Barriers to Reallocation and Economic Growth: the Effects of Firing Costs*

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Abstract

Recent empirical studies have underlined the existence of large reallocation flows across firms. In this paper, we study how factors that hinder this reallocation process influence aggregate productivity growth. We extend Hopenhayn and Rogerson's (1993) general equilibrium firm dynamics model to allow for endogenous innovation. We calibrate the model using U.S. data on firms' entry, exit, job creation, and job destruction. We then evaluate the effects of firing taxes on reallocation, innovation, and aggregate productivity growth.

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1 Introduction

Recent empirical studies have underlined the existence of large flows of productive resources across firms and their important role for aggregate productivity. Production inputs are constantly being reallocated as firms adjust to changing market environments and new products and techniques are developed. As documented recently by Micco and Pagés (2007) and Haltiwanger et al. (2014), labor market regulations may dampen this reallocation of resources. Using cross-country industry-level data, they show that restrictions on hiring and firing reduce the pace of job creation and job destruction. In the U.S., Davis and Haltiwanger (2014) find that the introduction of common-law exceptions that limit the firms' ability to fire their employees at will has a negative impact on job reallocation. The objective of the paper is to study the implications of this reduced job reallocation for aggregate productivity growth.

We investigate the consequences of employment protection on job reallocation and productivity growth using a model of innovation-based economic growth. We extend Hopenhayn and Rogerson's (1993) model of firm dynamics by introducing an innovation decision. Firms can invest in research and development (R&D) to improve the quality of products. Hence, by contrast with Hopenhayn and Rogerson (1993), job creation and job destruction are the result of both idiosyncratic productivity shocks and endogenous innovation. Following the seminal work of Grossman and Helpman (1991) and Aghion and Howitt (1992), we model innovation as a process of *creative destruction*: by innovating on existing products, entrants displace the incumbent producers. In addition to this Schumpeterian feature, we also consider the innovations developed by incumbent firms. We allow incumbent firms to invest in R&D to improve the quality of their own product.[1] In the model, productivity growth thus results from the R&D of both entering and incumbent firms. The model highlights the crucial role of reallocation for economic growth. As products of higher quality are introduced into the market, labor is reallocated towards the firms producing the higher quality products.[2]

[1]The importance of incumbents' innovation is emphasized in recent papers, such as Acemoglu and Cao (2015), Akcigit and Kerr (2015), and Garcia-Macia et al. (2015). Earlier papers that analyze incumbents' innovations in the quality-ladder framework include Segerstrom and Zolnierek (1999), Aghion et al. (2001), and Mukoyama (2003).

[2]Aghion and Howitt (1994) is an earlier study that highlight this aspect of the Schumpeterian growth

By limiting the reallocation of labor across firms, employment protection may modify the firms' incentives to innovate.

We model employment protection as a firing tax. We find that the effects of the firing tax on aggregate productivity growth depends on the interaction between the innovation of entrants and incumbents. Our results indicate that imposing a firing tax can have opposite effects on the entrants' and the incumbents' innovation. While the firing tax reduces the entrants' innovation, it may raise the incentives for incumbent firms to innovate. The firing tax reduces the entrants' innovation because it represents an additional cost that reduces the expected profits from entry. Moreover, the misallocation of labor further reduces the operational profit. For incumbents, the consequences of the firing tax is less clear-cut. In particular, the misallocation of labor has an ambiguous impact on the incumbents' incentive to innovate. Firms which are larger than their optimal size, that is firms that had a succession of negative transitory shocks or that were unsuccessful at innovating, have stronger incentives to invest in R&D in the presence of firing costs. For those firms, innovation has the added benefit of allowing them to avoid paying the firing tax. Conversely, firms that are smaller than their optimal size have weaker incentives to invest in R&D in the presence of firing costs. In addition, the incumbents' incentive to innovate is affected by the rate at which entrants innovate. By reducing the entry rate, firing costs lower the risk for incumbents of being taken over by an entrant. This decline in the rate of creative destruction tends to raise the incumbents' innovation. Overall, we find that the firing tax may raise the aggregate growth rate if the incumbents' innovation is sufficiently sensitive to the entry rate.

Past theoretical studies have shown that firing costs can have adverse consequences on macroeconomic performance. Using a general equilibrium model of firm dynamics, Hopenhayn and Rogerson (1993) have shown that employment protection hinders job reallocation and reduces allocative efficiency and aggregate productivity. They find that a firing cost that amounts to one year of wages reduces aggregate total factor productivity by 2%. In a more recent paper, Moscoso Boedo and Mukoyama (2012) consider a wider range of countries,

models in their analysis of unemployment.

and show that firing costs calibrated to match the level observed in low income countries can reduce aggregate total factor productivity by 7%. The existing literature, however, has mainly focused on the effects of employment protection on the *level* of aggregate productivity. This paper instead analyzes the effect on aggregate productivity *growth*. In focusing on the effects of barriers to labor reallocation on aggregate productivity growth, our analysis also goes one step beyond the recent literature on misallocation that follows the seminal work of Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). We highlight the fact barriers to reallocation affect not only the allocation of resources across firms with different productivity levels, but also the productivity process itself as it modifies the firms' incentives to innovate. Empirical studies that evaluate the contribution of reallocation on productivity, such as Foster et al. (2001) and Osotimehin (2013), are designed to analyze the sources of productivity growth, rather than the level; in that sense, our analysis is more comparable to that literature. Poschke (2009) is one of the few exceptions that studies the effects of firing costs on aggregate productivity growth. In Poschke (2009), firing costs act as an exit tax which lowers the exit rate of low productivity firms. We focus on a different channel and show that firing costs may also affect aggregate productivity growth through their effects on R&D and innovation.

Our paper is also related to firm-dynamics models with endogenous innovation, such as Klette and Kortum (2004), Lentz and Mortensen (2008), and Acemoglu et al. (2013). Our focus, however, is different. While our objective is to study the effects of employment protection, Lentz and Mortensen (2008) mainly focus on the structural estimation of the model, and Acemoglu et al. (2013) study the consequences of subsidies to R&D spending and the allocation of R&D workers across firms. Models by Akcigit and Kerr (2015) and Acemoglu and Cao (2015) extend the Klette-Kortum model and allow incumbents to innovate on their own products. Our model also exhibit that feature. Compared to these models, one important difference of our study is that we use labor market data to discipline the model parameters, consistently with our focus on labor market reallocation and labor market policy.

The paper is organized as follows. The next section sets up the model. Section 3 outlines our computational method and details of calibration. Section 4 describes the results. Section 5 concludes.

2 Model

We build a model of firm dynamics in the spirit of Hopenhayn and Rogerson (1993). We extend their framework to allow for endogenous productivity at the firm level. The innovation process of our model is built on the classic quality-ladder models of Grossman and Helpman (1991) and Aghion and Howitt (1992), and also on the recent models of Acemoglu and Cao (2015) and Akcigit and Kerr (2015).

There is a continuum of differentiated intermediate goods on the unit interval $[0, 1]$ and firms innovate by improving the quality of the intermediate goods. Final goods are produced from the intermediate goods in a competitive final good sector. We first describe the optimal aggregate consumption choice. We then turn to the firms, first describing the final goods sector and the demand for each intermediate good, and then the decisions of the intermediate goods firms. Finally, we present the potential entrants' decision in the intermediate goods sector and the aggregate innovation in general equilibrium.

2.1 Consumers

The utility function of the representative consumer has the following form:

$$\mathbf{U} = \sum_{t=0}^{\infty} \beta^t [\log(C_t) - \xi L_t],$$

where C_t is consumption at time t, L_t is labor supply at time t, $\beta \in (0, 1)$ is the discount factor, and $\xi > 0$ is the parameter of the disutility of labor. Similarly to Hopenhayn and Rogerson (1993), we adopt the indivisible-labor formulation of Rogerson (1988).

The consumer's budget constraint is

$$A_{t+1} + C_t = (1 + r_t)A_t + w_t L_t + T_t,$$

where

$$A_t = \int_{\mathcal{N}_t} V_t^j dj$$

is the asset holding. The asset in this economy is the ownership of firms.[3] Here, V_t^j indicates the value of a firm that produces product j at time t, and $\mathcal{N}_t$ is the set of products that are actively produced at time t. In the budget constraint, r_t is the net return of the asset, w_t is wage rate, and T_t is the lump-sum transfer of the firing tax to the consumer.

The consumer's optimization results in two first-order conditions. The first is the Euler equation:

$$\frac{1}{C_t} = \beta(1 + r_{t+1})\frac{1}{C_{t+1}}, \tag{1}$$

and the second is the optimal labor-leisure choice:

$$\frac{w_t}{C_t} = \xi. \tag{2}$$

2.2 Final-good firms

The final good Y_t is produced by the technology

$$Y_t = \left(\int_{\mathcal{N}_t} \mathbf{q}_{jt}{}^{\psi} y_{jt}{}^{1-\psi} dj \right)^{\frac{1}{1-\psi}}.$$

The price of Y_t is normalized to one, y_{jt} is the use of intermediate product j at time t, $\mathbf{q}_{jt}$ is the *realized quality* of intermediate product j.[4] The realized quality is the combination of the *potential quality* q_t and the transitory shock α_{jt}:

$$\mathbf{q}_{jt} = \alpha_{jt} q_{jt}.$$

We assume that α_{jt} is i.i.d. across time and products. We also assume that the transitory shock realizes at the product level, rather than at the firm level, so that the value of α_{jt} does not alter the ranking of the realized quality compared to the potential quality[5]

Let the *average potential quality* of intermediate goods be

$$\bar{q}_t \equiv \frac{1}{N_t} \left(\int_{\mathcal{N}_t} q_{jt} dj \right)$$

[3] We do not distinguish firms and establishments in this paper. Later we use establishment-level dataset in our calibration. Using the firm-level data yields similar results.

[4] Similar formulations are used by Luttmer (2007), Acemoglu and Cao (2015), and Akcigit and Kerr (2015), among others.

[5] If the shock is at the firm level, it is possible that the incumbent firm i's realized quality $\alpha_{it} q_{it}$ is larger than the new firm j's realized quality $\alpha_{jt} q_{jt}$ even if $q_{jt} > q_{it}$.

and the *quality index* Q_t be

$$Q_t \equiv \bar{q}_t^{\frac{\psi}{1-\psi}}.$$

Note that the quality index grows at the same rate as the aggregate output Y_t along the balanced-growth path.

The final goods sector is perfectly competitive, and the problem for the representative final good firm is

$$\max_{y_{jt}} \quad \left(\int_{\mathcal{N}_t} \mathbf{q}_{jt}{}^{\psi} y_{jt}{}^{1-\psi} dj \right)^{\frac{1}{1-\psi}} - \int_{\mathcal{N}_t} p_{jt} y_{jt} dj.$$

The first-order condition leads to the inverse demand function for y_{jt}:

$$p_{jt} = \mathbf{q}_{jt}{}^{\psi} y_{jt}{}^{-\psi} Y_t{}^{\psi}. \tag{3}$$

Final-good firms are introduced for ease of exposition; as in the standard R&D-based growth models, one can easily transform this formulation into a model without final goods, assuming that the consumers (and firms engaging in R&D activities) combine the intermediate goods on their own.[6] In this sense, the final-good sector is a veil in the model, and we ignore final-good firms when we map the model to the firm-dynamics data.

2.3 Intermediate-good firms

Each intermediate-good firm produces one differentiated product. The core of the model is the dynamics of the heterogeneous intermediate-good firms. Intermediate-good firms enter the market, hire workers, and produce. Depending on their productivity changes, they expand or contract over time, and they may be forced to exit. Compared to standard firm-dynamics models, the novelty of our model is that these dynamics are largely driven by endogenous innovations.

The intermediate firms conduct R&D activities to innovate. We consider two sources of innovations. One is the *innovation by incumbents*: an incumbent can invest in R&D in order to improve the potential quality of its own product. The other is the *innovation by entrants*: an entrant can invest in R&D to innovate a product that is either (i) not currently produced,

[6] See, for example, Barro and Sala-i-Martin (2004).

or (ii) currently produced by another firm. If the product in not currently produced, the entrant becomes a new monopolist in the industry. If the product is currently produced by another firm, the entrant displaces the incumbent monopolist. The previous producer is, as a result, forced to exit.[7]

Our main policy experiment is imposing a firing tax on intermediate-good firms. We assume that the firm has to pay the tax τw_t on each worker it fires,[8] including when it exits.[9]

2.3.1 Production of intermediate goods

Each product j is produced by the leading-edge monopolist who produces the highest quality for that particular product. The firm's production follows a linear technology

$$y_{jt} = \ell_{jt},$$

where ℓ_{jt} is the labor input of production workers. The monopolist decides on the production quantity given the inverse demand function, given by equation (3).

2.3.2 Innovation by incumbents

An incumbent producer can innovate on its own product. The probability for an incumbent to innovate on its product at time t is denoted x_{It}. A successful innovation increases the quality of the product from q_t to $(1+\lambda_I)q_t$, where $\lambda_I > 0$, in the following period. The cost of innovation, r_{It}, is assumed to be

$$r_{It} = \theta_I Q_t \frac{q_{jt}}{\bar{q}_t} {x_{It}}^{\gamma},$$

where $\gamma > 1$ and θ_I are parameters.

[7]Instead of assuming that the lower-quality producer automatically exits, we can resort to a market participation game with price competition as in Akcigit and Kerr (2015).

[8]Following the literature (e.g. Hopenhayn and Rogerson (1993)), we assume that the firing costs are incurred only when the firm contracts or exits (that is, only when job destruction occurs). As is well documented (see, for example, Burgess et al. (2000)), worker flows are typically larger than job flows. The implicit assumption here is that all worker separations that are not counted as job destruction are voluntary quits that are not subject to the firing tax.

[9]An alternative specification is to assume that the firm does not need to incur firing costs when it exits. See Samaniego (2006) and Moscoso Boedo and Mukoyama (2012) for discussions.

2.3.3 Innovation by entrants

A new firm can enter after having successfully innovated on either an intermediate good currently produced by an incumbent or on a good that is not currently produced. In order to innovate, a potential entrant has to spend a fixed cost ϕQ_t and a variable cost

$$r_{Et} = \theta_E Q_t {x_{Et}}^{\gamma}$$

to innovate with probability x_{Et}. Here, ψ and θ_E are parameters. As for the incumbents' innovation, a successful innovation increases the quality of product from q_t to $(1+\lambda_E)q_t$ in the following period. Here, we are allowing the innovation step for the entrants, λ_E, to be different from the incumbents' innovation step λ_I. We assume that the entrants' innovation is not targeted: each entrant innovates on a product that is randomly selected.

We assume free entry, that is, anyone can become a potential entrant by spending these costs. The free entry condition for potential entrants is

$$\max_{x_{E_t}} \left\{ -\theta_E Q_t {x_{Et}}^{\gamma} - \phi Q_t + \frac{1}{1+r_t} x_{Et} \bar{V}_{E,t+1} \right\} = 0, \tag{4}$$

where $\bar{V}_{E,t+1}$ is the expected value of an entrant at time $t+1$. The optimal value of x_{Et}, x^*_{Et}, is determined by

$$\frac{1}{1+r_t} \bar{V}_{E,t+1} - \gamma \theta_E Q_t {x_{Et}}^{\gamma-1} = 0 \tag{5}$$

and the value of aggregate innovation by entrants is $X_{Et} = m_t x^*_{Et}$, where m_t is the number of potential entrants at time t. From (4) and (5), x^*_{Et} satisfies

$$-\theta_E {x^*_{Et}}^{\gamma} - \phi + \gamma \theta_E {x^*_{Et}}^{\gamma} = 0$$

and thus x^*_{Et} is a constant number x^*_E that can easily be solved as a function of parameters. The solution is

$$x^*_E = \left(\frac{\phi}{\theta_E(\gamma-1)} \right)^{\frac{1}{\gamma}}.$$

Note that x^*_E is not affected by the firing tax. Thus all adjustments for entry in response to the changes in firing tax occur at the margin of m_t.

2.3.4 Exit

We assume that the firm can exit for two reasons: (i) the product line is taken over by another firm with a better quality; (ii) the firm is hit by an exogenous exit shock. In both cases, exit is not the results of an endogenous decision.

The exogenous exit shock, assumed to be constant, is denoted by $\delta > 0$. After this shock, the product that the firm has been producing becomes inactive until a new entrant picks it up.

2.4 Timing of events and value functions

The timing of events in the model is the following. Below, we omit the firm subscript j when there is no risk of confusion.

1. At the beginning of period t, all innovations that were paid last period realize. Incumbent firms have to exit from the product lines on which entrants have innovated, including when both the incumbent and the entrant innovate at the same time.

2. The transitory productivity shock realizes.

3. The firms (including the newly-entered firms) receive the exit shock with probability δ.

4. Exiting firms pay the firing cost.

5. Let us express the distribution of the firm size at this point by the stationary measure over the individual state $(q_t, \alpha_t, \ell_{t-1})$, where q_t is the potential quality, α_t is the transitory shock, and ℓ_{t-1} is the size in the previous period.

6. Firms decide on hiring, firing and innovation (this include the potential entrants' innovation), then the labor market clears and the production takes place. The consumer decides on consumption and saving.

Now we express the firm's optimization problem as a dynamic programming problem. The expected value for the firm at the beginning of the period (after stage 2 of the timing) is

$$Z_t(q_t, \alpha_t, \ell_{t-1}) = (1-\delta)V_t^s(q_t, \alpha_t, \ell_{t-1}) + \delta V_t^o(\ell_{t-1}).$$

The first term in the right-hand side is the value from surviving and the second term is the value from exiting due to the exogenous exit shock. The value of exiting is

$$V_t^o(\ell_{t-1}) = -\tau w_t \ell_{t-1}.$$

The value of staying is

$$\begin{aligned} &V_t^s(q_t, \alpha_t, \ell_{t-1}) \\ &\quad = \max_{\ell_t, x_{It}} \quad \Pi_t(q_t, \alpha_t, \ell_{t-1}, \ell_t, x_{It}) + \frac{1}{1+r_{t+1}} \left((1-\mu_t) S_{t+1}(x_{It}, q_t, \ell_t) - \mu_t \tau w_{t+1} \ell_t\right). \end{aligned}$$

Here, the $S_{t+1}(x_{It}, q_t, \ell_t)$ is the value of survival into the next period and μ_t is the probability that the incumbent firm is forced to exit because of the entrant innovation. The value of survival is

$$S_{t+1}(x_{It}, q_t, \ell_t) = (1-x_{It}) E_{\alpha_{t+1}}[Z_{t+1}(q_t, \alpha_{t+1}, \ell_t)] + x_{It} E_{\alpha_{t+1}}[Z_{t+1}((1+\lambda_I)q_t, \alpha_{t+1}, \ell_t)],$$

where the period profit is

$$\Pi_t(q_t, \alpha_t, \ell_{t-1}, \ell_t, x_{It}) = ([\alpha_t q_t]^{\psi} {\ell_t}^{-\psi} {Y_t}^{\psi} - w_t)\ell_t - \theta_I Q_t \frac{q_t}{\bar{q}_t} {x_{It}}^{\gamma} - \tau w_t \max\langle 0, \ell_{t-1} - \ell_t \rangle.$$

Because the economy exhibits perpetual growth, we first need to transform the problem into a stationary one before applying usual dynamic programming techniques.

3 Balanced-growth equilibrium

From this section, we focus on the balanced-growth path of the economy, where w_t, C_t, Y_t, Q_t grow at a common rate g. Our specification implies that $\bar{q}_t$ grows at a rate $g_q = (1+g)^{\frac{1-\psi}{\psi}} - 1$ along this path. Let us normalize all variables except q_t by dividing Q_t. For q_t, we normalize with $\bar{q}_t$. Denote all normalized variables with a hat (ˆ): for example, $\hat{Y}_t = Y_t/Q_t$, $\hat{C}_t = C_t/Q_t$, $\hat{q}_t = q_t/\bar{q}_t$, and so on.

3.1 Normalized Bellman equations

From the consumer's Euler equation (1),

$$\beta(1+r_{t+1}) = \frac{C_{t+1}}{C_t} = 1+g$$

holds. Therefore $(1+g)/(1+r) = \beta$ holds along the stationary growth path. This can be used to rewrite the firm's value functions as the following. (We use the hat notation for the value functions, in order to distinguish from the previous section.) The value at the beginning of the period is (given the stationarity, time subscripts are dropped)

$$\hat{Z}(\hat{q}, \alpha, \ell) = (1-\delta)\hat{V}^s(\hat{q}, \alpha, \ell) + \delta\hat{V}^o(\ell), \tag{6}$$

where

$$\hat{V}^o(\ell) = -\tau\hat{w}\ell.$$

The value of staying is

$$\hat{V}^s(\hat{q}, \ell) = \max_{\ell' \geq 0, x_I} \hat{\Pi}(\hat{q}, \alpha, \ell, \ell', x_I) + \beta\left((1-\mu)\hat{S}\left(x_I, \frac{\hat{q}}{1+g_q}, \ell'\right) - \mu\tau\hat{w}\ell'\right), \tag{7}$$

where

$$\hat{S}\left(x_I, \frac{\hat{q}}{1+g_q}, \ell'\right) = (1-x_I)E_{\alpha'}\left[\hat{Z}\left(\frac{\hat{q}}{1+g_q}, \alpha', \ell'\right)\right] + x_I E_{\alpha'}\left[\hat{Z}\left(\frac{(1+\lambda_I)\hat{q}}{1+g_q}, \alpha', \ell'\right)\right].$$

Note that in (7), ℓ is the previous period employment and ℓ' is the current period employment. The period profit can be rewritten as

$$\hat{\Pi}(q, \alpha, \ell, \ell', x_I) = ([\alpha\hat{q}]^{\psi}\ell'^{-\psi}\hat{Y}^{\psi} - \hat{w})\ell' - \theta_I\hat{q}{x_I}^{\gamma} - \tau\hat{w}\max\langle 0, \ell - \ell'\rangle. \tag{8}$$

Note that the Bellman equation, Equation (7), can be solved for given $\hat{Y}$, $\hat{w}$, g, and μ.

For the entrants, the free entry condition can be rewritten as:

$$\max_{x_E}\left\{-\theta_E {x_E}^{\gamma} - \phi + \beta x_E \hat{\hat{V}}_E\right\} = 0,$$

where x_E satisfies the optimality condition

$$\beta\hat{\hat{V}}_E = \gamma\theta_E {x_E}^{\gamma-1}.$$

3.2 General equilibrium under balanced growth

Let the decision rule for x_I be $\mathcal{X}_I(\hat{q},\alpha,\ell)$, and the decision rule for ℓ' be $\mathcal{L}'(\hat{q},\alpha,\ell)$. Denote the stationary measure of the (normalized) individual state variables as $f(\hat{q},\alpha,\ell)$ at the point of decision for innovation and hiring. Assume that innovating over a vacant line improves the quality of the product over a quality drawn from a given distribution $h(\hat{q})$. Let Ω denote the cumulative distribution function of α and ω denote the corresponding density function.

The stationary measure has to be the fixed point of the mapping $f \to \mathbf{T}f$, where $\mathbf{T}$ is defined by

$$\begin{aligned}
&\textstyle\int_0^{\alpha'}\int_0^{\ell'}\int_0^{\hat{q}'} \mathbf{T}f(\hat{q},\alpha,\ell)d\hat{q}d\alpha d\ell = \\
&\quad \Omega(\alpha')\Bigg[(1-\mu)M_s(\hat{q}',\ell') \\
&\quad +\mu(1-N)\int_{(1+\lambda_E)\hat{q}/(1+g_q)\le\hat{q}'}(1-\delta)h(\hat{q})d\hat{q} \\
&\quad +\mu\int_{(1+\lambda_E)\hat{q}/(1+g_q)\le\hat{q}'}\int\int(1-\delta)f(\hat{q},\alpha,\ell)d\hat{q}d\alpha d\ell\Bigg]
\end{aligned}$$

The first term of right hand side is the mass of surviving firms (defined below). The second line is the entry into inactive products. The last line refers to the products whose ownership changes because of entry. The mass of surviving firms is

$$\begin{aligned}
&M_s(\hat{q}',\ell') = \\
&\quad \int_{\hat{q}/(1+g_q)\le\hat{q}'}\int_{\mathcal{L}'(\hat{q},\alpha,\ell)\le\ell'}(1-\mathcal{X}_I(\hat{q},\alpha,\ell))(1-\delta)f(\hat{q},\alpha,\ell)d\hat{q}d\alpha d\ell \\
&\quad +\int_{(1+\lambda_I)\hat{q}/(1+g_q)}\int_{\mathcal{L}'(\hat{q},\alpha,\ell)\le\ell'}\mathcal{X}_I(\hat{q},\alpha,\ell)(1-\delta)f(\hat{q},\alpha,\ell)d\hat{q}d\alpha d\ell.
\end{aligned}$$

The total mass of actively produced products is

$$N \equiv \int\int\int f(\hat{q},\alpha,\ell)d\hat{q}d\alpha d\ell.$$

From the steady-state condition (inflow equals outflow)

$$\delta N = \mu(1-\delta)(1-N),$$

this can be computed easily as

$$N = \frac{\mu(1-\delta)}{\delta+\mu(1-\delta)}. \tag{9}$$

The aggregate innovation by incumbents is

$$X_I = \int\int\int \mathcal{X}_I(\hat{q}, \alpha, \ell) f(\hat{q}, \alpha, \ell) d\hat{q} d\alpha d\ell,$$

and the aggregate innovation by entrants is

$$X_E = m x_E^*.$$

The probability for an incumbent to lose a product, μ, is equal to entrants' innovation rate:

$$\mu = X_E.$$

The growth rate of $\bar{q}$ is given by

$$1 + g_q = \frac{\bar{q}'}{\bar{q}}.$$

Let

$$\bar{f}(\hat{q}) \equiv \int\int f(\hat{q}, \alpha, \ell) d\alpha d\ell.$$

Then the normalized value of entry in the stationary equilibrium can be calculated as:

$$\hat{\hat{V}}_E = \int \left[\int \hat{Z}\left(\frac{(1+\lambda_E)\hat{q}}{1+g_q}, \alpha, 0 \right) (\bar{f}(\hat{q}) + (1-N)h(\hat{q})) d\hat{q} \right] \omega(\alpha) d\alpha.$$

In the goods market, the final goods are used for consumption and R&D, and therefore

$$\hat{Y} = \hat{C} + \hat{R},$$

holds, where $\hat{R}$ is the normalized R&D spending which includes the potential entrants' fixed cost. In the labor market, the consumer's first-order condition for labor-leisure decision (2) is

$$\frac{\hat{w}}{\hat{C}} = \xi,$$

and thus

$$\frac{\hat{w}}{\hat{Y} - \hat{R}} = \xi$$

holds. Since $\hat{Y}$ is a function of intermediate-good production which utilizes labor, this equation (implicitly) clears the labor market.

4 Characterization of the model

The case without firing tax can easily be characterized analytically. It provides a useful benchmark and give intuitions for the determinants of innovation and growth in the model. Also, the economy without firing costs is later used for the purpose of calibration in the quantitative analysis. The case with firing tax is less straightforward to characterize. We provide a partial characterization of the model, that facilitates the numerical computation of the equilibrium.

4.1 Analytical characterization of the frictionless economy

The solution to the economy without firing tax can be boiled down to a system of nonlinear equations. The full characterization is in Appendix B. Here, we present several key results.

The first is the value function for the incumbents and the decision for the internal innovation.

Proposition 1 *Given $\hat{Y}$, μ, and g_q, the value function for the incumbents is of the form*

$$\hat{Z}(\hat{q},\alpha) = \mathcal{A}\alpha\hat{q} + \mathcal{B}\hat{q},$$

and the optimal decision for x_I is

$$x_I = \left(\frac{\beta(1-\mu)\lambda_I(\mathcal{A}+\mathcal{B})}{(1+g_q)\gamma\theta_I}\right)^{\frac{1}{\gamma-1}}$$

where

$$\mathcal{A} = (1-\delta)\psi\frac{\hat{Y}}{N}$$

and $\mathcal{B}$ solves

$$\mathcal{B} = (1-\delta)\beta(1-\mu)\left(1+\frac{\gamma-1}{\gamma}\lambda_I x_I\right)\frac{\mathcal{A}+\mathcal{B}}{1+g_q}.$$

Proof. See Appendix B. Note that N solves (9) for a given μ. ■

This result shows that x_I is constant across different firms regardless of the values of α and $\hat{q}$. This is consistent with *Gibrat's law*: the expected growth of a firm is independent of its size.

This property implies that the process for the firm productivity is a *stochastic multiplicative process with reset events.*[10] This process allows us to characterize the right tail of the firm productivity distribution as follows.

Proposition 2 *Suppose that the distribution of the quality for a innovation on vacant line, $h(\hat{q})$, is bounded. Then the right tail of the relative firm productivity $\hat{q}$ follows a Pareto distribution with the shape parameter ξ (that is, the density has a form of $F\hat{q}^{\xi+1}$) which solves*

$$1 = (1-\delta)\left[(1-\mu)x_I\gamma_i^{\kappa} + \mu\gamma_e^{\xi} + (1-\mu-(1-\mu)x_I)\gamma_n^{\kappa}\right].$$

where $\gamma_i \equiv (1+\lambda_I)/(1+g_q)$, $\gamma_e \equiv (1+\lambda_E)/(1+g_q)$, and $(1-\delta)(1-\mu-(1-\mu)x_I)$.

Proof. See Appendix B. ■

Because the firm size (in terms of employment) is log-linear in $\hat{q}$ for a given α, the right-tail of the firm size also follows the Pareto distribution with the same shape parameter κ.

Finally, the growth rate of the aggregate productivity is given by the following expression.

Proposition 3 *The growth rate of the aggregate productivity is given by*

$$g_q = (1-\delta)[(1+\lambda_I x_I)(1-\mu) + (1+\lambda_E)\mu] + \delta(1+\lambda_E)\bar{q}^h - 1,$$

where $\bar{q}_h$ is the average value of the distribution $h(\hat{q})$.

Proof. This can be shown by a simple accounting relation. Let the measure of q_t (without normalization) for active products be $z(q_t)$. The fraction $(1-\mu)x_I(1-\delta)$ of active lines experiences innovation by incumbents, and the fraction $(1-\mu-(1-\mu)x_I)(1-\delta)$ do not experience any innovation (but stay in the market). The fraction $\mu(1-\delta)$ of active products experiences innovation by entrants. Among the inactive products, the fraction $\mu(1-\delta)$ experiences innovation by entrants, but it is an upgrade from distribution $h(q_t/\bar{q}_t)$ rather than $z(q_t)/N$. Thus g_q can be calculated from

$$1 + g_q = (1-\delta)\left[(1+\lambda_I x_I)(1-\mu) + (1+\lambda_E)\mu + (1+\lambda)\mu\frac{1-N}{N}\frac{\bar{q}^h}{\bar{q}^z}\right].$$

[10] See, for example, **?**.

Here, $\bar{q}^h$ and $\bar{q}^z$ are averages of q_t with respect to distributions h and z. Thus $\bar{q}^h/\bar{q}^z = \int q_t h(q_t/\bar{q}_t)dq_t / \int q_t[z(q_t)/N]dq_t = \int \hat{q}h(\hat{q})d\hat{q} / \int \hat{q}[\hat{z}(\hat{q})/N]d\hat{q}$. Combining this with the expression for N in (9) and the fact that $\bar{q}^z = 1$ yields the above result. ■

Once the firing tax is introduced, x_I is no longer constant across firms, and therefore this formula is not valid. However, it is still useful to think of the effect of the policy on growth in these three components: the incumbents' innovation, the entrants' innovation on active products, and the entrants' innovation on non-active products.

4.2 Some characterization of the economy with firing tax

With firing tax, the employment decision by a firm is no longer static, and therefore the characterization is not as easy as in the case without firing tax. However, we can still make some progress so that the computational burden in the quantitative analysis can be eased. The main idea is to formulate the model in terms of the deviations from the frictionless outcome.

First, define the *frictionless level of employment without temporary shock* as

$$\ell^*(\hat{q};\hat{w},\hat{Y}) \equiv \arg\max_{\ell'}([\alpha\hat{q}]^{\psi}\ell'^{-\psi}\hat{Y}^{\psi} - \hat{w})\ell'$$

with $\alpha = 1$; that is,

$$\ell^*(\hat{q};\hat{w},\hat{Y}) = \left(\frac{1-\psi}{\hat{w}}\right)^{\frac{1}{\psi}}\hat{q}\hat{Y}.$$

Also define $\Omega(\hat{w},\hat{Y})$ by

$$\Omega(\hat{w},\hat{Y}) \equiv \frac{\ell^*(\hat{q};\hat{w},\hat{Y})}{\hat{q}}.$$

In addition, define *deviation of employment from frictionless level* by

$$\tilde{\ell} \equiv \frac{\ell}{\ell^*(\hat{q};\hat{w},\hat{Y})}.$$

Similarly, let

$$\tilde{\ell}' \equiv \frac{\ell'}{\ell^*(\hat{q};\hat{w},\hat{Y})}.$$

Then, the period profit (8) can be rewritten as

$$\hat{\Pi}(\hat{q},\alpha,\ell,\ell',x_I) = \left(\left[\frac{\alpha}{\Omega(\hat{w},\hat{Y})} \right]^{\psi} \tilde{\ell}'^{-\psi} \hat{Y}^{\psi} - \hat{w} \right) \hat{q}\Omega(\hat{w},\hat{Y})\tilde{\ell}' - \theta_I \hat{q} x_I{}^{\gamma} - \tau\hat{w} \max\langle 0, \hat{q}\Omega(\hat{w},\hat{Y})\tilde{\ell} - \hat{q}\Omega(\hat{w},\hat{Y})\tilde{\ell}' \rangle.$$

Thus this is linear in $\hat{q}$, and can be rewritten as $\hat{q}\tilde{\Pi}(\alpha,\tilde{\ell},\tilde{\ell}',x_I)$, where

$$\tilde{\Pi}(\alpha,\tilde{\ell},\tilde{\ell}',x_I) \equiv \left(\left[\frac{\alpha}{\Omega(\hat{w},\hat{Y})} \right]^{\psi} \tilde{\ell}'^{-\psi} \hat{Y}^{\psi} - \hat{w} \right) \Omega(\hat{w},\hat{Y})\tilde{\ell}' - \theta_I x_I{}^{\gamma} - \tau\Omega(\hat{w},\hat{Y})\hat{w} \max\langle 0, \tilde{\ell} - \tilde{\ell}' \rangle.$$

Because the period return function is linear in $\hat{q}$, it is straightforward to show that all value functions are linear in $\hat{q}$. Defining $\tilde{Z}(\alpha,\tilde{\ell})$ from $\hat{Z}(\hat{q},\alpha,\ell) = \hat{q}\tilde{Z}(\alpha,\tilde{\ell})$, (6) can be rewritten as

$$\tilde{Z}(\alpha,\tilde{\ell}) = (1-\delta)\tilde{V}^s(\alpha,\tilde{\ell}) + \delta\tilde{V}^o(\tilde{\ell}),$$

where $\tilde{V}^o(\tilde{\ell})$ is from $\hat{V}^o(\ell) = \hat{q}\tilde{V}^o(\tilde{\ell})$ and thus

$$\tilde{V}^o(\tilde{\ell}) = -\tau\hat{w}\Omega(\hat{w},\hat{Y})\tilde{\ell}$$

and $\tilde{V}^s(\tilde{\ell})$ is from $\hat{V}^s(\hat{q},\ell) = \hat{q}\tilde{V}^s(\tilde{\ell})$ with

$$\tilde{V}^s(\tilde{\ell}) = \max_{\tilde{\ell}' \geq 0, x_I} \tilde{\Pi}(\alpha,\tilde{\ell},\tilde{\ell}',x_I) + \beta \left((1-\mu)\frac{\tilde{S}(x_I,\tilde{\ell}')}{1+g_q} - \mu\tau\hat{w}\Omega(\hat{w},\hat{Y})\tilde{\ell}' \right) \tag{10}$$

Here, the expression $\tilde{S}(x_I,\tilde{\ell}')/(1+g_q)$ comes from $\hat{S}(x_I,\hat{q}/(1+g_q),\ell') = \hat{q}\tilde{S}(x_I,\tilde{\ell}')/(1+g_q)$. The linearity of the value functions implies that

$$\frac{\tilde{S}(x_I,\tilde{\ell}')}{1+g_q} = (1-x_I)E_{\alpha'}\left[\tilde{Z}\left(\alpha',(1+g_q)\tilde{\ell}'\right)\right]\frac{1}{1+g_q} + x_I E_{\alpha'}\left[\tilde{Z}\left(\alpha',\frac{(1+g_q)\tilde{\ell}'}{1+\lambda_I}\right)\right]\frac{1+\lambda_I}{1+g_q}$$

also holds. Here we used that

$$\hat{Z}(\hat{q}',\alpha',\ell') = \hat{q}'\tilde{Z}\left(\alpha',\frac{\ell'}{\ell^*(\hat{q}';\hat{w}',\hat{Y}')}\right) = \hat{q}'\tilde{Z}\left(\alpha',\frac{\ell^*(\hat{q};\hat{w},\hat{Y})}{\ell^*(\hat{q}';\hat{w}',\hat{Y}')}\frac{\ell'}{\ell^*(\hat{q};\hat{w},\hat{Y})}\right)$$

with $\hat{w}' = \hat{w}$, $\hat{Y}' = \hat{Y}$; and that $\ell^*(\hat{q};\hat{w},\hat{Y})/\ell^*(\hat{q}';\hat{w}',\hat{Y}') = \hat{q}/\hat{q}'$ yields

$$\hat{Z}(\hat{q}',\alpha',\ell') = \hat{q}'\tilde{Z}\left(\alpha',\frac{\hat{q}}{\hat{q}'}\tilde{\ell}'\right)$$

for $\hat{q}' = \hat{q}/(1+g_q)$ and $\hat{q}' = (1+\lambda_I)\hat{q}/(1+g_q)$.

The optimization problem in (10) has two choice variables, $\tilde{\ell}'$ and x_I. The first-order condition for x_I is

$$\gamma\theta_I x_I^{\gamma-1} = \Gamma_I$$

and thus x_I can be computed from

$$x_I = \left(\frac{\Gamma_I}{\gamma\theta_I}\right)^{1/(\gamma-1)},$$

where $\Gamma_I \equiv \beta(1-\mu)E_{\alpha'}\left[\tilde{Z}(\alpha',(1+g_q)\tilde{\ell}'/(1+\lambda_I))(1+\lambda_I) - \tilde{Z}(\alpha',(1+g_q)\tilde{\ell}')\right]/(1+g_q)$. From here, it is easy to see that x_I is uniquely determined once we know $\tilde{\ell}'$. Let the decision rule for $\tilde{\ell}'$ in the right-hand side of (10) be $\mathcal{L}'(\alpha,\tilde{\ell})$. Then the optimal x_I can be expressed as $x_I = \mathcal{X}_I(\alpha,\tilde{\ell})$. This implies that x_I is independent of $\hat{q}$.

5 Computation and calibration

The details of the computational methods are described in Appendix A. Our method involves similar steps to the standard general-equilibrium firm dynamics model. Similarly to Hopenhayn and Rogerson (1993) and Lee and Mukoyama (2008), we first make a guess on relevant aggregate variables (in our case $\hat{w}$, μ, g, and $\hat{Y}$), solve the optimization problems given these variables, and then update the guess using the equilibrium conditions. This procedure is also similar to how the standard Bewley-Huggett-Aiyagari models of heterogeneous consumers are typically computed. This separates our work from recent models of innovation and growth, such as Klette and Kortum (2004), Acemoglu et al. (2013), and Akcigit and Kerr (2015), as these models heavily rely on analytical characterization in continuous-time setting.

Following a strategy similar to Hopenhayn and Rogerson (1993), we calibrate the parameters of the model under the assumption that firing costs are equal to zero and use US data to compute our targets.[11] All firm dynamics data used are computed from the establishment-level statistics of the U.S. Census Business Dynamics Statistics (BDS) data.[12]

[11]See Appendix B for the analytical characterization of the model without firing costs.
[12]The data are publicly available at "http://www.census.gov/ces/dataproducts/bds/data.html".

[The calibration is preliminary] The model period is one year. The discount rate β is set to 0.947 in line with Cooley and Prescott (1995). The value of ψ is set to 0.2 which implies an elasticity of substitution across goods of 5. This value is in the range of Broda and Weinstein's (2006) estimates. Our value of 0.2 implies a markup of 25% which is in line with the estimates of de Loecker and Warzynski (2012). We set the curvature of the innovation cost γ to 2. As noted by Acemoglu et al. (2013), $1/\gamma$ can be related to the elasticity of patents to R&D spending, which has been found to be between 0.3 and 0.6.[13] These estimates imply that γ is between 1.66 and 3.33.

We set the value of disutility of labor ξ so that the employment to population ratio is equal to 0.6. The parameters of the innovation and productivity processes are pinned down from the job flows data. We assume that the transitory shock α is uniformly distributed, and can take three values $\exp(-\varepsilon), 1$ and $\exp(\varepsilon)$, with probability 1/3 for each value. The cost of innovating is assumed to be identical for entrants and incumbents: $\theta_E = \theta_I \equiv \theta$. We then set, θ, ε, and ϕ to match the growth rate of output of 2%, the job creation rates of entering and that of incumbent establishments. For now, we assume that the size of the innovation of entrant and incumbent is equal $\lambda_E = \lambda_I \equiv \lambda$. [In the future, we will also consider the case where they differ.] The innovation size λ is pinned down by the ratio of the number of expanding establishments over the number of contracting establishments. For given values of the growth rate of the economy and the job creation by entrants, a higher λ implies a lower innovation rate for incumbents and therefore a smaller proportion of expanding establishments. Finally, we assume that the distributions of productivity $\hat{q}$ and α on inactive product lines are independent. When they innovate on an inactive product line, the entrants draw the productivity upon which they innovate from a uniform distribution over $[0,\ 2\ \times \bar{q}^h]$. We set the mean $\bar{q}^h$ together with the exogenous exit rate δ to match the proportion of establishment with 0-5 employees and the tail index κ of the productivity distribution[14]. The parameter values are summarized in Table 1.

[13]See for example Griliches (1990).

[14]See Section 4.1 for the expression of the tail index.

Table 1: Calibration

	Parameter	New Calib
Discount rate	β	0.947
Disutility of labor	ξ	1.55
Demand elasticity	ψ	1/5
Innovation step	λ	0.20
Innovation curvature	γ	2.00
Innovation cost	θ_I,θ_E	0.16
Entry cost	ϕ	1.06
Exogenous exit rate	δ	0.021
Transitory shock	ϵ	0.20
Inactive lines	h mean	0.25
Firing tax	τ	0

Table 2: Comparison between the U.S. data and the model outcome

	Data	Model
growth rate g		2.12
R&D ratio R/Y		13.47
L	60	59.65
JC rate	17.0	16.68
JC from entry	6.4	6.23
JD rate	15.0	16.25
JD from exit	5.3	7.22
nb estab. exp./contract.	1.01	1.03
Tail index	1.06	1.064

Table 2 compares the baseline outcome and the targets. The growth rate of output is about 2.1%; this is also the growth rate of the quality index Q_t. The table shows that the other targets – the job creation and job destruction rates, the share of expanding establishments and the tail index of the productivity distribution – roughly match the data magnitude. We also report the R&D expenditures as a share of aggregate output though we do not use it as a target in the calibration. The R&D ratio, at about 14%, is larger than what we typically see from conventional measures of R&D spending. However, because our model intends to capture innovation in a broad sense and to include various methods of product and technology improvements that are not necessarily included in formal R&D measurements, it is more appropriate the compare the model amount of R&D to a broader statistics than the

Table 3: Size distribution of firms

	Data	Model
1-5	0.495	0.425
5-10	0.223	0.236
10-20	0.138	0.168
20-50	0.089	0.103
50-100	0.030	0.036
100+	0.024	0.032

conventional measure of R&D. The output share of R&D spending is in line with Corrado et al.'s (2009) estimate of the U.S. intangible investments in the 1990s.

Table 3 compares the size distribution of firms in the data and in the model. The two match well. The model hence captures the salient features of the U.S. establishment dynamics and size distribution.

6 Quantitative results

[Results are preliminary]

Table 4: The effects of firing costs

	$\tau = 0$	baseline $\tau = 0.3$	fixed entry $\tau = 0.3$	no exit tax $\tau = 0.3$
growth rate g	2.12	2.37	2.11	2.33
R&D ratio R/Y	13.47	13.02	13.68	13.06
innov rate entry μ	5.23	4.00	5.23	4.28
innov rate inc. x_I	52.00	58.49	52.10	57.50
Employment L	59.65	58.60	58.84	0.597
Average productivity $\hat{Y}/L$	100	97.43	99.6	97.96
JC rate	16.68	6.50	7.72	6.78
JC from entry	6.23	4.27	5.47	4.54
JD rate	16.25	6.14	7.35	6.41
JD from exit	7.22	6.01	7.22	6.29

Note: the average productivity $\hat{Y}/L$ is normalized to 100 in the benchmark with no firing costs.

We now turn to the effects of employment protection. Table 4 compares the baseline result with the firing tax of $\tau = 0.3$; that is, the cost of dismissal per worker amounts to 3.6

months of wages. [We plan to evaluate the effects of cross-country differences in firing costs].

The second column of the table gives the results of the baseline experiment. Imposing the firing tax leads to a decline in labor demand and hence to a reduction in employment. The firing cost also substantially reduces the labor market reallocation, measured by job creation and job destruction. The implications of the reduction in labor market flows for misallocation and the level of aggregate productivity have been emphasized by Hopenhayn and Rogerson (1993). Similarly, we find that the firing tax reduces the average level of productivity, $\hat{Y}/L$, by 2.6 percents.

In addition to this effect on the level of productivity, our model shows that firing costs also affect the growth rate of productivity. Firing costs reduce the entrants' incentives to innovate. The innovation rate by entrants falls by about 1.3 percentage points. The incentive for entrants' to innovate is reduced because of two factors. First, the firing tax has a direct effect on expected profits as it raises the cost of operating a firm. Second, firing costs prevent firms from reaching their optimal scale and this misallocation reduces the entrants' expected profits.

By contrast, the incumbents' innovation *increases* by about 6 percentage points as a result of the firing tax. The consequences of the firing tax on the incumbents' incentive to innovate are theoretically ambiguous. The misallocation of labor may reduce the incumbents' incentives to innovate since it prevents the firm from reaching its optimal size. But this only holds for firms that are below their optimal size. On the other hand, firms that are larger than their optimal size, either because of a negative transitory shock or because they have been unsuccessful at innovating, have now stronger incentives to invest in R&D. By innovating, firms also avoid paying the firing cost. In addition, the incumbents' incentives to innovate also depend on the entry rate. A lower entry rate reduces the risk for incumbents of being taken over by an entrant, which raises the return of the firm's R&D investment. Firing costs, by reducing the rate of creative destruction may therefore lead to an increase in the incumbents' innovation rate. In our baseline calibration, the effect of the reduction of the entry rate and the positive effect of misallocation dominate and firing costs lead to an increase in the incumbents' innovation.

To assess the importance of the entry rate for the incumbents' innovation, we consider an alternative experiment in which we maintain the entry rate to the level of the economy without firing costs. The results are reported in the third column of Table 4. When the entry rate is held constant, the incumbents' innovation only slightly increases. This result indicates that the decline in the entry rate is key to understanding the increase in the incumbents' innovation. Without the decline in entry, the effects of the firing costs would have limited consequences on the incumbents' innovation as the positive and negative consequences of misallocation seem to roughly offset each other.

In the last experiment, we assume that incumbents do not have to pay the firing costs when they exit. As shown in the fourth column of the table, this assumption mainly affects the entrants' innovation. Compared to the baseline experiment, the value of entry declines less as firing costs do not increase the cost of operating a firm as much if firms do not have to pay the firing costs when they exit. As a result, entry falls less, which leads to a smaller increase in the incumbents' innovation.

All in all, our results illustrate the importance of including the incumbents' innovation in the analysis. We find that firing costs are likely to affect the innovation of entrants and incumbents in opposite directions. The aggregate growth rate could therefore increase or decrease as a result. In our baseline experiment, the positive effect on the incumbents dominates quantitatively and the growth rate is higher when firms face firing costs.

7 Conclusion

In this paper, we constructed a general equilibrium model of firm dynamics with endogenous innovation. The firms not only decide on production and employment, but also entry and expansion through innovation. Therefore, the productivity shocks that firms face are endogenous in our framework, and they react to economic incentives.

Our framework allows us to examine how barriers to reallocation affect firm dynamics and aggregate economic growth. This paper examined a particular type of barriers: a firing tax.

We found that a firing tax can have opposite effects on entrants' innovation and incumbents' innovation. A firing tax reduces job reallocation and entrants' innovation, while it may enhance incumbents' innovation.

Because the process of innovation inherently involves randomness, the incentive to innovate affects the risks that each firm faces from their own innovation. It also affects the risks that firms face from other firms' innovation, in the form of creative destruction. When there are barriers to reallocating productive resources, there is a natural feedback process between the misallocation of resources and innovation: misallocation affects incentive to innovate, and this in turn changes the process of shocks that affects misallocation. It is a promising future research topic to further investigate this interaction both theoretically and empirically.

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Appendix

A Details of computation

The computation is done by first guessing the relevant aggregate variables, perform optimization and derive the value function trough iteration, and then updating the guess.

The procedure is as follows.

1. Before finding the iteration, several variables can be computed from parameters. First, calculate x_E^* from
$$x_E^* = \left(\frac{\phi}{\theta_E(\gamma-1)}\right)^{\frac{1}{\gamma}}.$$
2. Then $\hat{\hat{V}}_E$ can be computed from
$$\hat{\hat{V}}_E = \frac{\gamma\theta_E}{\beta}{x_E}^{\gamma-1}.$$
3. Start the iteration. Guess $\hat{Y}$, $\hat{w}$, m, and g. We use the following guess:
$$0 \leq \hat{Y} \leq 2 \times target(L),$$
$$0 \leq \hat{w} \leq 2 \times (1-\psi),$$
$$0 \leq g \leq 2 \times \lambda((1-\mu)x_I + \mu),$$
Given m, we can calculate the value of μ by $\mu = X_E = mx_E^*$. Now we are ready to solve the Bellman equation for the incumbents.

   We have two choice variables, $\tilde{\ell}'$ and x_I. The first-order condition for x_I is
$$\gamma\theta_I x_I^{\gamma-1} = \Gamma_I$$
and thus x_I can be computed from
$$x_I = \left(\frac{\Gamma_I}{\gamma\theta_I}\right)^{1/(\gamma-1)},$$
where $\Gamma_I \equiv \beta(1-\mu)E_{\alpha'}\left[\tilde{Z}(\alpha',(1+g_q)\tilde{\ell}'/(1+\lambda_I))(1+\lambda_I) - \tilde{Z}(\alpha',(1+g_q)\tilde{\ell}')\right]/(1+g_q)$. We can see that x_I is uniquely determined once we know $\tilde{\ell}'$. Let the decision rule for $\tilde{\ell}'$ be $\mathcal{L}'(\alpha,\tilde{\ell})$. Then $x_I = \mathcal{X}_I(\alpha,\tilde{\ell})$.

4. Once all decision rules are computed, with iterative procedure we can find $f(\hat{q}, \alpha, \tilde{\ell})$ by iterating over density.

5. Now, we check if the first guesses are consistent with the solution from the optimization. The values of $\hat{w}$ and

$$\begin{aligned} \hat{Y} &= \left(\int \int \int [\alpha \hat{q}]^{\psi} [\ell^*(\hat{q}; \hat{w}, \hat{Y}) \mathcal{L}'(\alpha, \tilde{\ell})]^{1-\psi} f(\hat{q}, \alpha, \tilde{\ell}) d\hat{q} d\alpha d\tilde{\ell} \right)^{\frac{1}{1-\psi}} \\ &= \left(\int \int \alpha^{\psi} [\Omega(\hat{w}, \hat{Y}) \mathcal{L}'(\alpha, \tilde{\ell})]^{1-\psi} \int \hat{q} f(\hat{q}, \alpha, \tilde{\ell}) d\hat{q} d\alpha d\tilde{\ell} \right)^{\frac{1}{1-\psi}} \end{aligned}$$

and

$$\frac{\hat{w}}{\hat{Y} - \hat{R}} = \xi,$$

where

$$\begin{aligned} \hat{R} &= \int \int \int \theta_I \hat{q} \mathcal{X}_I(\alpha, \tilde{\ell})^{\gamma} f(\hat{q}, \alpha, \tilde{\ell}) d\hat{q} d\alpha d\tilde{\ell} + m(\phi + \theta_E x_E^{\gamma}) \\ &= \theta_I \int \int \mathcal{X}_I(\alpha, \tilde{\ell})^{\gamma} \int \hat{q} f(\hat{q}, \alpha, \tilde{\ell}) d\hat{q} d\alpha d\tilde{\ell} + m(\phi + \theta_E x_E^{\gamma}) \end{aligned}$$

In order to check the value of g_q, the condition $\frac{1}{N} \int \int \int \hat{q} f(\hat{q}, \alpha, \tilde{\ell}) d\alpha d\ell dq = 1$ is used. Intuitively, when g_q is too small, the stationary density $f(\hat{q}, \alpha, \tilde{\ell})$ implies the values of $\hat{q}$ that are too large.

For m, because a large m implies a large μ, which in turn lowers $\tilde{Z}$. Thus it affects the computed value of $\hat{\tilde{V}}_E$, through $\tilde{Z}$. Recall that

$$\hat{\tilde{V}}_E = \frac{\gamma \theta_E}{\beta} {x_E}^{\gamma - 1}$$

has to be satisfied, and this has to be equal to

$$\begin{aligned} \hat{\tilde{V}}_E &= \int \left[\int \hat{Z}((1+\lambda_E)\hat{q}/(1+g_q), \alpha, 0)(\bar{f}(\hat{q}) + (1-N)h(\hat{q})) d\hat{q} \right] \omega(\alpha) d\alpha \\ &= \int \left[\int \hat{q} \tilde{Z}(\alpha, 0)(1+\lambda_E)/(1+g_q)(\bar{f}(\hat{q}) + (1-N)h(\hat{q})) d\hat{q} \right] \omega(\alpha) d\alpha \\ &= \int \tilde{Z}(\alpha, 0) \omega(\alpha) d\alpha \left[N + (1-N) \int h(\hat{q}) d\hat{q} \right] (1+\lambda_E)/(1+g_q) \end{aligned}$$

because $\int \hat{q} \bar{f}(\hat{q}) d\hat{q} = N$.

6. Go back to Step 3, until convergence.

B Analytical characterization of the case without firing tax

This section characterizes the model without firing tax and boils it down to a system of nonlinear equations. The derivations also serve as proofs for Propositions 1 and 2.

B.1 Model solution

Note first that for a given μ, the number of actively produced product, N, is calculated by (9). Recall that μ is an endogenous variable, and it is determined by the entrants' innovation:

$$\mu = m {x_E}^*.$$

As we have seen, ${x_E}^*$ is a function of parameters

$$x_E^* = \left(\frac{\phi}{\theta_E(\gamma - 1)} \right)^{\frac{1}{\gamma}},$$

and thus μ (and also N) is a function of m. In particular, note that N is an increasing function of m.

Because there are no firing taxes, the previous period employment, ℓ, is not a state variable anymore. The measure of individual states can be written as $f(\hat{q}, \alpha)$, and because $\hat{q}$ and α are independent, we can write $f(\hat{q}, \alpha) = \hat{z}(\hat{q})\omega(\alpha)$. In particular, note that $\int \hat{q}\hat{z}(\hat{q})d\hat{q} = N$, because $\hat{q}$ is the value of q_t normalized by its average. We also assume that $\omega(\alpha)$ is such that $\int \alpha\omega(\alpha)d\alpha = 1$.

Without firing costs, the labor can be adjusted freely. Thus the intermediate-good firm's decision for ℓ' is essentially static:

$$\max_{\ell'} \quad \hat{\pi} \equiv ([\alpha\hat{q}]^{\psi}\ell'^{-\psi}\hat{Y}^{\psi} - \hat{w})\ell'. \tag{11}$$

From the first-order condition,

$$\ell' = \left(\frac{1-\psi}{\hat{w}} \right)^{\frac{1}{\psi}} \alpha\hat{q}\hat{Y} \tag{12}$$

holds. Because $y = \ell'$, we can plug this into the definition of $\hat{Y}$:

$$\hat{Y} = \left(\int \int [\alpha\hat{q}]^{\psi} y^{1-\psi} \hat{z}(\hat{q})\omega(\alpha) d\hat{q} d\alpha \right)^{\frac{1}{1-\psi}}.$$

This yields

$$\hat{Y} = \hat{Y}\left(\frac{1-\psi}{\hat{w}}\right)^{\frac{1}{\psi}} N^{\frac{1}{1-\psi}}$$

and therefore

$$\hat{w} = (1-\psi)N^{\frac{\psi}{1-\psi}}. \tag{13}$$

Recall that N is a function of the endogenous variable m. Thus $\hat{w}$ is also a function of m.

The equations (12) and (13) can also be combined to

$$\ell' = \alpha\hat{q}\hat{Y}N^{-\frac{1}{1-\psi}}. \tag{14}$$

Integrating this across all active firms yield

$$\int\int \ell'\hat{z}(\hat{q})\omega(\alpha)d\hat{q}d\alpha = N^{-\frac{1}{1-\psi}}\hat{Y}\int\int \alpha\hat{q}\hat{z}(\hat{q})\omega(\alpha)d\hat{q}d\alpha = N^{-\frac{\psi}{1-\psi}}\hat{Y}.$$

The left-hand side is the aggregate employment L. Thus,

$$L = N^{-\frac{\psi}{1-\psi}}\hat{Y}.$$

One way of looking at this equation is that $\hat{Y}$ can be pinned down once we know L and N (and thus L and m). Plugging (13) and (14) into (11) yields

$$\hat{\pi} = \psi\alpha\hat{q}\frac{\hat{Y}}{N}.$$

Now, let us characterize the innovation decision of a intermediate-good firm. Recall that the value functions are (with our simplifications)

$$\hat{Z}(\hat{q},\alpha) = (1-\delta)\hat{V}^s(\hat{q},\alpha),$$

where

$$\hat{V}^s(\hat{q},\alpha) = \max_{x_I} \quad \psi\alpha\hat{q}\frac{\hat{Y}}{N} - \theta_I\hat{q}{x_I}^{\gamma} + \beta(1-\mu)\hat{S}(x_I, \hat{q}/(1+g_q)) \tag{15}$$

and

$$\hat{S}(x_I,\hat{q}/(1+g_q)) = (1-x_I)\int \hat{Z}(\hat{q}/(1+g_q),\alpha')\omega(\alpha')d\alpha' + x_I\int \hat{Z}((1+\lambda_I)\hat{q}/(1+g_q),\alpha')\omega(\alpha')d\alpha'.$$

We start from making a guess that $\hat{Z}(\hat{q},\alpha)$ takes the form

$$\hat{Z}(\hat{q},\alpha) = \mathcal{A}\alpha\hat{q} + \mathcal{B}\hat{q},$$

where $\mathcal{A}$ and $\mathcal{B}$ are constants. With this guess, the first-order condition in (15) for x_I is

$$\gamma\theta_I\hat{q}x_I{}^{\gamma-1} = \frac{\beta(1-\mu)\lambda_I(\mathcal{A}+\mathcal{B})\hat{q}}{1+g_q}.$$

Thus

$$x_I = \left(\frac{\beta(1-\mu)\lambda_I(\mathcal{A}+\mathcal{B})}{(1+g_q)\gamma\theta_I}\right)^{\frac{1}{\gamma-1}} \tag{16}$$

and x_I is constant across $\hat{q}$ and α. Using this x_I,

$$\hat{Z}(\hat{q},\alpha) = (1-\delta)\left(\psi\alpha\hat{q}\frac{\hat{Y}}{N} - \theta_I\hat{q}x_I{}^{\gamma} + \beta(1-\mu)\frac{1+x_I\lambda_I}{1+g_q}(\mathcal{A}+\mathcal{B})\hat{q}\right).$$

Thus, the guess is verified with

$$\mathcal{A} = (1-\delta)\psi\frac{\hat{Y}}{N}$$

and $\mathcal{B}$ is a value that solves

$$\mathcal{B} = (1-\delta)\left(-\theta_I x_I{}^{\gamma} + \beta(1-\mu)\frac{1+x_I\lambda_I}{1+g_q}(\mathcal{A}+\mathcal{B})\right) = (1-\delta)\beta(1-\mu)\left(1+\frac{\gamma-1}{\gamma}\lambda_I x_I\right)\frac{\mathcal{A}+\mathcal{B}}{1+g_q},$$

where x_I is given by (16). Therefore, we found that x_I (and the coefficients of $\hat{Z}(\hat{q},\alpha)$ function) is a function of endogenous aggregate variables μ, g_q, $\hat{Y}$, and N. We have already seen that we can pin down μ and N if we know m, and $\hat{Y}$ can be pinned down if we know m and L. How about g_q?

To calculate g_q, let us start from the measure of q_t (without normalization) for active products, $z(q_t)$. As we have seen above, the transitory shock α does not affect the innovation decision, and thus can be ignored when calculating the transition of q_t. The fraction $(1-\mu)x_I(1-\delta)$ of active lines experiences innovation by incumbents, and the fraction $(1-\mu-(1-\mu)x_I)(1-\delta)$ do not experience any innovation (but stay in the market). The fraction $\mu(1-\delta)$ of active products experiences innovation by entrants. Among the inactive products, the fraction $\mu(1-\delta)$ experiences innovation by entrants, but it is an upgrade from distribution $h(q_t/\bar{q}_t)$ rather than $z(q_t)/N$. Thus g_q can be calculated from

$$1+g_q = (1-\delta)\left[(1+\lambda_I x_I)(1-\mu) + (1+\lambda_E)\mu + (1+\lambda_E)\mu\frac{1-N}{N}\frac{\bar{q}^h}{\bar{q}^z}\right].$$

The first term is the productivity increase of the surviving incumbents, the second term is the entry into active products, and the last is the entry into inactive products. Here, $\bar{q}^h$ and $\bar{q}^z$ are averages of q_t with respect to distributions h and z. Thus $\bar{q}^h/\bar{q}^z = \int q_t h(q_t/\bar{q}_t)dq_t / \int q_t[z(q_t)/N]dq_t = \int \hat{q}h(\hat{q})d\hat{q}/\int \hat{q}[\hat{z}(\hat{q})/N]d\hat{q}$. Using the expression for N in (9) and the fact that $\bar{q}^z = 1$,

$$g_q = (1-\delta)[(1+\lambda_I x_I)(1-\mu) + (1+\lambda_E)\mu] + \delta(1+\lambda_E)\bar{q}^h - 1.$$

Thus, g_q can be written as a function of μ and x_I, and therefore m and L.

From above procedure, we found that once we pin down m and L, we can determine all endogenous variables in the economy. The values of m and L can be pinned down by two additional conditions: the labor-market equilibrium condition and the free-entry condition. To see this, let us first be explicit about each variable's (and each coefficient's) dependence on m and L: $\hat{w}(m)$, $N(m)$, $\hat{Y}(m,L)$, $x_I(m,L)$, $g_q(m,L)$, $\mathcal{A}(m,L)$, and $\mathcal{B}(m,L)$. Also note that the total R&D, $\hat{R}$, can be written as

$$\hat{R} = \int \theta_I \hat{q} x_I(m,L)^\gamma \hat{z}(\hat{q})d\hat{q} + m(\phi + \theta_E {x_E}^\gamma) = \theta_I N(m) x_I(m,L)^\gamma + m(\phi + \theta_E {x_E}^\gamma)$$

and therefore we can write $\hat{R}(m,L)$.

The labor-market equilibrium condition is

$$\frac{\hat{w}(m)}{\hat{Y}(m,L) - \hat{R}(m,L)} = \xi$$

and the free-entry condition is

$$\begin{aligned}
\frac{\gamma\theta_E {x_E}^{\gamma-1}}{\beta} = \hat{\bar{V}}_E &= \int \left[\int \hat{Z}((1+\lambda_E)\hat{q}/(1+g_q),\alpha)(\hat{z}(\hat{q}) + (1-N)h(\hat{q}))d\hat{q}\right]\omega(\alpha)d\alpha \\
&= \int \frac{\mathcal{A}(m,L)+\mathcal{B}(m,L)}{1+g_q(m,L)}(1+\lambda_E)\hat{q}(\hat{z}(\hat{q}) + (1-N)h(\hat{q}))d\hat{q} \\
&= \frac{\mathcal{A}(m,L)+\mathcal{B}(m,L)}{1+g_q(m,L)}(1+\lambda_E)[N(m) + (1-N(m))\bar{q}^h].
\end{aligned}$$

These two equations pin down the values of m and L.

B.2 Productivity distribution

The invariant distribution $\hat{z}(\hat{q})$ can easily be computed. The next-period mass at relative quality $\hat{q}$ is (i) upgrade by incumbents' innovation: $(1-\delta)(1-\mu)x_I\hat{z}((1+g_q)\hat{q}/(1+\lambda_I))d\hat{q}$, (ii) upgrade by entrants' innovation: $(1-\delta)\mu\hat{z}((1+g_q)\hat{q}/(1+\lambda_E))d\hat{q}$, (ii) natural downgrade from non-innovating products: $((1-\delta)(1-\mu-(1-\mu)x_I)\hat{z}((1+g_q)\hat{q})d\hat{q}$, and (iii) entry from inactive products, $(1-\delta)\mu(1-N)h(\hat{q}/(1+\lambda_E))dq$. The sum of these has to be equal to $\hat{z}(\hat{q})d\hat{q}$ along the stationary growth path.

In fact, it is possible to characterize the right tail of the distribution analytically, when the distribution $h(\hat{q})$ is bounded. Let the density function of the stationary distribution be $s(\hat{q}) \equiv \hat{z}(\hat{q})/N$. Because $h(\hat{q})$ is bounded, there is no direct inflow from the inactive product lines at the right tail.

Take the point $\hat{q}$ and interval Δ around that point. The outflow from that interval is $s(\hat{q})\Delta$, as entire firms there will either move up, move down, or exit.

The inflow is from two sources. First is the mass of firms who innovated. Innovation is either done by incumbents or entrants. Let $\gamma_i \equiv (1+\lambda_I)/(1+g_q) > 1$ be the (adjusted) improvement of $\hat{q}$ upon innovation by incumbents. The probability of innovation by either incumbents is $(1-\delta)(1-\mu)x_I$. Thus the mass of inflow into the above interval is $(1-\delta)(1-\mu)x_I s(\hat{q}/\gamma_i)\Delta/\gamma_i$. Similarly, letting $\gamma_e \equiv (1+\lambda_E)/(1+g_q) > 1$ be the improvement of $\hat{q}$ upon innovation by entrants, the mass of inflow due to entrants' innovation is $(1-\delta)\mu s(\hat{q}/\gamma_e)\Delta/\gamma_e$.

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The second inflow is the firms that didn't innovate or exit. With probability $(1-\delta)(1-\mu-(1-\mu)x_I)$, there are no innovations (nor exit). Let $\gamma_n \equiv 1/(1+g_q) < 1$ be the (adjusted) improvement (in this case the "negative improvement") when there is no innovation. Then the mass of inflow into above interval is $(1-\delta)(1-\mu-(1-\mu)x_I)s(\hat{q}/\gamma_n)\Delta/\gamma_n$.

In the stationary distribution, inflow equals outflow, and therefore

$$s(\hat{q})\Delta = (1-\delta)\left[(1-\mu)x_I s\left(\frac{\hat{q}}{\gamma_i}\right)\frac{\Delta}{\gamma_i} + \mu s\left(\frac{\hat{q}}{\gamma_e}\right)\frac{\Delta}{\gamma_e} + (1-\mu-(1-\mu)x_I)s\left(\frac{\hat{q}}{\gamma_n}\right)\frac{\Delta}{\gamma_n}\right],$$

or

$$s(\hat{q}) = (1-\delta)\left[(1-\mu)x_I s\left(\frac{\hat{q}}{\gamma_i}\right)\frac{1}{\gamma_i} + \mu s\left(\frac{\hat{q}}{\gamma_e}\right)\frac{1}{\gamma_e} + (1-\mu-(1-\mu)x_I)s\left(\frac{\hat{q}}{\gamma_n}\right)\frac{1}{\gamma_n}\right],$$

Guess that the right-tail of the density function is Pareto and has the form $s(x) = Fx^{-(\kappa+1)}$. ξ is the shape parameter and the expected value of x exists only if $\kappa > 1$.

Plugging this guess into the above yields

$$F\hat{q}^{-(\kappa+1)} = (1-\delta)\left[(1-\mu)x_I F\left(\frac{\hat{q}}{\gamma_i}\right)^{-(\kappa+1)}\frac{1}{\gamma_i} + \mu F\left(\frac{\hat{q}}{\gamma_e}\right)^{-(\kappa+1)}\frac{1}{\gamma_e} + (1-\mu-(1-\mu)x_I)F\left(\frac{\hat{q}}{\gamma_n}\right)^{-(\kappa+1)}\frac{1}{\gamma_n}\right],$$

or

$$1 = (1-\delta)\left[(1-\mu)x_I\gamma_i^{\kappa} + \mu\gamma_e^{\kappa} + (1-\mu-(1-\mu)x_I)\gamma_n^{\kappa}\right].$$

Thus, κ is the solution of this equation.