than i only have one message. Clearly, every block consists of entire cells, and an i-block with more than one cell is formed when sender i changes her message at a boundary. In the above example, there are two blocks with two cells (a 1-block with the (L, L) and (H, L) cells and a 2-block with the (H, L) and (H, H) cells), and two blocks with one cell. Note also that any two blocks can only overlap for at most one cell, because every boundary only allows one block to survive.

Proposition 1: Suppose that for all θ, each sender i's utility function is symmetric about its maximum $\theta + b_i(\theta)$. Given candidate play Γ, let k_i^Γ denote the size of the largest i-block, c^Γ denote the size of the largest cell, and let $x_i^\Gamma = k_i^\Gamma + 2(c^\Gamma + \max_{\theta\in\Theta} |b_i(\theta)| - \min_{\theta\in\Theta} |b_i(\theta)|)$. If the sum of the two largest x_i^Γ is less than 1, then there exists an equilibrium where:

- play is described by Γ;
- each player's messages can be ranked so that a weakly higher message is sent at a higher state.

Proof: All omitted proofs can be found in Appendix A.

Proposition 1 implies that usually[13], if senders' preferences are symmetric and if their

[13] Since the overlap between any two i-blocks is at most a cell, the only caveat is that the two largest i-blocks cannot cover nearly the entire state space. This corresponds to a scenario where almost the entire space is comprised of an interval where only one sender is used, and a second interval where only another sender is used.

biases are small, then any sufficiently informative candidate play corresponds to play in an equilibrium where senders' messages are monotonic.[14] The same holds if preferences are slightly asymmetric: in that case, the upper bound on the sum of the two largest x_i^Γ would be slightly lower than 1. Thus, in the case of relatively symmetric preferences and small biases[15], computing reasonably informative coordination-free equilibria is simple.

5 Robustness to Small Disagreement

This section introduces the possibility of small noise in the senders' observations of the state, as considered in the literature. In this paper, it is assumed that senders may have different beliefs about the exact form of the noise. The proposed criterion, *disagreement-robustness*, requires that in such a setting, whenever there is common knowledge that the noise is small, there exists a profile close to the noiseless equilibrium where each player's strategy is nearly optimal. Optimality is required in the interim: given any signal s_i, sender i sends an almost optimal message, and given any message vector m, the receiver takes an almost optimal action.

Theorem 1 in Section 5.3 shows that generic[16] disagreement-robust equilibria are complete[17] and coordination-free, and vice versa. If players have a small tolerance for suboptimality (for example, if not following the noiseless equilibrium carries a small cost), complete coordination-free equilibria always survive disagreement about small noise. By contrast, if a generic equilibrium profile is not coordination-free or complete, then it not only fails to be nearly optimal with small disagreement, but also does not have a close-by profile that is nearly optimal. In this case, small disagreements about noise can lead players to revise their strategies, and the resulting profile may differ significantly from the original noiseless equilibrium.

While disagreement-robust equilibria must be complete, it is possible to accommodate non-complete equilibria by limiting the type of noise considered. *Local robustness*, proposed in Section 5.4, requires robustness only to noise where each sender's signal is close to the actual state with certainty, as opposed to high probability. Theorem 2 shows that with mild restrictions on the receiver's action after out-of-equilibrium message vectors and the form

[14] Clearly, sufficiently informative candidate play exists when the biases are small enough.

[15] Note that the existing literature on fully revealing and almost fully revealing equilibria in one dimension examines exactly these settings.

[16] as defined in Section 5.1

[17] Recall that an equilibrium is called complete if every message vector in $\times_{i=1}^n M_i^\Gamma$ occurs in equilibrium.

of equilibrium, local robustness delivers the same result as disagreement-robustness save for the completeness requirement.

5.1 Strongly Natural Equilibria

Disagreement-robustness does not directly imply that an equilibrium is coordination-free. Rather, it implies that every boundary between cells is "natural" in the sense that coming from either direction, some sender would change her message at the boundary *even if other senders keep sending the same messages past the boundary.* Note that all examples from Section 3 feature boundaries that are not natural. This subsection shows that such equilibria normally are coordination-free equilibria.

The following definition formalizes the idea of natural boundaries.

Definition: Suppose two cells in Γ share a boundary θ, with m sent in the left cell and m' sent in the right cell. Then:

- θ is a *left-natural boundary for sender* i if $\exists m''_i \in M_i^{\Gamma} \backslash \{m_i\}$ such that $u_i(a^{\Gamma}(m_i, m_{-i}), \theta) = u_i(a^{\Gamma}(m''_i, m_{-i}), \theta)$ and $a^{\Gamma}(m_i, m_{-i}) \neq a^{\Gamma}(m''_i, m_{-i})$;[18] if additionally (m''_i, m_{-i}) occurs at some state in Γ, then θ is *strongly left-natural*;
- θ is a *right-natural boundary for sender* i if $\exists m''_i \in M_i^{\Gamma} \backslash \{m'_i\}$ such that $u_i(a^{\Gamma}(m''_i, m'_{-i}), \theta) = u_i(a^{\Gamma}(m'_i, m'_{-i}), \theta)$ and $a^{\Gamma}(m''_i, m'_{-i}) \neq a^{\Gamma}(m'_i, m'_{-i})$;[19] if additionally (m''_i, m'_{-i}) occurs at some state in Γ, then θ is *strongly right-natural*;
- θ is a *(strongly) natural boundary* if it is (strongly) left-natural for at least one sender and (strongly) right-natural for at least one sender.

Thus, a boundary θ is left/right-natural if coming from the left/right, some sender would change her message at θ even if others do not, and it is natural if it is both left- and right-natural. A natural boundary fails to be strongly natural when it is natural only thanks to an out-of-equilibrium message vector. That is, a boundary is strongly natural if, coming from both directions, some sender optimally changing her message at the boundary (when

[18] By single-crossing, optimality of the sender implies that $a^{\Gamma}(m_i, m_{-i}) \neq a^{\Gamma}(m''_i, m_{-i})$ can be replaced by $a^{\Gamma}(m_i, m_{-i}) < a^{\Gamma}(m''_i, m_{-i})$.

[19] By single-crossing, optimality of the sender implies that $a^{\Gamma}(m''_i, m'_{-i}) \neq a^{\Gamma}(m'_i, m'_{-i})$ can be replaced by $a^{\Gamma}(m''_i, m'_{-i}) < a^{\Gamma}(m'_i, m'_{-i})$.

others do not) always results in an on-path message vector. Equilibria where all boundaries are strongly natural will be key to the analysis.

Of course, for boundaries between cells to arise, cells must actually exist. Equilibria where almost the entire state space is partitioned into cells are of particular interest.

Definition: An equilibrium Γ has *weak interval structure* if $\lambda(\Theta \backslash \cup_{I \text{ is a cell in } \Gamma} I) = 0$.
Definition: An equilibrium Γ has *interval structure* if $\Theta^{\circ} \subseteq \cup_{I \text{ is a cell in } \Gamma} I$.[20]

Weak interval structure requires that almost all of Θ be in some cell, but allows for certain irregularities. For example, if θ_b is a boundary between two cells, $m^{\Gamma}(\theta_b)$ may not correspond to the message vector sent in either cell. Interval structure rules out these situations. For instance, Example 2 has weak interval structure, but not interval structure since 0.15 and 0.65 are not part of a cell. Example 3 has interval structure.

Definition: An equilibrium is *(strongly) natural* if it has interval structure, and every boundary between two cells is (strongly) natural.

The remainder of this subsection relates strongly natural equilibria to coordination-free equilibria. Proposition 2 shows that one direction of this relation is simple.

Proposition 2: Every coordination-free equilibrium is strongly natural.
Proof: If an equilibrium Γ is coordination-free, then at every boundary θ, $u_i(a^{\Gamma}(m_i, m_{-i}), \theta) = u_i(a^{\Gamma}(m_i', m_{-i}), \theta)$ for some sender i, where (m_i, m_{-i}) is sent in the left cell and (m_i', m_{-i}) is sent in the right cell. Moreover, by definition, $a^{\Gamma}(m_i, m_{-i}) \neq a^{\Gamma}(m_i', m_{-i})$ and both actions occur in Γ, so θ is a strongly natural boundary. Thus Γ is strongly natural. ■

Unfortunately, the converse of Proposition 2 does not hold: some strongly natural equilibria are not coordination-free. However, the argument below shows that such equilibria exist only "by coincidence." For ease of exposition, assume in the rest of this subsection that the endpoints of the state space are in fact part of cells.

First, Lemma 1 gives some basic properties of strongly natural equilibria.

Lemma 1: (a) In any natural equilibrium Γ, $\theta^{\Gamma}(m)$ is connected for all m.

[20] Again, allowing the endpoints of Θ to be outside a cell enables all one-sender equilibria to be deemed to have interval structure.

(b) Given $\eta > 0$, there is a finite upper bound on number of cells in strongly natural equilibria.

The intuition for Lemma 1a is simple: by the definition of natural boundary, m ceases to be optimal at the endpoints of any cell where m is sent. Single crossing then implies that $\theta^{\Gamma}(m)$ is connected. Thus, one desirable property of natural equilibria is that every message vector sent in equilibrium points to an interval of states. The proof of Lemma 1b notes that for every induced action except the leftmost one, there must be another induced action at least η to the left of it such that the message vectors inducing these two actions differ in one component. An inductive argument, starting with the interval $[0, \eta]$, is then used to obtain the result.

Suppose profile Γ is strongly natural; by Lemma 1b, the number of cells is bounded above. Label the cells $0, ..., K$ from left to right, and let m^k denote the message vector sent in cell k. For $k = 1, ..., K$, let:

- θ_k denote the boundary between cell $k-1$ and cell k;
- i_{kR} (resp. i_{kL}) denote a sender for whom the boundary between cell $k-1$ and k is right-natural (resp. left-natural);
- $R_k < k$ be such that $u_{i_{kR}}(a^{\Gamma}(m^k), \theta_k) = u_{i_{kR}}(a^{\Gamma}(m^{R_k}), \theta_k)$ (R_k exists because θ_k is right-natural); and
- $L_k \geq k$ be such that $u_{i_{kL}}(a^{\Gamma}(m^{k-1}), \theta_k) = u_{i_{kL}}(a^{\Gamma}(m^{L_k}), \theta_k)$ (L_k exists because θ_k is left-natural).

Refer to $\{i_{kR}\}_{k=1}^{K}, \{i_{kL}\}_{k=1}^{K}, \{R_k\}_{k=1}^{K}$ and $\{L_k\}_{k=1}^{K}$ as the *structure* of Γ.[21] Because K is bounded above, the number of possible structures is finite.

Refer to $\{i_{kR}\}_{k=1}^{K}$ and $\{R_k\}_{k=1}^{K}$ as a *right-structure*, and to $\{i_{kL}\}_{k=1}^{K}$ and $\{L_k\}_{k=1}^{K}$ as a *left-structure*.

A strongly natural equilibrium's right-structure and θ_1 fully determine the action induced at every state (except at the boundaries θ_k where the induced action may be $a^{\Gamma}(m^{k-1})$ or $a^{\Gamma}(m^k)$). This is because given $a^{\Gamma}(m^{R_k})$ and θ_k, i_{kR}'s utility function and R_k uniquely determine $a^{\Gamma}(m^k)$, and given θ_k and $a^{\Gamma}(m^k)$, the receiver's utility function and the prior

[21] A given profile may be described by more than one structure, as there may happen to be multiple i_{kR}'s or i_{kL}'s at a boundary k.

density uniquely determine θ_{k+1}. θ_1 needs to be such that cell K's right endpoint is at 1. When $R_k = k-1$ for all k, this procedure is simply the forward induction used to compute single-sender and coordination-free equilibria.

Suppose Γ is coordination-free. Here, for all $k = 1, ..., K$, it must be that $R_k = k-1$, $L_k = k$ and $i_{kR} = i_{kL}$. Thus, the left-structure is redundant with the right-structure, and does not impose any additional condition for Γ to be an equilibrium.

By contrast, if a strongly natural equilibrium is not coordination-free, then at some boundary, multiple senders change their message. By Lemma 1a, in a natural equilibrium, no two message vectors that both occur can induce the same action. This implies that any such boundary cannot be both left- and right-natural for the same sender.[22] Thus in this case, the left-structure imposes a supplementary indifference condition. Since the right-structure has already fixed all boundaries and actions, intuitively, extra conditions imposed by the left-structure are only satisfied in special cases.

I now show formally that, given any such equilibrium Γ, generic perturbations to the game eliminate Γ and nearby equilibria of the same structure. The specific perturbations considered are in the size of the state space (*i.e.* a rescaling of utility functions and the prior) and in the senders' preferences. Readers satisfied with the intuition given above can proceed directly to Section 5.2.

Given receiver preference $v(a, \theta)$, sender preferences $u_i(a, \theta)$ and prior $f(\theta)$, consider the *x-scaled game* where $\Theta = [0, x]$, with preferences $v(a, \theta)$, $u_i(a, \theta)$ and prior $\frac{f(\theta)}{F(x)}$. Note that this is equivalent to the game where $\Theta = [0, 1]$, the receiver preference is $v(xa, x\theta)$, sender preferences are $u_i(xa, x\theta)$ and the prior is $f(x\theta)\frac{x}{F(x)}$.

Definition: Given a game, an event A occurs *for generic scales* if $\lambda(\{x \in (0, 1) : A$ does not occur in the x-scaled game$\}) = 0$.

The first part of the argument for Proposition 3 below shows that given a right-structure, the set of sizes of the leftmost cell yielding a profile with the said right-structure has measure 0 for generic scales. That is, generically, the set of equilibria with a given right-structure is small. In the uniform-quadratic case, this step is not needed (and all later references to generic scales can be dropped) because there is a one-to-one relation between the size of the leftmost cell and the right endpoint of the rightmost cell.

[22] Otherwise, we would have $u_i(a^\Gamma(m^k), \theta_k) = u_i(a^\Gamma(m^{R_k}), \theta_k)$ and $u_i(a^\Gamma(m^{k-1}), \theta_k) = u_i(a^\Gamma(m^{L_k}), \theta_k)$, with $a^\Gamma(m^{R_k}) < a^\Gamma(m^{k-1}) < a^\Gamma(m^k) < a^\Gamma(m^{L_k})$. This is not possible by single-peakedness.

Next, consider generic perturbations to the senders' utility functions. A generic set of perturbations is defined as the set of almost all utility functions that satisfy the same basic properties as utilities in the game, and that differ slightly from the unperturbed utility functions on a fixed set of actions and states. "Almost all" is defined in the sense that there is a small set of points that the graph of the perturbed utilities does not pass through. In the proof of Proposition 3, these points form a set that the graph of utilities must intersect in order for a nearby equilibrium with the same structure to exist. This set is small because, by the first part of the proof, the set of equilibria with a given right-structure is small.

Definition: A *generic set of ε-perturbations* is the set of all sender utility functions $v_i(a, \theta)$ such that, for some $\varepsilon > 0$, $\{a_i, b_i, x_i, y_i\}_{i=1}^n$ where $y_i > x_i$ and $b_i > a_i$ and sets $S_i \subset \mathbb{R}^3$ with one-dimensional Hausdorff measure 0:

- $v_i(a, \theta)$ is single-peaked, Lipschitz continuous and satisfies single-crossing;
- $v_i(a, \theta) = u_i(a, \theta)$ whenever $\theta \notin (x_i, y_i)$ or $a \notin (a_i, b_i)$;
- $|v_i(a, \theta) - u_i(a, \theta)| < \varepsilon$ whenever $(a, \theta) \in (a_i, b_i) \times (x_i, y_i)$;
- for each i, the intersection of the graph of $v_i(a, \theta)$ with S_i is empty.

For an equilibrium to be deemed generic, there must be an equilibrium with the same structure and similar size of the first interval that survives both scale and preference perturbations.

Definition: A strongly natural equilibrium Γ with $\theta_1^\Gamma = x$ is *non-generic* if for generic scales, either:

- no equilibrium that can be described by the same structure as Γ exists; or
- for any $\gamma > 0$, there exists a generic set of ε-perturbations P with $\varepsilon \in (0, \gamma)$ for which $\exists \delta > 0$ such that no equilibrium Γ' with $\theta_1^{\Gamma'} \in (x - \delta, x + \delta)$, where Γ and Γ' can be described by the same structure, exists under any perturbation in P.

Proposition 3 shows that non-coordination-free equilibria are non-generic by considering a set of perturbations that affects the left-structure, but not the (non-redundant) right-structure. It shows that for a given size z of the leftmost cell, there is a player whose perturbed utility must pass through a particular point for an equilibrium with the designated structure to exist. Because the set of admissible sizes of the leftmost cell has measure 0 for generic scales, it follows that for generic perturbations, equilibria with the desired structure and size of the left cell near z do not exist.

Proposition 3: All generic strongly natural equilibria are coordination-free.

5.2 Noise

Throughout this section, it is assumed that given state θ, each sender i independently observes signal $s_i \in \Theta$, whose density is measurable on $\Theta \times \Theta$. In the noiseless game, we simply have $s_i = \theta$. The following definition, inspired from the Ky Fan metric, formalizes what is meant by "small noise."

Definition: Noise has *size less than* ε if:

1. for all $i \in N$ and $\theta \in \Theta$, $\Pr(|s_i - \theta| < \varepsilon|\theta) \geq 1 - \varepsilon$; and
2. for all $i \in N$ and $s_i \in \Theta$, $\Pr(|s_i - \theta| < \varepsilon|s_i) \geq 1 - \varepsilon$.

Noise is small when at any state, each sender's signal is close to the state with high probability, and when after any signal, each sender puts a high probability on the state being close to the signal. Note that this definition does not rule out the presence of atoms in the noise distribution. For example, the class of noise sequences considered by Battaglini (2002), where each sender observes the true state with probability approaching 1 and observes the realization of a continuous random variable otherwise, has size converging to 0, as does a sequence of noise along which senders observe the closest element of a finer and finer finite grid. A sequence of noise with size converging to 0 converges in probability to the trivial signal structure $s_i = \theta$.

5.3 Disagreement-Robustness

Even when all players agree that noise is small, they may differ slightly in their assessment of the noise. For example, a sender may believe that she is a little more accurate than her peers, and a receiver may believe that in some particular instance, certain senders are likely to slightly overestimate the state while others are liable to underestimate it. An equilibrium that can withstand small disagreements is therefore particularly appealing.

The definition of disagreement-robustness proposes that an equilibrium Γ "withstands small disagreements" if, given common knowledge that noise is small enough, there is a "nearby" profile Γ' where it is common knowledge that:

(i) every sender's strategy is almost optimal given any signal; and

(ii) the receiver's strategy is almost optimal given any message vector.[23]

Thus, Γ' is an interim δ-equilibrium for small δ.

If Γ has interval structure and finitely many messages, Γ' is considered close to Γ if for signals far enough away from any cell boundaries, senders' messages in Γ' are the same as in Γ, and the receiver's actions after each message vector are close in Γ and in Γ'; the definition is more involved if Γ has a more complex structure.

The following formally define "almost optimal" strategies and "nearby" profile.

Definition: Player i's strategy r_i is a *δ-best response* to opponent strategies r_{-i} if after any history h_i (signal s_i if i is a sender and message vector if i is the receiver), $E[u_i(r_i, r_{-i})|h_i] \geq E[u_i(BR_i(r_{-i}), r_{-i})|h_i] - \delta$.

The sense in which a strategy is almost optimal is simply that it is a δ-best response for small δ.

Definition: Given a profile Γ, profile Γ' is *δ-close to* Γ if:

1. $M_i^{\Gamma'} = M_i^{\Gamma}$;

2. $m_i^{\Gamma}(s_i) = m_i^{\Gamma'}(s_i)$ whenever $[s_i - \delta, s_i + \delta]$ is contained in a cell in Γ;

3. $\exists \varepsilon > 0$ such that the receiver's best response to $\{m_j^{\Gamma}\}_{j=1}^n$ and $\{m_j^{\Gamma'}\}_{j=1}^n$ given noise Ξ, respectively a^{Ξ} and $a^{\Xi\prime}$, satisfies $|a^{\Xi}(m) - a^{\Xi\prime}(m)| < \delta$ whenever the size of Ξ is less than ε, for all $m \in \times_{i=1}^n M_i^{\Gamma}$;[24] and

4. $|a^{\Gamma}(m) - a^{\Gamma'}(m)| < \delta$ for all $m \in \times_{i=1}^n M_i^{\Gamma}$.

[23] Almost optimality of a player's strategy is evaluated using that player's belief about the noise.

[24] Only Ξ such that $a^{\Xi}(m)$ and $a^{\Xi\prime}(m)$ are well-defined are considered.

Points 1 and 4 in the definition of δ-closeness simply require that the senders use the same messages in Γ' as in Γ, and that the receiver takes a nearby action after every message vector. Point 2 restricts the senders' strategies by requiring the use of the same messages in Γ and Γ' in cells and at least δ away from boundaries. However, this condition has no power when dealing with sender strategies that do not feature intervals: it is difficult to directly determining whether two sender profiles with complicated structures are "close."[25] Point 3 addresses this issue by using the receiver's best response to evaluate how close sender profiles are to each other. Noise is used because, in some cases, two sender profiles could generate the same receiver actions without noise while generating far apart actions with noise; such profiles ought to be considered distinct. Note that when δ is taken arbitrarily small, as it will be in the definition of disagreement-robustness, the power of point 3 is limited to cases where Γ has an "unusual" structure. Indeed, when weak interval structure is satisfied and each message vector is sent on a set of positive measure, there always exists $\delta' \leq \delta$ such that if points 1, 2 and 4 are satisfied for δ', then point 3 is satisfied for δ. By contrast, it will be shown that if Γ does not have weak interval structure, then the receiver's optimal action after some message vector must be very sensitive to arbitrarily small noise. Point 3 restricts the set of profiles considered close to such equilibria by requiring that in any such profile, the receiver's optimal action responds to noise in a similar way.[26]

The table is now set to state the definition of disagreement-robustness.

Definition: An equilibrium Γ in the noiseless game is *disagreement-robust* if for all $\delta > 0$, there exists $\varepsilon > 0$ such that, whenever there is common knowledge that noise has size less than ε, there exists a δ-close strategy profile Γ' where:

1. $m_i^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j \in N \setminus \{i\}}$ and $a^{\Gamma'}$ evaluated under sender i's belief about the noise;
2. $a^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j=1}^{n}$ evaluated under the receiver's belief about the noise; and

[25] For example, suppose that within some interval, strategy m_i^{Γ} assigns m_i within the set of irrational numbers and m_i' elsewhere. Strategy $m_i^{\Gamma'}$ is identical to m_i^{Γ} everywhere except on the said interval, where it assigns m_i within the set of transcendental numbers and m_i' elsewhere. It is unclear by simple inspection how "close" m_i^{Γ} and $m_i^{\Gamma'}$ should be considered.

[26] Also note that by the Lipschitz continuity of the receiver's utility in her action, one can restate point 3 in terms of δ-best responses rather than exact best responses. Then although the smallest δ that makes the definition hold for a profile may change, it would not vary by more than a certain factor. This modification would therefore be immaterial in the context of disagreement-robustness.

3. points 1 and 2 are common knowledge.[27]

Such a strategy profile will be referred to as a *δ-supporting profile*.

Note that Γ' is an interim δ-equilibrium where each player's payoffs are evaluated under her own beliefs. One interpretation for requiring approximate best responses instead of exact ones is that players incur a small cost when deviating from their noiseless plan of action. For example, if a player is a committee, it may not be worth convening a meeting to determine a new strategy when an event makes the original messages slightly suboptimal on a small set of signals (for senders) or only slightly changes the optimal actions (for the receiver). Alternatively, the noiseless equilibrium may represent a widely followed convention; if no matter what a player's particular assessment of the small noise is, she has little to gain by drawing up a new strategy, then she may well decide to forego computing her optimal course of action, both *ex ante* and in the interim. Thus, an equilibrium is disagreement-robust when for any positive tolerance of suboptimality, given any noise appraisals close enough to zero, everyone can be expected to play a strategy close to her noiseless equilibrium strategy.

Another interpretation of δ-best responses is that players may not exactly know other players' utilities. Thus, players may assume that others will play some strategy close to the noiseless one (and best-respond to such a strategy) unless she knows they have a strong reason not to.[28]

Theorem 1 provides necessary and sufficient conditions for disagreement-robustness for generic equilibria.

Theorem 1:

(a) If an equilibrium Γ is disagreement-robust, then it is natural and complete (and therefore strongly natural).

(b) If an equilibrium Γ is coordination-free and complete, then it is disagreement-robust.

[27] Point 3 does not have an impact on Theorem 1. Indeed, points 1 and 2 are sufficient for ruling out equilibria that are not natural. Furthermore, given δ, for ε small enough, any complete and coordination-free equilibrium Γ is always its own δ-supporting profile. It is therefore trivial that point 3 holds whenever it is common knowledge that noise has size less than ε. I leave it in the definition to emphasize this property of coordination-free equilibria. A similar observation holds for Theorem 2.

[28] See Jackson *et al.* (2010) for a general discussion of ε-equilibria. Their main result, that selection criteria based on ε-equilibria of perturbed games have no power, does not apply to the setting considered here: interim optimality with a continuum of types and discontinuous strategies. It does highlight that conditions similar to disagreement-robustness are likely to be weak when applied to settings with finitely many types.

Given that strongly natural equilibria that are not coordination-free are non-generic, Theorem 1 states that disagreement-robustness essentially selects coordination-free and complete equilibria. Note that all equilibria in the one-sender model are disagreement-robust. Section 5.4 provides a result that does not require completeness by restricting the set of allowable noise.

The intuition behind Theorem 1a has two main parts. First, a disagreement-robust equilibrium must have weak interval structure. The idea is that otherwise, the receiver's optimal response to some message vectors would be very sensitive to small noise.

Note that given a profile Γ and a δ-supporting profile Γ', a^{Γ} must be close to $a^{\Gamma'}$, which is close to the best response to $\{m_j^{\Gamma'}\}_{j=1}^n$ under noise. By point 3 in the definition of δ-closeness, the latter is close to the best response to $\{m_j^{\Gamma}\}_{j=1}^n$ under noise. Therefore, the receiver's best response to $\{m_j^{\Gamma}\}_{j=1}^n$ with no noise (which is simply a^{Γ}) must be close to her best response to the same sender strategies with any small noise. Now, if Γ does not have weak interval structure, then the set of states U that do not belong to any cell in Γ has positive measure. Some message vector m is therefore sent on a subset of U with positive measure.[29] However, because any state within that subset has arbitrarily close states where m is not sent, there exists arbitrarily small noise such that the receiver puts negligible weight on those states. For such noise, the optimal action after m differs significantly from $a^{\Gamma}(m)$, which means that Γ could not be receiver-robust.

The second part of the intuition is that equilibria where a boundary is not natural are prone to unraveling: if a sender thinks that other senders will use a slightly different location for an unnatural boundary between two messages, then she should coordinate and "move her boundary" as well. But then, due to disagreement, all senders may believe that all others will erroneously move their boundaries by ε. Then each sender, wishing to coordinate, will move her own boundary by ε. This process could then continue until the boundary is more than δ away.

The proof executing the latter argument proceeds as follows. Suppose a boundary θ_b is not left-natural and consider the following beliefs about noise: each sender believes that she observes the true state while all other senders observe $s_i = \max\{\theta - \varepsilon, 0\}$, the receiver believes that all senders observe $s_i = \theta$, and these beliefs are common knowledge. Let m be the message vector sent to the left of the boundary. Then in a δ-supporting profile, for δ small enough, m must be sent in a neighborhood to the left of $\theta_b - \delta$. Given the beliefs and

[29] The proof shows that every message vector in $\times_{i=1}^n M_i^{\Gamma}$ is sent on a set of positive measure if $|\{i \in N : |M_i^{\Gamma}| \geq 2\}| \geq 2$.

for ε small enough, m must then also be sent between $\theta_b - \delta$ and $\theta_b - \delta + \varepsilon$: upon observing a signal in that range, each sender believes opponents will send m_{-i}, and in turn must send m_i, their best response by at least δ for δ small enough. Because θ_b is not left-natural, this argument can be iterated past $\theta_b + \delta$, which means that no δ-supporting profile can exist. This argument implies that disagreement-robust equilibria must be natural, and therefore strongly natural because they are complete.

Theorem 1b is simple: any coordination-free equilibrium Γ is its own δ-supporting profile. The key for this result is that at all states, any sender i's message in Γ when there is no noise is either optimal or very close to being optimal for all m_{-i} occurring in a small enough neighborhood - there is no strong reason to coordinate with other senders.[30] Thus, every sender follows a δ-best response by playing the original profile as long as she believes that noise is small. Intuitively, the same goes for the receiver since every message vector is sent exactly in a cell, and the minimum measure of cells in any coordination-free Γ is strictly positive. Finally, away from boundaries, senders' strategies are strictly optimal when there is no noise, and remain so with small noise as other senders send the same messages with high probability. All of these facts are common knowledge since it is common knowledge that ε is small.

5.4 Local Robustness

Completeness guarantees that all actions following message vectors in $\times_{i=1}^{n} M_i^{\Gamma}$ are determined by equilibrium play. While this property may improve the robustness of an equilibrium, some plausible equilibria do not have it. For example, profiles with 2 senders, 2 messages per sender, and 3 intervals where messages are (L, L),(H, L) and (H, H) are ruled out. Disallowing out-of-equilibrium message vectors in $\times_{i=1}^{n} M_i^{\Gamma}$ may therefore be too restrictive. This subsection provides modifications to disagreement-robustness in order to accommodate equilibria with out-of-equilibrium message vectors.

Disagreement-robust equilibria are complete because the receiver's optimal action after any out-of-equilibrium[31] message vector is very sensitive to noise. However, some out-of-equilibrium message vectors cannot arise if the support of the noise lies entirely near the true state. In the above example, (L, H) remains out-of-equilibrium with such noise because

[30] Note that the latter necessarily fails in any equilibrium that is not natural.

[31] in the noiseless game

all states where sender 1 sends L are far from states where sender 2 sends H. The following definition of small noise restricts attention to this type of noise:

Definition: Noise has *local size less than* ε if for all $i \in N$ and $\theta \in \Theta$, $|s_i - \theta| < \varepsilon$.

Clearly, all noise with local size less than ε have size less than ε. Local robustness considers only noise with local size less than ε, and is otherwise identical to disagreement-robustness:

Definition: An equilibrium Γ in the noiseless game is *locally robust* if for all $\delta > 0$, there exists $\varepsilon > 0$ such that, whenever there is common knowledge that noise has local size less than ε, there exists a δ-close strategy profile Γ' where:

1. $m_i^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j \in N \setminus \{i\}}$ and $a^{\Gamma'}$ evaluated under sender i's belief about the noise;
2. $a^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j=1}^{n}$ evaluated under the receiver's belief about the noise; and
3. points 1 and 2 are common knowledge.

The following assumption is imposed on the receiver's action following out-of-equilibrium message vectors.

Assumption A: Suppose equilibrium Γ has an out-of-equilibrium vector $m \in \times_{i=1}^{n} M_i^{\Gamma}$. Then $\exists \gamma > 0$ such that all otherwise identical profiles Γ' where $a^{\Gamma'}(m) \in [a^{\Gamma}(m) - \gamma, a^{\Gamma}(m) + \gamma]$ are also equilibria.

Assumption A requires that the equilibrium be robust to small changes in actions following out-of-equilibrium message vectors. Given that those actions are not determined by play, equilibria that do not survive small changes in them can reasonably be considered fragile. Note that on its own, Assumption A is not powerful: for example, it does not rule out fully revealing equilibria (when the bias is small relative to the state space) supported by the receiver choosing a far away action in response to diverging messages, or any of the examples from Section 3.[32]

[32] Consider sender trembles to messages within ε of being optimal. Requiring that for ε small enough, such trembles do not lead to the occurrence of an out-of-equilibrium $m \in \times_{i=1}^{n} M_i^{\Gamma}$ would imply Assumption A.

The main role of Assumption A is to ensure that all natural equilibria are strongly natural, *i.e.* that profiles are not "artificially" natural because the receiver's action after an out-of-equilibrium message vector induces a sender to want to change her message (even when others do not) exactly at a boundary.

Lemma 2: Any natural equilibrium satisfying Assumption A is strongly natural.

Proof: Suppose not, and assume without loss of generality that θ_b is left-natural but not strongly left-natural for sender i. Let m be sent to the left of θ_b. Then $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) = u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$, for some $m_i' \in M_i^\Gamma$, but (m_i', m_{-i}) is an out-of-equilibrium vector. By single-crossing, decreasing $a^\Gamma(m_i', m_{-i})$ infinitesimally would make m_i' strictly optimal immediately to the left of θ_b, where i is supposed to send m_i, which violates Assumption A. □

Another issue that arises from restricting attention to local noise is that local noise fails to rule out message vectors m that are sent at only one state θ (thus fully revealing it): it could be that away from θ, local noise never results in m. As explained later, the proof of Theorem 2 shows that there cannot be a positive measure of such fully revealed states in a locally robust equilibrium. Assumption B rules out the remaining possibility, that a measure 0 but nonempty set of states is fully revealed.

Assumption B: If the set of fully revealed states in Θ° has measure 0, then it is empty.

Once again, on its own (or combined with Assumption A), Assumption B does not rule out fully revealing equilibria or any example from Section 3. It does rule out cases where only a select few states are fully revealed, which is intuitive since no aspect of the game highlights any states in particular.[33]

Theorem 2 uses Assumptions A and B to derive the counterpart of Theorem 1 for local robustness.

Theorem 2: Suppose an equilibrium Γ satisfies Assumptions A and B.

(a) If Γ is locally robust, then it is strongly natural.

What if one requires that for any such tremble, the receiver's optimal action remains close? As ε gets small, trembles leading to any given out-of-equilibrium $m \in \times_{i=1}^n M_i^\Gamma$ can then only occur within a small interval: otherwise, different trembles would lead to very different optimal responses for the receiver. This also implies Assumption A, except in the case where Γ specifies, for an out-of-equilibrium m, that $a^\Gamma(m) = \theta$, that some sender i is indifferent between $a^\Gamma(m)$ and $a^\Gamma(m_i', m_{-i})$ at θ, and that (m_i', m_{-i}) be sent arbitrarily close to θ.

[33] For example, there are no atoms in the prior, utility functions are continuous, etc.

(b) If Γ is coordination-free, then it is locally robust.

The proof of Theorem 2b is the same as that of Theorem 1b: with local noise, no out-of-equilibrium message vectors from a coordination-free equilibrium Γ can occur in the noisy game when all senders follow Γ, which as before is its own δ-supporting profile.

To show Theorem 2a, however, the proof of Theorem 1a needs to be adapted because, as mentioned earlier, the argument from Theorem 1a does not rule out message vectors occurring at a single state. Therefore, to show that locally robust equilibria Γ have weak interval structure, it is necessary to rule out the possibility that full revelation occurs on a set of states with positive measure. For this to happen, it must be that some sender i has a continuum of equilibrium messages. Then for any $\varepsilon > 0$, there must be an interval of size ε where i has a continuum of equilibrium messages as well. Therefore, for any $\varepsilon > 0$, there is a continuum of message vectors, where senders other than i send the same vector m_{-i}, that can arise under some local noise less than ε. The argument uses Assumption A to show that for ε small enough, these message vectors must occur on-path in noiseless Γ. But, by the argument from Theorem 1a, each of them is sent either at one state or on a set of positive measure. Since there can be at most finitely many of the former (since actions after message vectors containing the same m_{-i} must be separated by at least η) and countably many of the latter, a contradiction is reached.

The step showing that unnatural boundaries are vulnerable to unraveling carries through thanks to Assumption A. Finally, Assumption B implies that Γ has interval structure, and is therefore strongly natural by Lemma 2.

6 Stage-Unique Equilibria

This section presents another desirable property of coordination-free equilibria. Even when there is no noise, coordination among senders can be an issue: at some states, there may be multiple equilibria for senders' play, even fixing the receiver's strategy. When the set of such states has positive measure in a cheap talk equilibrium, then an exogenous event leading the senders to review their strategies[34] can easily move play to a new equilibrium by changing the receiver's optimal action after certain message vectors.

[34] For example, an unexpected meeting among senders may occur, or there could be a "change of culture" among senders.

By contrast, if the receiver's strategy induces a unique equilibrium for senders at almost all states, then senders "know what to play" almost everywhere, and the receiver can credibly commit to her strategy: it remains optimal as long as senders play an equilibrium amongst themselves. This section shows that complete equilibria in the cheap talk game with such a property essentially correspond to coordination-free equilibria. A weaker criterion admitting all non-complete coordination-free equilibria is then considered.

Definition: Let Γ is an equilibrium of the cheap talk game. The (Γ, θ)-*sender game* is a normal-form game where:

- the set of players is N;
- sender i's pure strategy space is M_i^Γ; and
- sender i's payoff is $u_i(a^\Gamma(m_1, ..., m_n), \theta)$.

Definition: An equilibrium Γ is *stage-unique* if for **almost** every $\theta \in \Theta$, the (Γ, θ)-sender game has a unique Nash equilibrium (*i.e.* $\lambda(\{\theta \in \Theta :$ the (Γ, θ)-sender game does not have a unique Nash equilibrium$\}) = 0)$.

Proposition 4: Suppose a complete equilibrium Γ satisfies Assumption B. Then Γ is stage-unique if and only if it is strongly natural.[35]

Because generic strongly natural equilibria are coordination-free, Proposition 4 says that generic complete equilibria (where no state is fully revealed) satisfying stage-uniqueness are coordination-free, and any complete coordination-free equilibrium is stage-unique.[36]

The proof of Proposition 4a starts by showing that in a complete stage-unique equilibrium, the set of inducible actions does not have an accumulation point. If it did have such a point, then infinitely many message vectors would be optimal on sets smaller than η, since by single-crossing, these sets must be (possibly trivial) intervals. The left endpoints of such intervals must correspond to states where the message vector ceases to be optimal for a left-biased sender, so the sender can induce an action at least η to the left of the endpoint.

[35] In fact, if Γ satisfies Assumptions A and B and is stage-unique, then it is strongly natural (completeness is not required). The proof in that case is very similar to the proof of Proposition 2a, and is therefore omitted.

[36] The set of non-complete coordination-free equilibria contains both ones that are stage-unique and ones that are not.

The proof then shows that this implies another accumulation point at least η to the left of the original one. Furthermore, by completeness[37], there is a sequence of actions induced in equilibrium converging to the new accumulation point. The argument can then be repeated until it reaches a contradiction by virtue of an accumulation point being required outside the state space.

The above implies that the number of actions induced in equilibrium is finite. Furthermore, stage-uniqueness requires that at most two message vectors can induce the same action: otherwise, the sets on which certain message vectors are optimal would have to overlap for a positive measure. Therefore, each player has a finite number of equilibrium messages. The fact that, for each message vector m, the measure of $\theta^{\Gamma}(m)$ must equal the measure of the set where m is the Nash equilibrium of the sender game implies weak interval structure. The proof shows that states remaining outside cells must be fully revealed, so Assumption B implies interval structure. Boundaries between cells must be natural because otherwise, there would be an interval on the left and/or right of the boundary where the message vectors from both cells are equilibria of the sender game.

As noted previously, natural and complete equilibria are strongly natural, and generic strongly natural equilibria are coordination-free. Thus, the following statements are roughly equivalent:

- Γ is disagreement-robust;
- Γ is complete and stage-unique, and satisfies Assumption B;
- Γ is complete and coordination-free.

6.1 Weak Stage-Uniqueness

As with disagreement-robustness, it is possible to weaken stage-uniqueness in order to accommodate non-complete equilibria.

Definition: An equilibrium Γ is *weakly stage-unique* if for **almost** every $\theta \in \Theta$, among message vectors occurring on the equilibrium path in Γ, the (Γ, θ)-sender game has a unique Nash equilibrium.

[37] or Assumption A

Weak stage-uniqueness is a sensible concept in situations where senders have a good reason not to coordinate on an out-of-equilibrium message vector, such as when the receiver has an outside action giving low payoffs to senders, and threatens its use after such message vectors. For example, the receiver may have the ability to fire senders, and may want to do so after observing unexpected messages.

Assumption A, introduced in conjunction with local robustness, is useful here as well: it ensures that the set of states where a message vector constitutes a Nash equilibrium of the sender game does not end due to out-of-equilibrium message vectors. With Assumption A, all arguments used to prove Proposition 4 carry through, yielding the following result.

Proposition 5: Suppose an equilibrium Γ satisfies Assumptions A and B. Then Γ is weakly stage-unique if and only if it is strongly natural.

7 Discussion

7.1 Best Coordination-Free Equilibrium for the Receiver

Broadly speaking, in a coordination-free equilibrium, the receiver wants to minimize the size of intervals. Intuitively, this implies that if all senders are biased in the same direction, the best coordination-free equilibrium involves only the sender with the smallest bias being informative.[38] This conclusion does not always hold exactly, however. For example, if there are two senders with slightly different biases, an equilibrium Γ' where the more biased sender changes her message at some boundaries may be better for the receiver than the most informative equilibrium Γ where only the less biased sender is used. This is because, if the difference in biases is sufficiently small, Γ' may feature as many intervals as Γ. The size of the intervals would differ in the two equilibria, so if the receiver puts a greater weight on the states where Γ' features smaller intervals, she may prefer Γ' to Γ.

A similar conclusion holds if there are senders biased in both directions: using two senders, the least biased from each direction, would allow the receiver to approach her best outcome. The senders would be used in such a way that the size of the intervals is kept low. If the senders' preferences are not too asymmetric, starting from the left, at every boundary θ between cells C on the left and C' on the right, the left-biased sender is made

[38] Assuming that senders' preferences are such that the magnitude of their biases can be ranked.

indifferent when feasible;[39] otherwise, the right-biased sender is made indifferent. Note that C' is smaller in the first case than in the second when the senders' preferences are not too asymmetric. Once again, depending on the exact parameters of the game, it may be better to use slightly more biased senders at certain boundaries. Parameters of the game also dictate the order in which the two (or more) senders are used.

The above observations show that despite the lack of an exact general form for the best coordination-free equilibrium for the receiver, the conclusions below follow:

1. If the receiver starts with one sender, then consulting an additional sender biased in the same direction is only likely to help if the second sender is not much more biased than the first, and will not help much unless the second sender is less biased than the first. This implication contrasts sharply with results from the existing literature, where as long as both senders' biases are small relative to the state space, the receiver can achieve or approach full revelation, even if the second sender's bias is much larger than the first's.

2. On the other hand, adding a second sender biased in the opposite direction benefits the receiver, unless this sender's bias is so large that she is never used. This concords with the idea that it is valuable to consult experts with conflicting standpoints.

3. Having a large number of senders helps mainly insofar as the smallest bias in each direction is brought down. Full revelation is not approached if there are lower bounds on the biases of every sender in the population. Of course, this result arises because noise is negligible in the setting considered here; it is not surprising that bringing in "many similar people" helps little when any one of them essentially knows that state.

7.2 Multi-Dimensional State Space

Battaglini (2002) shows that with an unbounded state space and multiple dimensions, there is an intuitive construction yielding full revelation: senders report information in the direction of common interest with the receiver. However, Ambrus and Takahashi (2008) point out that these equilibria, or indeed any fully revealing equilibrium satisfying a continuity property, often do not exist when the state space is bounded. Indeed, without strong functional form assumptions, little is known about the forms of equilibria with a bounded multi-dimensional state space even with *one* sender: boundary conditions make any construction very difficult.

[39] For example, suppose the left-biased sender has bias $-b$ in a uniform-quadratic setting. Then she can be made indifferent at the boundary only if C has size greater than $4b$. Otherwise, at θ, she prefers inducing the midpoint of C to inducing anything to the right of θ.

Given the above issues, it may be premature to seek an equilibrium refinement in multi-dimensional cheap talk games. Still, the idea that a sender's messages from both sides of a boundary should be near optimal for all combinations of other senders' message that can occur locally can generate intuitive restrictions. For example, in a two-dimensional case with two players, intersecting boundaries have this property. If the message vectors sent in the four quadrants are, going clockwise, $(m_1, m_2), (m'_1, m_2), (m'_1, m'_2), (m_1, m'_2)$, then regardless of whether sender 2 plays m_2 or m'_2, both m_1 and m'_1 are near optimal for sender 1 around the intersection point. On the other hand, if the boundaries coincide for some non-trivial length, then the same property may not hold.

However, given the difficulty of characterizing such equilibrium profiles (or even determining their existence), this direction is not pursued further.

8 Conclusion

This paper has shown that complete and coordination-free equilibria remain interim almost optimal for all players whenever there is common knowledge of small noise in senders' observation of the state. Any other generic equilibrium does not, and furthermore fails to have a nearby interim almost-optimal profile. By limiting the class of allowed noise to local noise, one can dispense with the completeness requirement. Finally, an alternative set of conditions relating to coordination among senders without noise leads to a similar conclusion.

As mentioned in Section 4, coordination-free equilibria have a similar structure to one-sender equilibria, in that the size of the first interval and the identity of the indifferent sender at each boundary determine play. This property implies that fixing senders' biases and increasing the size of the state space cannot make the size of the revealed intervals vanishingly small. The amount of information loss therefore remains nontrivial, unlike in the fully revealing and almost fully revealing equilibria examined by the existing literature.

Appendix A: Proofs

Proof of Proposition 1: Given candidate play Γ, assign messages as follows: in the leftmost cell, all senders send 1, and at every boundary where a sender's message changes, that sender's message increases by 1. Note first that this message assignment rules out the following scenario: in a cell where the assigned message vector is $m = (m_1, ..., m_n)$, a sender (without loss of generality, sender 1) wants to deviate to m_1', and $m' = (m_1', m_2, ..., m_n)$ occurs on the equilibrium path. To see this, assume without loss of generality that $a^\Gamma(m') > a^\Gamma(m)$. Then it must be that in the cell immediately to the right of the one where m is sent, the message vector is $m'' = (m_1'', m_2, ..., m_n)$ for some m_1'' possibly equal to m_1', so $a^\Gamma(m') \geq a^\Gamma(m'')$. Since within the cell where m is sent, sender 1 prefers $a^\Gamma(m)$ to $a^\Gamma(m'')$, by single-crossing, she also prefers $a^\Gamma(m)$ to $a^\Gamma(m')$ and cannot desire a deviation.

Therefore, the only concern is to place the receiver's actions after off-path message vectors without inducing a deviation. The first useful observation is that for any off-path message vector m, there are at most two senders whose deviation can induce m. To see that, normalize messages by subtracting a constant to each sender's messages such that $m = (0, ..., 0)$. If a sender can induce m by deviating from a negative message when all others send 0, then when that sender sends 0, all other senders' messages must be nonnegative. Thus only one sender can do so. The same holds for deviation from a positive message, so at most two senders can induce m through a deviation.

Now suppose sender i can deviate to induce m. The set of states from which she can do this must constitute an i-block, which has size at most k_i^Γ. It follows that the set of her optimal actions when the state is in the i-block is an interval with size at most $k_i^\Gamma + \max_{\theta\in\Theta} |b_i(\theta)| - \min_{\theta\in\Theta} |b_i(\theta)|$. By preference symmetry, there is an interval with size less than x_i^Γ such that whenever $a^\Gamma(m)$ is outside the interval, i does not wish to induce it from the i-block.[40] The desired result follows. ■

The following definition and Lemma 0 are useful for several later proofs.

Definition: An equilibrium is *(strongly) boundary-natural* if it has weak interval structure and every boundary between two cells is (strongly) natural.

Lemma 0: Suppose that equilibrium Γ is boundary-natural. Then for every $m \in$

[40] For example, suppose i is biased toward the right, and denote the left and right endpoints of the i-block by θ_L and θ_R. Then the right endpoint of the interval must be to the left of $\theta_R + c^\Gamma + 2\max_{\theta\in\Theta} |b_i(\theta)|$, while the left endpoint of this interval must be to the right of $\min\{\theta_L, \theta_L - c^\Gamma + 2\min_{\theta\in\Theta} |b_i(\theta)|\} = \theta_L - c^\Gamma + 2\min_{\theta\in\Theta} |b_i(\theta)|$ (for i to be informative, $c^\Gamma > 2\min_{\theta\in\Theta} |b_i(\theta)|$, and if i is not informative, all of her messages can be assumed to be a single one). Thus the interval has size less than x_i^Γ.

$\times_{i=1}^{n} M_i^\Gamma$ that occurs in at least one cell in Γ, $\theta^\Gamma(m)$ is connected.

Proof: Let R be a set containing exactly one point from every cell in a boundary-natural equilibrium. Then any element of $\Theta^\circ \setminus \cup_{I \text{ is a cell in } \Gamma} I$ is either between two points in R (and therefore a boundary between two cells) or an accumulation point of R. Thus every cell has both a left and a right boundary with other cells.

Suppose that $\theta^\Gamma(m)$ is not connected for some m occurring in a cell, and assume without loss of generality that $m^\Gamma(\theta) = m^\Gamma(\theta') = m$ for some θ in a cell C and θ' to the right of C. Let θ_b be the right boundary of C. Because Γ is boundary-natural, for some sender i, we have $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) = u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$ and $a^\Gamma(m_i, m_{-i}) < a^\Gamma(m_i', m_{-i})$, with $m_i' \in M_i^\Gamma$. But by single-crossing, this implies that at $\theta' > \theta_b$, $u_i(a^\Gamma(m_i, m_{-i}), \theta') < u_i(a^\Gamma(m_i', m_{-i}), \theta')$, so sender i has a profitable deviation at θ'. $\square$

Proof of Lemma 1: (a) By the definition of interval structure, $\theta^\Gamma(m)$ is either empty (and trivially connected) or has positive measure. The latter implies that m occurs in a cell in Γ, so by Lemma 0 (see above), $\theta^\Gamma(m)$ is connected. $\square$

(b) Note that for any m sent in a cell C other than the leftmost one, $a^\Gamma(m) > a^\Gamma(m') + \eta$ for some m' differing from m in one component: otherwise, at the left boundary of C, denoted θ, no sender can be indifferent between $a^\Gamma(m)$ and $a^\Gamma(m')$. This is because, by single-crossing, $a^\Gamma(m) > a^\Gamma(m')$, so by Lemma 1a, $a^\Gamma(m') < \theta$. Thus if $a^\Gamma(m')$ and $a^\Gamma(m)$ are within η of each other, then any sender's ideal point given state θ is either to the right or to the left of both.

It follows that in $(0, \eta]$, there can be at most one inducible action a. The maximum number of actions in $(\eta, 2\eta]$ is then n, since for each sender, a can support at most one induced action in $(\eta, 2\eta]$. Thus at most $n+1$ actions in $(0, 2\eta]$ can be induced. Repetition of this argument implies a finite upper bound for the number of cells. $\square$

Proof of Proposition 3: By the argument in the main text, it is sufficient to show that any strongly natural equilibrium where some boundary is not left- and right-natural for the same sender is non-generic.

For a given right-structure, consider the function $\varphi(\theta_1)$ that computes the right endpoint of cell K given the first boundary θ_1, for $\theta_1 \in [0, 1]$; let $\varphi(x) = 1$ if the right endpoint cannot be determined because it would need to be greater than 1. Note that φ is continuous, so the pre-image $\varphi^{-1}(y)$ must be measurable for all y. It follows that the set of $y < 1$ for which $\lambda(\varphi^{-1}(y)) > 0$ must have measure 0. Thus, for generic scales, any given utility functions and any right-structure, the set of sizes of cell 0 that generate an equilibrium with said right-structure has measure 0.

Consider a strongly natural equilibrium Γ where θ_k is not both left- and right-natural for any single sender. Then for each size of cell 0 generating an equilibrium with the appropriate right-half-structure, the following additional condition must be satisfied: $u_{i_{kL}}(a^{\Gamma}(m^{k-1}), \theta_k) = u_{i_{kL}}(a^{\Gamma}(m^{L_k}), \theta_k)$. This condition is not redundant because the only condition imposed by the right-half-structure at θ_k concerns sender $i_{kR} \neq i_{kL}$.

Now fix a generic scale, and pick δ such that for all pairs of equilibria Γ' and Γ'' with the same structure as Γ and with $\theta_1^{\Gamma'}, \theta_1^{\Gamma''} \in (\theta_1^{\Gamma} - \delta, \theta_1^{\Gamma} + \delta)$, we have $\theta_{k-1}^{\Gamma'} < \theta_k^{\Gamma''} < \theta_{k+1}^{\Gamma'}$. Such $\delta > 0$ exists because all boundaries are continuous in the first boundary. There is then an interval $(x_{i_{kL}}, y_{i_{kL}})$ containing $\theta_k^{\Gamma'}$ for every Γ' with the same structure as Γ with $\theta_1^{\Gamma'} \in (\theta_1^{\Gamma} - \delta, \theta_1^{\Gamma} + \delta)$, such that $(x_{i_{kL}}, y_{i_{kL}})$ does not contain any other boundary of such Γ'. There also exists an interval $(a_{i_{kL}}, b_{i_{kL}})$ containing $a^{\Gamma}(m^{L_k})$ for every such Γ'; note that $a_{i_{kL}} > \theta_{k+1}^{\Gamma'}$ for all such Γ', which implies $a_{i_{kL}} > a^{\Gamma'}(m^{k-1})$ for all such Γ'. It follows that for each such Γ' to exist when perturbing sender i_{kL}'s utility function within $(a_i, b_i) \times (x_i, y_i)$, we must have $u_{i_{kL}}(a^{\Gamma'}(m^{k-1}), \theta_k^{\Gamma'}) = v_{i_{kL}}(a^{\Gamma'}(m^{L_k}), \theta_k^{\Gamma'})$, where u denotes the original utility function and v denotes the perturbed utility function: such perturbations do not affect the forward induction on the right-structure that determines all actions and boundaries. Therefore, whenever the graph of $v_{i_{kL}}$ does not intersect $(a^{\Gamma'}(m^{L_k}), \theta_k^{\Gamma'}, u_{i_{kL}}(a^{\Gamma'}(m^{k-1}), \theta_k^{\Gamma'}))$, equilibrium Γ' with the same structure as Γ does not exist.

Thus, for every possible $\theta_1^{\Gamma'}$, there is a point $p(\theta_1^{\Gamma'}) \in \mathbb{R}^3$ such that if $p(\theta_1^{\Gamma'})$ is not in the graph of $v_{i_{kL}}$, Γ' does not exist. Because the prior density is bounded and all utility functions are Lipschitz continuous, $p(.)$ is also Lipschitz continuous. Then because the set of possible $\theta_1^{\Gamma'}$ has measure 0 in $\mathbb{R}$, the corresponding set $S_{i_{kL}}$ of $p(\theta_1^{\Gamma'})$ has one-dimensional Hausdorff measure 0. The procedure above can be repeated for every i_{kL} that is compatible with Γ (recall that a given equilibrium may be described by multiple structures); for all other i, define (x_i, y_i) in the interior of a cell (and any (a_i, b_i)) so that no right-structure forward induction is affected, and let $S_i = \varnothing$. The resulting $\{a_i, b_i, x_i, y_i, S_i\}_{i=1}^n$, combined with any $\varepsilon > 0$, defines a generic set of perturbations under which no equilibrium with the same structure as Γ with cell 0 size between $\theta_1^{\Gamma} - \delta$ and $\theta_1^{\Gamma} + \delta$ exists. As such a generic set can be determined for each generic scale, by definition, Γ is non-generic. ■

Proof of Theorem 1a: Suppose Γ is disagreement-robust and $|\{i \in N : |M_i^{\Gamma}| \geq 2\}| \geq 2$ (if $|\{i \in N : |M_i^{\Gamma}| \geq 2\}| < 2$, then Γ is trivially natural and complete). The proof proceeds in six steps:

Step 1: For any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with size less than ε, $|a^{\Gamma}(m) -$

$a^{\Xi}(m)| < \delta$ *for all* $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

By the definitions of disagreement-robustness and δ-closeness, we know that for any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with size less than ε, $\exists \Gamma'$ such that:

- $a^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j=1}^{n}$ under Ξ;
- $|a^{\Gamma}(m) - a^{\Gamma'}(m)| < \delta$ for all $m \in M_i^{\Gamma}$; and
- $|a^{\Xi}(m) - a^{\Xi\prime}(m)| < \delta$ where a^{Ξ} and $a^{\Xi\prime}$ respectively denote the receiver's best response to $\{m_j^{\Gamma}\}_{j=1}^{n}$ and $\{m_j^{\Gamma'}\}_{j=1}^{n}$ given noise Ξ.

Note that the first point implies that $\exists \gamma(\delta)$ such that $|a^{\Gamma'}(m) - a^{\Xi\prime}(m)| < \gamma(\delta)$, with $\lim_{\delta \to 0} \gamma(\delta) = 0$ because u_R is continuous and strictly concave in a, and Θ is compact. Therefore, $|a^{\Gamma}(m) - a^{\Xi}(m)| < 2\delta + \gamma(\delta)$ for all $m \in M_i^{\Gamma}$ and Ξ with size less than ε.

Rewriting δ in lieu of $2\delta + \gamma(\delta)$ yields the result. $\Diamond$

Step 1 is used in each of steps 2 through 5.

Step 2: Γ *is complete.*

Suppose instead that $m = (m_1, ..., m_n) \in \times_{i=1}^{n} M_i^{\Gamma}$ does not occur in Γ. Then consider noise Ξ where:

(i) at some $\theta \neq a^{\Gamma}(m)$, each sender i observes $s_i = \theta$ with probability $1 - \varepsilon$, and $s_i = \theta_i$ with probability ε for some θ_i where sender i's message in Γ is m_i;

(ii) at all other states, each sender observes the true state.

Clearly, Ξ has size at most $\varepsilon \frac{D}{d}$. For any ε, θ is the only state at which m can occur in then noisy game, and m can indeed occur at θ. Thus $a^{\Xi}(m) = \theta$. By step 1, taking $\delta < |a^{\Gamma}(m) - \theta|$ implies that Γ cannot be disagreement-robust. $\Diamond$

Implication: By completeness and senders' strict preference between messages yielding the same utility, if $a^{\Gamma}(m_i, m_{-i}) = a^{\Gamma}(m_i', m_{-i})$ for some $m_{-i} \in \times_{j \neq i} M_j^{\Gamma}$ and $m_i, m_i' \in M_i^{\Gamma}$, then $m_i = m_i'$.

Step 3: Every message vector in $\times_{i=1}^{n} M_i^{\Gamma}$ *occurs in* Γ *on a set of positive measure.*

Suppose that $m = (m_1, ..., m_n) \in \times_{i=1}^{n} M_i^{\Gamma}$ occurs on a set of measure 0 in Γ. Then for any θ where $m^{\Gamma}(\theta) = m$, $\exists \theta_0(\theta) \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^{\Gamma}(\theta_0(\theta)) \neq m_i$. Let $i^{\Gamma}(\theta)$ be the smallest such i.

Because at least two senders have at least two equilibrium messages in Γ, by step 2, each sender must send each of her equilibrium messages at a minimum of two states. Let

θ' be such that $m_1^\Gamma(\theta') = m_1$ and $m^\Gamma(\theta') \neq m$, and let $\theta'' \neq a^\Gamma(m)$ be such that $m^\Gamma(\theta'') = (m_1', m_2, ..., m_n)$ for some $m_1' \in M_1^\Gamma$. Consider noise Ξ where:

(i) at states θ where $m^\Gamma(\theta) = m$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i^\Gamma(\theta)$ observes $s_i = \theta_0(\theta)$;

(ii) at state θ'', sender 1 observes $s_i = \theta''$ with probability $1 - \varepsilon$, and $s_i = \theta'$ with probability ε;

(iii) if neither (i) or (ii) applies, the true state is observed.

Clearly, Ξ has size at most $\varepsilon \frac{D}{d}$. At all states θ where $m^\Gamma(\theta) = m$, the receiver will observe m with probability 0; at θ'', she observes m with probability ε; and at all other states, she cannot observe m. Because $\lambda(\theta^\Gamma(m)) = 0$, upon observing m, the receiver puts probability 1 on $\theta = \theta''$.[41] Picking $\delta < |a^\Gamma(m) - \theta''|$ completes the argument. $\Diamond$

Step 4: There exists $\tau > 0$ such that every message vector in $\times_{i=1}^n M_i^\Gamma$ occurs in Γ on a set of measure at least τ.[42]

Suppose this is not the case. Then for any $\varepsilon > 0$, $\exists m = (m_1, ..., m_n) \in \times_{i=1}^n M_i^\Gamma$ such that $\lambda(\theta^\Gamma(m)) < \varepsilon$. Then for any θ where $m^\Gamma(\theta) = m$, $\exists \theta_0(\theta) \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^\Gamma(\theta_0(\theta)) \neq m_i$. Let $i^\Gamma(\theta)$ be the smallest such i, and assume without loss of generality that $a^\Gamma(m) \geq 0.5$. Consider the following noise Ξ:

(i) at states θ where $m^\Gamma(\theta) = m$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i^\Gamma(\theta)$ observes $s_i = \theta_0(\theta)$;

(ii) unless (i) applies, at states $\theta \in [0, 0.4]$, each sender i observes $s_i = \theta$ with probability $1 - \varepsilon$, and s_i distributed uniformly over $[0,1] \backslash \theta^\Gamma(m)$ with probability ε;

(iii) if neither (i) or (ii) applies, the true state is observed.

Ξ has size at most $\varepsilon \frac{F(0.4)}{d}$. The receiver sees m with probability 0 when the state is in $\theta^\Gamma(m)$, with positive probability (by step 2) when the state is in $[0, 0.4]$, and with probability 0 otherwise. Thus, $a^\Xi(m) < 0.4$, so taking $\delta < 0.1$ implies the result. $\Diamond$

Step 4 implies that $\times_{i=1}^n M_i^\Gamma$ is finite, so M_i^Γ is as well for all $i \in N$.

Step 5: Γ has weak interval structure.

Suppose not. Let $S = \cup_{I \text{ is a cell in } \Gamma} I^\circ$. By step 3, for some $m \in \times_{i=1}^n M_i^\Gamma$, $\lambda((\Theta \backslash S) \cap \{\theta \in \Theta : m^\Gamma(\theta) = m \text{ and } \theta < a^\Gamma(m)\}) > 0$. Fix such $m = (m_1, ..., m_n)$, and let $C = (\Theta \backslash S) \cap \{\theta \in$

[41] Heuristically, the density that m is observed and that the state is in $\{\theta \in \Theta : m^\Gamma(\theta) = m\}$ is at most $D\lambda(\{\theta \in \Theta : m^\Gamma(\theta) = m\}) = 0$, while the density that m is observed and that the state is θ'' is at least $d\varepsilon > 0$.

[42] This step simplifies the presentation of step 6, but as the proof of Theorem 2a indicates, it is not necessary. Nevertheless, I include this result since it is used in Appendix B.

$\Theta : m^{\Gamma}(\theta) = m$ and $\theta < a^{\Gamma}(m)\}$. Note that for any $\varepsilon > 0$ and any $\theta \in C$, $\exists\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^{\Gamma}(\theta') \neq m_i$. Let $i^{\Gamma}(\theta)$ be the smallest such i, and consider the following noise Ξ, similar to the noise considered in step 3:

(i) at states $\theta \in C$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i^{\Gamma}(\theta)$ observes $s_i = \theta$, while if not, sender $i^{\Gamma}(\theta)$ observes $s_i = \theta'$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ where $m_i^{\Gamma}(\theta') \neq m_i$;

(ii) otherwise, the true state is observed.

Clearly, Ξ has size at most ε, and upon observing m, the receiver's optimal action, which is independent of ε, satisfies $a^{\Gamma'}(m) > a^{\Gamma}(m)$. Thus taking $\delta < a^{\Gamma'}(m) - a^{\Gamma}(m)$ completes the proof. $\Diamond$

Step 6: Γ *is natural.*

Suppose instead, without loss of generality, that boundary θ_b is not left-natural, and that m is sent to its left. It follows that for all i, m_i is the unique best-response (within M_i^{Γ}) to m_{-i} at θ_b (recall the implication of step 2). Let δ be less than half the size of either cell bordering θ_b and be small enough such that for all i and $m_i'' \in M_i^{\Gamma} \backslash \{m_i\}$, $u_i(a^{\Gamma}(m_i, m_{-i}), \theta) > u_i(a^{\Gamma}(m_i'', m_{-i}), \theta) + \delta$ for all $\theta \in [\theta_b - \delta, \theta_b + 2\delta]$. Such δ exists because θ_b is not left-natural and $\times_{i=1}^{n} M_i^{\Gamma}$ is finite.

Note that by the definition of δ-closeness, for $\varepsilon < \lambda(\text{left cell}) - 2\delta$, in any δ-supporting profile, it must be that for all i and for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, $m_i(s_i) = m_i$.

As mentioned in the main text, consider the beliefs about noise: each sender believes that she observes the true state while all other senders observe $s_i = \max\{\theta - \varepsilon, 0\}$, the receiver believes that all senders observe $s_i = \theta$, and these beliefs are common knowledge.

Now suppose sender j observes $s_j \in [\theta_b - \delta, \theta_b - \delta + \varepsilon)$. She believes that all senders $i \neq j$ observed $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, and therefore will send m_i. Her best response (by at least δ) is therefore m_j. Since this holds for all senders, for all i and for all $s_i \in [\theta_b - \delta, \theta_b - \delta + \varepsilon)$, $m_i(s_i) = m_i$.

Iterating the above argument, it follows by the definition of δ that for all i and for all $s_i \in [\theta_b - \delta, \theta_b + 2\delta]$, $m_i(s_i) = m_i$. There can therefore not be a δ-supporting profile for Γ. Thus Γ is boundary-natural; Lemma 0 and the combination of steps 3 and 5 imply that it is natural. $\blacksquare$

Proof of Theorem 1b: This proof shows that any coordination-free equilibrium profile Γ is a δ-supporting profile for itself, for any positive δ.

Clearly, Γ is δ-close to itself.

Since any coordination-free equilibrium is natural (Proposition 2), $\theta^{\Gamma}(m)$ is connected (Lemma 0). Moreover, any two actions a_1 and a_2 induced in a coordination-free equilibrium must be separated by at least η, or no sender would be indifferent between them at any state in $[a_1, a_2]$. The number of intervals is therefore finite.

Given that noise is less than ε, after every s_i, every sender i places probability at least $1-\varepsilon$ on the state being in $[s_i-\varepsilon, s_i+\varepsilon]$, and therefore probability at least $(1-\varepsilon)^n$ on all senders' signals being in $[s_i-2\varepsilon, s_i+2\varepsilon]$. That, for ε small enough, all senders are playing δ-best responses at least 2ε away from boundaries is clear: utilities are Lipschitz continuous, and therefore bounded as well. By the same token, close to a boundary, the same applies to the sender changing her message: she is nearly indifferent between her two messages and almost sure about what others are sending. For other senders i close to the boundary, there are two potential message vectors sent by $N\backslash\{i\}$. But since the message sent by i near the boundary is either optimal or close to optimal in a noiseless setting regardless of which of the two vectors is sent by $N\backslash\{i\}$, the same argument applies.

Relying on the fact that in the noiseless game, actions in Γ are strictly optimal except at boundaries due to single-crossing, a similar argument shows that as $\varepsilon \to 0$, the measure of the set of states where Γ is optimal for all senders approaches 1.

For the receiver, let $S_{\varepsilon} = \{\theta \in [0,1] : m^{\Gamma}(\theta') = m^{\Gamma}(\theta) \text{ for all } \theta' \in [\theta-\varepsilon, \theta+\varepsilon]\}$. Thus, for any $\gamma > 0$, $\exists \varepsilon(\gamma) > 0$ such that $\lambda(S_{\varepsilon(\gamma)}) > 1-\gamma$. So for any $m \in \times_{i=1}^{n} M_i^{\Gamma}$, under any noise less than $\varepsilon(\gamma)$:

- when $\theta \in S_{\varepsilon(\gamma)} \cap \theta^{\Gamma}(m)$, the receiver sees m with probability at least $(1-\varepsilon(\gamma))^n$;
- when $\theta \in S_{\varepsilon(\gamma)} \backslash \theta^{\Gamma}(m)$, the receiver sees m with probability at most $\varepsilon(\gamma)$.

Suppose $\lambda(\theta^{\Gamma}(m)) = \tau > 0$. Then upon seeing m, the receiver puts probability at least

$$\frac{(1-\varepsilon(\gamma))^n(\tau-\gamma)d}{(1-\varepsilon(\gamma))^n(\tau-\gamma)d + \gamma D + \varepsilon(\gamma)D}$$

on the state being in $\theta^{\Gamma}(m)$. Clearly, as $\gamma \to 0$, the above quantity converges to 1. Moreover, within $S_{\varepsilon(\gamma)} \cap \theta^{\Gamma}(m)$, the density with which m is reported converges uniformly to f as $\gamma \to 0$.

By the finiteness of intervals, $\min_{m \in \times_{i=1}^{n} M_i^{\Gamma}} \lambda(\theta^{\Gamma}(m)) > 0$. It follows that for any $\delta > 0$, it is possible to pick $\gamma > 0$, such that under any noise less than $\varepsilon(\gamma)$, the receiver's optimal action upon seeing any $m \in \times_{i=1}^{n} M_i^{\Gamma}$ is within δ of $a^{\Gamma}(m)$.

Finally, in the special case of an equilibrium with $|\{i \in N : |M_i^\Gamma| \geq 2\}| = 1$ where some message is only sent at 0 or 1, note that upon observing that message, the receiver knows the sender's signal. Since noise is small, the true state is nearby. ■

Proof of Theorem 2a: By Lemma 2, it is sufficient to show that Γ is natural. The proof, which parallels the proof of Theorem 1a, follows the steps below:

Step 1: For any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with local size less than ε, $|a^\Gamma(m) - a^\Xi(m)| < \delta$ for all $m \in \times_{i=1}^n M_i^\Gamma$.

Same argument as for step 1 of Theorem 1a. ◊

Step 2: Every message vector in $\times_{i=1}^n M_i^\Gamma$ that occurs in Γ at two or more states occurs on a set of positive measure.

Suppose instead that $m \in \times_{i=1}^n M_i^\Gamma$ occurs on a set of measure zero including $\theta^* \neq a^\Gamma(m)$. Such θ^* exists whenever m is sent at two or more states. Then for any θ where $m^\Gamma(\theta) = m$, $\exists \theta_0(\theta) \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^\Gamma(\theta_0(\theta)) \neq m_i$. Let $i^\Gamma(\theta)$ be the smallest such i, and consider the following noise Ξ:

(i) at states $\theta \in \theta^\Gamma(m) \backslash \{\theta^*\}$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i^\Gamma(\theta)$ observes $s_i = \theta'$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ where $m_i^\Gamma(\theta') \neq m_i$;

(ii) otherwise, the true state is observed.

Clearly, Ξ has size at most ε, and upon observing m, the receiver's optimal action, which is independent of ε, is equal to $\theta^* \neq a^\Gamma(m)$.[43] Thus taking $\delta < |\theta^* - a^\Gamma(m)|$ completes the proof. ◊

Step 3: Γ has weak interval structure.

The same argument as for step 5 of Theorem 1a combined with step 2 above implies that Γ can only fail to have weak interval structure if a positive measure of states is fully revealed. Let S denote this set. Then for any $\varepsilon > 0$, there exists an interval I of size ε such that $\lambda(S \cap I) > 0$. Within $S \cap I$, a continuum of equilibrium message vectors is sent, so we may assume without loss of generality that sender 1 has a continuum of equilibrium messages within $S \cap I$; let T denote this set of messages.

Pick δ less than γ from Assumption A, and let ε be the corresponding bound on noise from the definition of local robustness. Define I and T accordingly.

[43] Heuristically, the density that m is observed and that the state is in $\{\theta \in \Theta : m^\Gamma(\theta) = m\}$ is at most $D\lambda(\{\theta \in \Theta : m^\Gamma(\theta) = m\}) = 0$, while the density that m is observed and that the state is θ^* is at least $d > 0$.

Suppose $\theta \in S \cap I$, and let $m^{\Gamma}(\theta) = (m_1, m_2, ..., m_n)$. Consider $m' = (m'_1, m_2, ..., m_n)$ where $m'_1 \in T$. I now argue that m' must be sent in equilibrium. Suppose this is not the case. Then consider the following noise Ξ:

(i) when the state is θ, sender 1 observes with equal probability θ and $\theta' \in S \cap I$ where $m_1^{\Gamma}(\theta') = m'_1$, while all other senders observe θ;

(ii) at all other states, everyone observes the true state.

Clearly, Ξ has local size at most ε, and $a^{\Xi}(m') = \theta$.

By step 1, $|a^{\Gamma}(m') - \theta| < \delta$, but then $\theta \in [a^{\Gamma}(m') - \gamma, a^{\Gamma}(m') + \gamma]$. It follows that Assumption A is violated, since there are actions $a \in [a^{\Gamma}(m') - \gamma, a^{\Gamma}(m') + \gamma]$ such that if the receiver's actions following m' were a, then sender 1 would have a profitable deviation at θ.

Thus m' must be sent in equilibrium. Because T contains a continuum of messages, there is a continuum of such m', and each must be sent either at a single state or on a set of positive measure. Because the number of actions such m' can induce is finite (as they must be separated by at least η), only finitely many can be sent at a single state. Of course, only countably many can be sent on a set of positive measure, which implies a contradiction. $\Diamond$

Step 4: Γ is natural.

A similar argument as for step 6 of Theorem 1a applies. For the same reason as before, no two message vectors that differ in one component and that both occur in the noiseless equilibrium induce the same action. Assumption A ensures that the same holds if only one of them occurs in the noiseless equilibrium. Thus, using the notation of step 6, it still holds that for all i, m_i is the unique best-response (within M_i^{Γ}) to m_{-i} at θ_b.

However, because it has not been shown that $\times_{i=1}^{n} M_i^{\Gamma}$ is finite, the existence of δ as used in the proof of step 6 needs to be established. The worry here is that while at θ_b, no sender is indifferent between sending m_i and some other message $m'_i \in M_i^{\Gamma}$ in response to m_{-i}, there may be arbitrarily close such indifference points if M_i^{Γ} is infinite. Assumption A implies that this problem cannot arise when (m'_i, m_{-i}) is an out-of-equilibrium message vector. Furthermore, the number of actions that such (m'_i, m_{-i}) can induce on path is finite since they must be separated by at least η. Thus, δ does exist, and Γ is boundary-natural.

By Assumption B and weak interval structure, no state (except possibly 0 or 1) is fully revealed. Therefore, all messages are sent on a set of positive measure. By Lemma 0, Γ has interval structure, and is therefore natural by definition. ■

Proof of Theorem 2b: Same as the proof of Theorem 1b. ■

Proof of Proposition 4a: Let $NE^{\Gamma}(m)$ be the set of states θ at which m is a Nash equilibrium of the (Γ, θ)-sender game. Note that $\theta^{\Gamma}(m) \subseteq NE^{\Gamma}(m)$.

Step 1: For any $m \in \times_{i=1}^{n} M_i^{\Gamma}$, both $NE^{\Gamma}(m)$ and the closure of $\theta^{\Gamma}(m)$ are connected.

Suppose $NE^{\Gamma}(m)$ is not connected. Then for some $\theta < \theta' < \theta''$, m is a Nash equilibrium of the sender game at θ and θ'', but not θ'. Thus for some $m_i' \neq m_i$, $u_i(a^{\Gamma}(m), \theta) \geq u_i(a^{\Gamma}(m_i', m_{-i}), \theta)$ and $u_i(a^{\Gamma}(m), \theta') < u_i(a^{\Gamma}(m_i', m_{-i}), \theta')$, which implies $a^{\Gamma}(m_i', m_{-i}) > a^{\Gamma}(m)$. But then, by single-crossing, we must have $u_i(a^{\Gamma}(m), \theta'') < u_i(a^{\Gamma}(m_i', m_{-i}), \theta'')$, which contradicts m being a Nash equilibrium at θ''.

Now if the closure of $\theta^{\Gamma}(m)$ were not connected, because $\theta^{\Gamma}(m) \subseteq NE^{\Gamma}(m)$ and $NE^{\Gamma}(m)$ is connected, there would be a nontrivial interval contained in $NE^{\Gamma}(m)$ but not $\theta^{\Gamma}(m)$. This cannot be the case because everywhere within that interval, the sender game would have at least two Nash equilibria. $\Diamond$

Before moving to step 2, consider the following lemma.

Lemma 3: $\{\theta : a^{\Gamma}(m) = \theta$ for some m sent in equilibrium $\Gamma\}$ does not have an accumulation point.

Proof: Suppose it does. Then there exists a sequence of message vectors $\{m^k\}_{k=1}^{\infty}$ such that $\{a^{\Gamma}(m^k)\}_{k=1}^{\infty}$ converges. Assume for notational simplicity that for all k, $a^{\Gamma}(m^k) < a^{\Gamma}(m^{k+1})$.

We know that $a^{\Gamma}(m) \in NE^{\Gamma}(m)$ for all m sent in Γ, and by stage-uniqueness, $a^{\Gamma}(m) \in (NE^{\Gamma}(m))^{\circ}$ whenever $NE^{\Gamma}(m)$ is a nontrivial interval. Drop from the above sequence all m^k such that $a^{\Gamma}(m^l) \in NE^{\Gamma}(m^k)$ for some $l \neq k$. Note that for any k, at most two of m^k, m^{k+1}, m^{k+2} can be dropped: if m^k and m^{k+2} are dropped, then necessarily $NE^{\Gamma}(m^k)$ and $NE^{\Gamma}(m^{k+2})$ are nontrivial intervals, and their intersections with $NE^{\Gamma}(m^{k+1})$ each contain at most one point. This point cannot be $a^{\Gamma}(m^k) \in (NE^{\Gamma}(m^k))^{\circ}$ or $a^{\Gamma}(m^{k+2}) \in (NE^{\Gamma}(m^{k+2}))^{\circ}$, so $NE^{\Gamma}(m^{k+1})$ does not contain any $a^{\Gamma}(m^l)$ for $l \neq k+1$, and thus m^{k+1} is not dropped.

The resulting sequence of messages is still associated with a converging sequence of actions. Now drop the early elements so that $a^{\Gamma}(m^{k+1}) < a^{\Gamma}(m^k) + \eta$ for all k.

Assume without loss of generality that no message vectors are dropped from the original sequence. Note that the left endpoint of $NE^{\Gamma}(m^k)$ is to the right of $a^{\Gamma}(m^{k-1})$ whenever $k > 1$.

Denote the left endpoint of $NE^{\Gamma}(m^k)$ by θ^k, and note that $a^{\Gamma}(m^k) < \theta^k + \eta$ for each $k > 1$.[44] At θ^k, at least one left-biased sender i is indifferent between inducing $a^{\Gamma}(m^k)$ and $a^{\Gamma}(m_i', m_{-i}^k)$, for some $m_i' \in M_i^{\Gamma}$. The reason why not all such i are right-biased is

[44] $k > 1$ ensures that $\theta^k \neq 0$.

as follows: otherwise, since $a^\Gamma(m^k) < \theta^k + \eta$, indifference implies $a^\Gamma(m_i', m_{-i}^k) > a^\Gamma(m^k)$, so by single-crossing, i would prefer sending m_i to m_i' in response to m_{-i}^k immediately to the left of θ^k. Since i is left-biased, $a^\Gamma(m_i', m_{-i}^k) \leq a^\Gamma(m^k) - \eta$. Because there are finitely many senders, there exists an infinite sequence of k's each satisfying the above for the same sender. Assume without loss of generality that the original sequence has this property for sender 1. Then there exists another sequence of message vectors $\{m^{k,2}\}_{k=1}^\infty$ such that $u_1(a^\Gamma(m^k), \theta^k) = u_1(a^\Gamma(m^{k,2}), \theta^k)$ for all k, where $a^\Gamma(m^{k,2}) \leq a^\Gamma(m^k) - \eta$.

Note that the same argument would hold even if the original sequence were not increasing: what matters is that the $a^\Gamma(m^k)$'s are close to each other, a feature that any converging sequence has.

The continuity of all senders' utility functions implies that $\{a^\Gamma(m^{k,2})\}_{k=1}^\infty$ is again a converging sequence, and it converges to a point at least η to the left of the original accumulation point. By completeness, these all occur in equilibrium, so the argument can be iterated. But eventually, actions to the left of 0 are needed, yielding the desired contradiction. □

Step 2: Γ *has interval structure.*

For all $m \in \times_{i=1}^n M_i^\Gamma$, by step 1, $NE^\Gamma(m)$ is either a nontrivial interval, a single point, or empty. Since Γ is stage-unique and strategies in Γ are measurable, $\lambda(NE^\Gamma(m)) = \lambda(\theta^\Gamma(m))$. Moreover, for any $m, m' \in \times_{i=1}^n M_i^\Gamma$, $NE^\Gamma(m) \cap NE^\Gamma(m')$ contains at most one point. It follows that at most two equilibrium message vectors can induce the same action: at most one sent on a nontrivial interval, and at most one sent at a single point. Therefore, by Lemma 3, the number of equilibrium message vectors is finite. Hence, Γ has weak interval structure, and the finitely many states (if any) outside cells must be fully revealed. By Assumption B, Γ has interval structure. ◊

Step 3: Γ *is natural (and therefore strongly natural).*

Suppose not. Then some boundary θ_b is either not left-natural or not right-natural. Assume the former, and denote the message vectors sent in the left and right cells by m and m' respectively. By definition, for all senders i, $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) > u_i(a^\Gamma(m_i'', m_{-i}), \theta_b)$ for all $m_i'' \in M_i^\Gamma$, so by continuity and finiteness of the number of equilibrium messages, there exists some $\varepsilon > 0$ such that $u_i(a^\Gamma(m_i, m_{-i}), \theta_b + \gamma) > u_i(a^\Gamma(m_i'', m_{-i}), \theta_b + \gamma)$ for all $\gamma \in (0, \varepsilon)$. But this implies that at all $\theta \in (\theta_b, \theta_b + \varepsilon)$, both m and m' are Nash equilibria of the (Γ, θ)-sender game. ■

Proof of Proposition 4b: Suppose not. Then at some θ in the interior of a cell, $m = m^\Gamma(\theta)$ and $m' \neq m$ are both Nash equilibria of the (Γ, θ)-sender game. Assume without loss of generality that $a^\Gamma(m) < a^\Gamma(m')$. Naturality and completeness imply that there exists a cell where $a^\Gamma(m')$ is induced. By the definition of natural boundary, at the left boundary θ_b of that cell, for some sender i, $u_i(a^\Gamma(m_i'', m_{-i}'), \theta_b) = u_i(a^\Gamma(m_i', m_{-i}'), \theta_b)$, with $a^\Gamma(m_i'', m_{-i}') < a^\Gamma(m_i', m_{-i}')$ and $m_i'' \in M_i^\Gamma$. But since $\theta < \theta_b$, single-crossing implies that $u_i(a^\Gamma(m_i'', m_{-i}'), \theta) > u_i(a^\Gamma(m_i', m_{-i}'), \theta)$, so m' cannot be a Nash equilibrium of the (Γ, θ)-sender game. ■

Appendix B: Strengthening Battaglini's Robustness Concept

The following strengthens Battaglini's (2002) robustness concept by applying it to in-equilibrium message vectors, and by requiring robustness to "all" small noise. The resulting criterion implies weak interval structure and completeness; however, it imposes few more conditions, so unappealing equilibria (such as Examples 2 and 3) survive.

Definition: An equilibrium Γ in the noiseless game is *robust* if for all $\delta > 0$, there exists $\varepsilon > 0$ such that under any noise Ξ with size less than ε: a^Ξ, the receiver's best response to $\{m_j^\Gamma\}_{j \in N}$, satisfies $|a^\Xi(m) - a^\Gamma(m)| < \delta$ for all m.[45][46]

Note that robustness focuses on the receiver's optimal action after each message vector, holding the senders' strategies constant, just like in Battaglini's (2002) definition. Note that step 1 of the proof of Theorem 1a implies that any disagreement-robust equilibrium is robust.

Robustness requires the receiver's strategy, when she believes that there is small noise and that other players play the noiseless equilibrium strategies, to remain close to her strategy in the no-noise equilibrium. In other words, the receiver would not change her play much were she to start slightly second-guessing senders - she may expect them to play noiseless equilibrium strategies because they believe there is no noise and are not aware that they are being second-guessed.

[45] Ξ is assumed to be such that the players' best responses are well defined.

[46] An alternative definition could instead require, for out-of-equilibrium vectors m, that $a^{\Gamma'}(m) \in A^\Gamma(m)$, where $A^\Gamma(m)$ is the set of locations of $a^\Gamma(m)$ that do not affect on-path play. Such a definition would not change any results, as this paper assumes that all senders are biased; the definition in the text is used for simplicity. However, with multiple unbiased senders, this alternative definition may be preferable, as it would allow reasonable equilibria where more than one unbiased senders report the true state.

Robustness implies that if more than one sender is informative (*i.e.* sends more than one message) in an equilibrium, then there is a lower bound $\tau > 0$ for the measure of the set of states at which each message vector is sent. The intuition is that if a message vector m is sent very infrequently, then given the possibility of noise, the receiver, upon seeing m, may believe that it most likely resulted from noise rather than from the state being in $\theta^{\Gamma}(m)$. Indeed, if the lower bound τ did not exist, then for any ε, there would exist a message vector m such that for every state in $\theta^{\Gamma}(m)$, there exists a state within ε that is outside $\theta^{\Gamma}(m)$. It is then possible to have noise less than ε such that when the state is in $\theta^{\Gamma}(m)$, senders will with high probability (much greater than $1 - \varepsilon$) report a vector other than m because someone's signal is such that she does not send m_i. Note that completeness and the existence of at least two informative senders guarantee that there are states outside $\theta^{\Gamma}(m)$ where some sender i is supposed to send m_i; thus, with noise, m_i can occur with nontrivial probability even though signals in $\theta^{\Gamma}(m)$ do not.

Proposition 6: An equilibrium Γ is robust if and only if at least one of the following holds:

1. Γ has weak interval structure, and $\exists\tau > 0$ such that every message vector in $\times_{i=1}^{n} M_i^{\Gamma}$ occurs in Γ on a set of states of measure at least τ; or

2. M_i^{Γ} is a singleton for $n - 1$ players.

Proof: "Only if": Steps 2 through 5 of the proof of Theorem 1a directly apply. $\Diamond$
"If": The receiver part of the proof of Theorem 1b directly applies. ■

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