Prize allocation and incentives in team contests

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Abstract

We study a contest between teams that compete for multiple indivisible prizes. Team output is a CES function of all the team members' efforts. We use a generalized Tullock contest success function to allocate prizes between teams. We study how different intra-team prize allocation rules impact team output. We consider an egalitarian rule that gives all members the same chance of receiving a prize, and a list rule that sets ex-ante the order in which members receive a prize. The convexity of the cost of effort function and the complementarity of individual efforts determine which rule maximizes team output and success. Our results speak to many reald world situations, such as elections, contests for the allocation of local public goods and the internal organization of firms.

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1 Introduction

In many economic and political settings, contests are used to allocate indivisible prizes between teams of individuals. Each team competes against other teams to win as many prizes as possible. Prizes are allocated to teams on the basis of their output. The prizes won are then allocated between team members. To maximize its output, a team needs to provide the right incentives to its members. The design of an efficient intra-team prize allocation mechanism thus requires to take into account both inter-team and intra-team strategic considerations.

Our leading example comes from parliamentary elections. Political parties are best described as teams of individual candidates (see Downs (1957)). In elections under proportional representation in multiple-seat electoral districts, political parties propose to voters a list of politicians who compete as a team to win as many legislative seats as possible. Legislative seats are allocated to parties proportionally to their vote shares. These vote shares depend in turn on the list's 'electoral output' that team members generate during the electoral campaign. Within parties, each legislative seat won by a party is an individual prize that politicians on the list strive to win. Under *closed-list* proportional representation —the electoral rule used in, for instance, Argentina, Germany (partially), Iceland, Israel and Spain— voters cannot vote for individual candidates, but only for the entire party list and seats are allocated to politicians strictly following the order of the party list.[1] Our model allows us to understand the conditions under which an ordered list of candidates is the optimal intra-team prize allocation rule.[2]

This interplay between intra-team incentives and inter-team competition is also relevant in other contexts. For instance, consider two departments within a firm. These departments can be viewed as teams. The firm's CEO wants to design the hiring and promotion policy to provide incentives to their various employees. As performance typically correlates positively with the need to expand the size of a department, the number of positions and promotions going to each department will be proportional to the department's relative performance. Then, employees in a department will exert effort to boost their department performance. Each of them hopes to then be given one of the

[1] Even in flexible proportional representation systems, in which voters can also cast a vote for their favourite individual candidate(s) on the party electoral list, the vast majority of legislative seats are allocated in the order of the electoral list. For example, in the last election of March 2017 in The Netherlands, which uses PR with party lists but also the need for voters to express such a preferential vote, a full one hundred percent of the one hundred and fifty members of Parliament were elected following the order of their parties' electoral lists.

[2] These party lists also serve other purposes, such as selection and gender/sex/origin balance. In this paper, we focus purely on the incentive effects of these lists.

prizes their department won. In this context, what intra-departmental prize allocation rule should the firm use to maximize the outputs of the teams? Once again, to answer this question, we need to study the interplay between incentives within departments and competition between departments.

As a last example, consider a central government that needs to choose where to build a new hospital, school or military base. As the local public good generates both direct and indirect jobs and economic activity for the region in which they are located, each region in the country would like to see the public good built on their territory. The different regions thus lobby or compete against each other to secure the local public good. Yet, once a region has won this contest, the exact location of the local public good must still be decided among the region's different cities or municipalities. The regions can thus be described as teams and, once again, to pin down which intra-team allocation rule maximises team output, we need to study the interplay between incentives within regions and competition between regions.

In line with the above examples, we develop a model of a contest between teams of identical individuals who compete, via the provision of costly individual effort, for multiple indivisible prizes.[3] The team's production function aggregates the efforts of all team members and is assumed to be a constant elasticity of substitution technology. Individual efforts can thus be substitutes or complements. We also parametrize the cost function of team members to allow for different degrees of convexity. The number of prizes a team wins is proportional to team output. To model the inter-team part of the contest, we propose a novel allocation mechanism for the prizes. We extend the classical Tullock (1980) imperfectly discriminating contest success function to the case of many prizes. Each prize is won by a team with a probability equal to the ratio of its own output over the sum of all the teams' outputs. The number of prizes won by a team thus follows a binomial distribution.

In the main body of the paper, to model the intra-team allocation of prizes, we focus on the case of non observable (or non-contractible) efforts. Then, the allocation mechanism cannot depend directly on the efforts of individual members and specifies the probability that a given team member wins a prize as a function of the total number of prizes won by his team. We compare two intra-team allocation rules: a list rule that allocates prizes according to a pre-specified list – as under closed-list proportional representation – and an egalitarian allocation rule that treats all members equally by randomly distributing the prizes won. The list rule is thus biased and discriminatory as

[3] In a companion paper, we consider the case of teams of agents with heterogeneous abilities, which impacts their cost of effort function and productivity, which in turn impacts team output.

it gives an ex-ante advantage to some members over others, even though all members are identical.

We show that which intra-team allocation rule is associated to highest team output depends on the complementarity (substitutability) of the individual efforts in the team production technology and the convexity of the individual cost function. In particular, the egalitarian rule dominates the list rule when individual efforts are complements and the marginal cost function is convex. When efforts are substitutes and the marginal cost of effort is concave, the list rule dominates the egalitarian one.

We then turn to the question of finding the optimal mechanism among all possible intra-team allocation rules. We first restrict attention to monotonic rules, under which the individual probability of winning a prize must be (weakly) increasing in the number of prizes won by the team. We show that, depending on the same condition on complementarity of effort and convexity of cost, the optimal monotonic rule is either the list or the egalitarian allocation rule. This justifies our decision to focus on these two rules to start with.

To the best of our knowledge, our findings offer the first theoretical justification for the rule that specifies, for elections under proportional representation in multiple-seat districts, that the seats a party has won are to be distributed to the candidates on the list in the order of the list. Indeed, our findings imply that this intra-team allocation rule is the optimal incentive provision mechanism when efforts are substitutable enough and the cost of effort is not too convex. This contrasts sharply with the typical negative views about the incentive effects of closed-list proportional representation both in the economics and in the political science literature (see for instance, Persson and Tabellini (2000, 2003), Persson, Tabellini and Trebbi (2003) and Tavits (2007)). Even though the list allocation rule is associated with the free-riding and demotivating effects the political economy literature has focused on, we show that, in our model, it is the best allocation rule if the team's electoral output exhibits enough substitutability (and/or the cost of effort is not too convex), as low effort provision by the first and last politicians on a party list are more than compensated by high effort by the candidates in the median positions of the list.

We close our analysis of optimal rules by noting that, intuitively, removing the monotonicity constraint opens the way to intra-team allocation rules that can provide even better incentives than the list rule. Indeed, by removing the monotonicity constraint, we allow the rule designer to allocate incentives more freely across the different team members. thanks to the non-negativity constraint on effort provision. In particular, giving non-monotonic incentives to some team member frees incentive tokens that can be redistributed to other members, and the extra effort generated by such redistribution can over-compensate the drop in effort by the members who are given negative incentives. Further, because of the additional flexibility that removing the monotonicity constraint generates, we show that the egalitarian rule is the optimal rule for a smaller set of parameter values.

We then turn to various extensions. First, we allow individual efforts to be fully contractible: the allocation of prizes can depend on the effort exerted by team members. This implies that each team member can be put exactly on their participation constraint. In this case, the egalitarian rule is always the optimal monotonic rule. Then, we extend the model to the case of more than two teams and to the case of a biased inter-team contest, when one team has an advantage over the other teams.

The rest of the paper is structured as follows. The next section reviews the relevant literature. Section 3 presents the model. Section 4 solves for the equilibrium under the egalitarian and list allocation rules and compares individual effort choices and team output under the two intra-team allocation rules. In Section 5, we discuss optimal mechanisms. Section 6 considers extensions. The last section concludes and offers avenues for further research.

2 Related Literature

Our paper first contributes to the literature on contests. It offers a bridge between two different strands of the literature: team contests and contests for multiple prizes. In team contests, several teams compete in order to win one prize, which may be of a public or private nature, or a mix of both. The focus of this strand of the literature is on the sharing rule that determines how to split the single available (private part of the) prize across the winning team's members, so as to maximize team output. Important contributions include the seminal work of Nitzan (1991) and the contributions of Lee (1995), Esteban and Ray (2001), Ueda (2002), Baik and Lee (2001), Nitzan and Ueda (2011), Baik and Lee (2012) and Balart et al. (2015).[4] Our contribution is to extend the literature on team contests to the case in which teams compete for many indivisible prizes.

Turning to the literature on multiple prizes, our contribution is to extend it to the case in which it is teams, as opposed to individuals, which compete for these prizes.[5] In particular, our Binomial-Tullock mechanism offers a novel way of allocating the prizes among the different teams competing in the contest. One classical predecessor to our inter-team allocation mechanism is the probabilistic voting mechanism developed by Enelow and Hinich (1977) and Lindbeck and Weibull

[4]For a recent survey on sharing rules in collective rent seeking, see Flamand and Troumpounis (2015).

[5]For a recent survey of the contest literature on contests with multiple prizes, see Sisak (2009).

(1987) and used recently by Galasso and Naniccini (2016) in their analysis of candidate selection issues under closed-list proportional representation. In that model, the probability of winning an extra prize is independent of the number of prizes a team has already won, whereas in our model the teams' probability of winning an extra prize decreases with the number of prizes it already won, a feature that we believe is desirable as it is more realistic. In the tournament literature, some contributions, such as that of Nalebuff and Stiglitz (1983), also consider the case of multiple prizes. One major advantage of our mechanism is its analytical tractability that allows for closed-form solutions.

Our paper contributes to the literature on incentives in teams, and in particular to the literature that links incentives and discrimination or non-equal treatment of ex-ante identical team members. Winter (2004) analyzes whether agents who are identical in their qualifications should receive asymmetric rewards to improve incentives and efficiency. Winter (2004) relies on an O-Ring technology where all agents must succeed in their task for the team to be successful. Also, Winter (2004) models effort as a binary choice and thus cannot discuss the role of the convexity of the cost function. In contrast, as we rely on a standard CES production function, we have a more continuous way to parametrize complementarity. We also let efforts be a continuous variable which allows us to analyze the effect of the convexity of the cost function. Bose et al. (2010) also study the unequal treatment of identical agents in teams. In their model, complementarities in team production lead to higher output when effort decisions are taken sequentially as opposed to simultaneously. Thus, they find that, within each team, it is optimal to treat differently the two team members in the presence of strategic complementarities.Our team output function is a CES, and all team members choose their efforts simultaneously. In that case, complementarities call for equal treatment. In most applications, the use of a sequential mechanism does not appear realistic.

Closer to our basic model setup is Ray et al. (2007) who also use a CES function to model team production. Yet, their model focuses on one team only and there is a continuous, fully divisible prize to be won. In accordance to our results, they find that unequal sharing rules are efficient when efforts are substitutes. Our findings suggest that their result extends to the case of team contests for multiple indivisible prizes. Our analysis also goes further than Ray et al.(2007), as we derive the optimal monotonic and non-monotonic allocation rules under contractible and non-contractible effort, when efforts are complements as well as substitutes.

Given that our setup coincides with the incentive provision version of an electoral competition game under closed-list proportional representation in multiple-seat districts, we also contribute to furthering the research on the incentive effects of electoral rules; see for example Myerson (1993 and 1999), Persson and Tabellini (2000, 2003), Persson Tabellini and Trebbi (2003), Tavits (2007), Buisseret and Prato (2017) and Crutzen and Sahuguet (2017).[6] In this literature, proportional representation, especially with closed-list, has received quite some bad press.[7] This is mainly because of the fact that, as we also show in our theory, incentives to work hard are blunted for the team members at the top and bottom of the list. Yet, our results show that, as long as individual efforts are not too complementary and/or the cost of effort is not too convex, the attribution of seats following the order of a pre-defined list is the optimal mechanism parties should use under closed-list proportional representation. Buisseret and Prato (2017) and Crutzen and Sahuguet (2017) both offer some more specific cautionary tales about the unconditional validity of the bad press closed-list proportional representation has been subject to. For example, Crutzen and Sahuguet (2017) show that the candidates' individual incentives to exert effort can be stronger under closed-list proportional representation than under plurality rule, contrary to what the current wisdom in comparative politics is, once we let party leaders attribute the different slots on the party list in a competitive fashion. Buisseret and Prato (2017) point out that party lists under proportional representation can lead to a better alignment of the individual candidates' objectives with parties and the electorate at large.

3 Model

3.1 Individual and team efforts

Two teams are competing in a contest for n identical prizes. Each prize has value V. Each team is composed of n identical members who can at most win one prize. Member i of team j exerts costly effort $e_{ij} \geqslant 0$ to improve his team's chances of winning prizes. Team members are identical in the sense that they all have the same effort productivity, that we normalize to unity, and they all face the same cost of effort function, which is increasing and convex:

$$c(e_{ij}) = e_{ij}^{\beta}/\beta, \text{ with } \beta > 1. \tag{1}$$

[6]See also Kunicova and Rose-Ackermann (2005), Chang and Golden (2007), Schleiter and Voznaya (2014) and Rafler (2016).

[7]With the exception of Myerson (1993 and 1999), Buisseret and Prato (2017) and Crutzen and Sahuguet (2017).

Team j's aggregate output is denoted by E_j. We assume that the production function aggregating individual efforts exhibits constant elasticity of substitution:

$$E_j = \left[\sum_{i=1}^{n} (e_{ij})^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \text{ with } \sigma < 1. \tag{2}$$

3.2 Allocation of prizes between teams

The allocation of the n prizes between teams depends on the aggregate output of each team. We assume that the allocation function is what we call hereafter a Binomial-Tullock imperfectly discriminating contest success function. This technology is a natural generalization of the Tullock (1980) contest success function to multiple prizes. As in a Tullock contest, the probability that team j wins a given prize is given by the ratio-form contest success function $p_j = \frac{E_j}{E_1+E_2}$. Thus, prizes are awarded to team j using independent draws from a Bernoulli distribution with parameter p_j. The probability that team j wins k prizes follows a binomial distribution and is given by:

$$P_j(k) = C_k^n \left(\frac{E_j}{E_1+E_2}\right)^k \left(1 - \frac{E_j}{E_1+E_2}\right)^{n-k}. \tag{3}$$

3.3 Allocation of prizes within teams

Individual efforts are not contractible . The allocation of prizes cannot depend directly on these efforts[8], but only on the number of prizes won by the team. An allocation rule specifies, for each number of prizes won by the team, the probability that a given team member wins a prize. Even though individual effort does not enter directly into the prize allocation mechanism, team members still increase their chances of getting a prize by exerting more effort.

We contrast two allocation rules: the egalitarian allocation rule which treats all team members equally, and the list allocation rule which treats team members differently as it gives priority to some team members in the allocation of prizes. We focus on these rules for two reasons. First they are natural rules. Second, they turn out to be the optimal monotonic rules, as we show in Section 5.

Under the egalitarian rule, for any number k of prizes won by a team, all group members have the same probability k/n of winning a prize. Then, member i in team j chooses his level of effort to solve:

$$Max_{e_{ij}} \left[V \sum_{k=1}^{n} P_j(k) \frac{k}{n} - \frac{e_{ij}^{\beta}}{\beta}\right] \tag{4}$$

[8] In Section 6.1, we relax this assumption and analyze the case where individual effort is perfectly contractible and prize allocation can depend directly on individual efforts.

Under the list allocation rule, as under closed-list proportional representation, the n team members are ordered on a list, which determines the order of the allocation of prizes won by the team. Thus, the member in mth position on the list wins a prize if their team wins at least m prizes. The list order is independent of effort decisions and based on effort-independent characteristics[9] of the different team members. This allocation rule treats similar members in different ways and is thus biased and discriminatory.

Team member in mth position on the list in team j solves

$$Max_{e_{ij}} \left[V \sum_{k=m}^{n} P_j(k) - \frac{e_{mj}^{\beta}}{\beta} \right] \tag{5}$$

Notice that the summation goes from m to n and not from 1 to n, as the team member in mth position on the list only gets a prize when his team wins at least m prizes.

4 Equilibrium efforts

Under the egalitarian rule, member i in team j chooses his level of effort to solve equation (4) above. Using the formula for the expectation of the binomial distribution, we can rewrite the objective function (4) as:

$$V \frac{E_j}{E_1 + E_2} - \frac{e_{ij}^{\beta}}{\beta} \tag{6}$$

Thus, the problem under the egalitarian rule is equivalent to a team contest over one fully divisible prize of individual value nV with an egalitarian intra-team sharing rule, as in Nitzan (1991).

Proposition 1: Under the egalitarian allocation rule, in a symmetric Nash-Equilibrium:

Team output E_E^* is given by:

$$E_E^* = \left(\frac{V}{4}\right)^{\frac{1}{\beta}} n^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}} \tag{7}$$

Individual effort e_E^* is given by :

$$e_E^* = \left(\frac{1}{4E_E^*} V n^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1}{\beta-1}} = \left(\frac{1}{4}\right)^{1/\beta} \left(\frac{V}{n}\right)^{\frac{\beta-1}{\beta}} \tag{8}$$

Proof. See appendix. ■

[9] Examples of such characteristics include seniority or age.

Individual effort increases in the value of a prize V and decreases with the number of prizes n., while team output increases in both V and n.

Under the list allocation rule, team member in position m on the list in team j maximizes:

$$V\sum_{k=m}^{n}P_j(k)-\frac{e_{mj}^{\beta}}{\beta}=V\sum_{k=m}^{n}C_k^n\left(\frac{E_j}{E_1+E_2}\right)^k\left(1-\frac{E_j}{E_1+E_2}\right)^{n-k}-\frac{e_{mj}^{\beta}}{\beta} \tag{9}$$

We then have:

Proposition 2: Under the list allocation rule, in a symmetric Nash-Equilibrium, team output E_L^* is given by:

$$E_L^*=\left\{\sum_{k=1}^{n}\left[kC_k^n\left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}\left(\frac{V}{4}\right)^{\frac{1}{\beta}} \tag{10}$$

Individual effort e_m^* of the team member in the mth position on the list is given by:

$$e_m^*=\left[E_L^{*\sigma-1}mC_m^n\left(\frac{1}{2}\right)^{n+1}V\right]^{\frac{1}{\beta+\sigma-1}} \tag{11}$$

Proof. See appendix. ■

The main difference between the two allocation rules lies in the way they treat the different team members. Under the egalitarian rule all team members receive the same incentives and exert the same effort in equilibrium. Under the list allocation rule, incentives and individual equilibrium efforts vary with the position on the list. In particular the members located around the median list position face the largest marginal benefit of effort, as these positions are associated to the steepest slope of the binomial distribution. To the contrary, the first and last members on the list have little incentive to exert effort, as the slope of the binomial distribution is essentially flat. Indeed, as we illustrate in Figure 1 (for the case of a contest for 30 prizes), the shape of the distribution of the equilibrium winning probabilities associated with the different positions on the list, represented by the blue bars (scaled so as to have the two curves of roughly equal size), implies that the distribution of individual equilibrium efforts is bell-shaped. This heterogeneity in incentives is the main reason why closed-list proportional representation in multiple-seat districts has received so much bad press in the political economy and the political science literature.

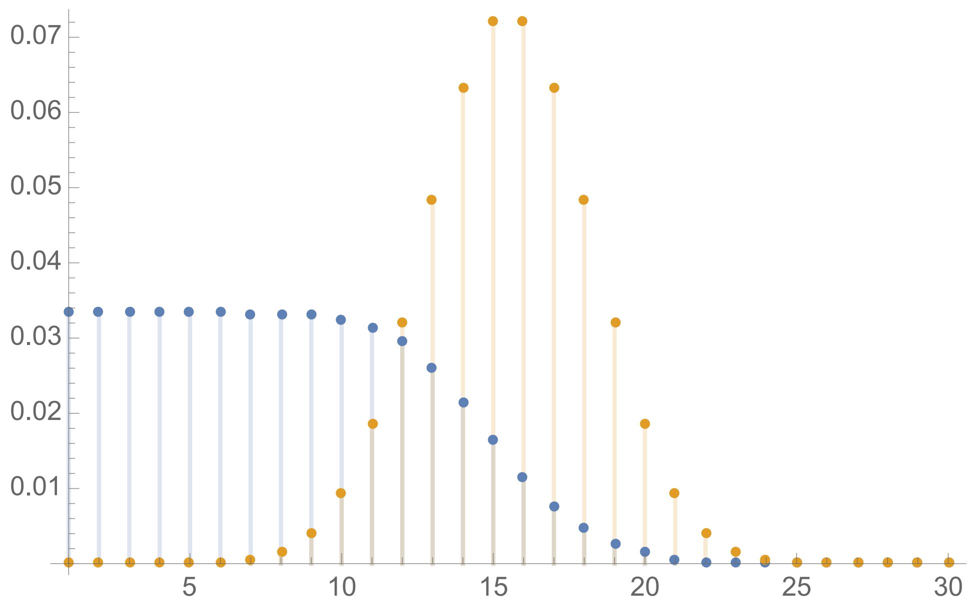

Figure 1: Binomial-Tullock probabilities and effort provision

4.1 Comparison of the allocation rules

We now compare team output and team success under the two allocation rules. We show that the parameter β, which represents the convexity of the cost function, and parameter σ, which parametrizes the degree of complementarity in the team output production function, play a central role. Indeed, we have:

Proposition 3: The list allocation rule leads to higher team output than the egalitarian rule if and only if $\beta < 2 - 2\sigma$. The two allocation rules yield the same team output if and only if $\beta = 2 - 2\sigma$.

Proof. See appendix. ■

The intuition behind this result is as follows. As we saw above, individual incentives are uniform under the egalitarian rule and bell-shaped under the list rule. When will the bell-shaped distribution of individual efforts yield higher team output? Suppose that the cost of effort function is close to being linear (β is close to 1). If individual efforts are highly substitutable (σ close to 0), team output is equal to the sum of efforts, and this sum is what matters, not so much the level of the different individual efforts. Thus, inducing differences in effort can be optimal. When efforts are complementary ($\sigma > 1/2$), inducing differences in individual effort is suboptimal as the decision of the low effort providers depresses team output. What about the convexity of the individual cost of effort function? Suppose for simplicity that $\sigma = 0$. When this function is very convex ($\beta > 2$), the marginal cost is also convex. Then, asymmetric incentives are bad for team performance. Indeed, starting from equal marginal benefits of effort, increasing the marginal benefit of one team member and decreasing the benefit of another one will have a positive effect if the marginal cost increases more slowly for the individual with stronger incentives than for the one with weaker incentives. Yet, when the marginal cost of effort is convex, this is simply not possible.

4.2 Teams choose their allocation rule independently

The result applies to a symmetric contest between teams that (are required to) use the same intra-team prize allocation rule. Yet, in reality teams choose strategically their allocation rule. We thus add a stage to our game to check how our results are affected in this extended game. In the first stage, each team chooses an allocation rule to maximize team success. In the second stage, after observing the allocation rule chosen by both teams, individual team members choose their effort.

Solving for the (pure-strategy) subgame perfect equilibrium, we find that the condition driving the teams' choice of the allocation rule is the same as in Proposition 3 above:

Proposition 4: In the pure-strategy subgame perfect equilibrium of the two-stage game, the list allocation rule is chosen if and only if $\beta < 2 - 2\sigma$.

Proof. See appendix. ■

This result shows that to maximize team output and thus the expected number of prizes won by the team, the optimal allocation rule is still pinned down by condition $\beta < 2 - 2\sigma$. Thus, given β and σ, both teams have a dominant strategy (strictly dominant when $\beta \neq 2 - 2\sigma$) in the first stage of the game.

To wrap up, propositions 3 and 4 show that the convexity of the marginal cost and the complementarity of the team production function drive the choice of allocation rule. When the marginal cost of effort is convex, giving powerful incentives to a few individuals is not productive, as even these individual are not going to choose to exert much effort. With convex marginal costs, it is thus more efficient to give all team members the same incentives and treat them in an equal, symmetric way. When the marginal cost is not too convex or even concave, it is efficient to provide powerful incentives to few individuals who will exert very high levels of effort. In that case, a rule that treats some members differently is optimal. The degree of complementarity plays a similar role. When efforts are substitutes, there is no cost in getting very different effort levels within the team. When efforts are complementary, it is better to induce similar efforts and once again the egalitarian rule becomes the optimal rule.

Going back to our leading example, proportional representation in multiple-seat districts, the above findings suggest that when efforts are not too complementary within each party and/or the individual cost of effort is not too convex, the use of closed lists can be optimal. In particular, it gives better incentives than any arrangement which treats all politicians who are running for election in an ex-ante fair and egalitarian way. Thus, propositions 3 and 4 provide an incentive argument for the use of closed lists over an egalitarian rule under proportional representation. In the next section, we go further and show that closed lists are the optimal incentive mechanism (when efforts are substitutes and the cost of effort is not too convex) when we impose a monotonicity constraint.

5 Optimal allocation rule

An allocation rule can be represented as a $n*(n+1)$ matrix of weights $[\lambda_{ik}]$. Entry λ_{ik} corresponds to the probability that team member i gets a prize when the team has won k prizes. Probabilities need to be between 0 and 1, and the number of prizes distributed cannot be larger than the number of prizes won by the team. These feasibility constraints thus require $0 \leq \lambda_{ik} \leq 1$ and $\sum_{i=1}^{n} \lambda_{ik} = k$.

The egalitarian allocation rule can be represented as a matrix in which each column has equal entries $\lambda_{ik} = k/n$. The list allocation rule can be represented as a matrix with $\lambda_{ik} = 0$ if $i > k$ and $\lambda_{ik} = 1$ if $i \leq k$.

For instance, with three prizes, the matrices corresponding to the egalitarian rule and the list allocation rule are $\begin{bmatrix} 0 & 1/3 & 2/3 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ respectively.

To simplify the exposition, we only present the case of perfect substitutes ($\sigma = 0$). The extension to the case of complements is straightforward, albeit algebraically more tedious.

Member i of team j maximizes $V \sum_{k=1}^{n} \lambda_{ik} P_j(k) - \frac{e_{ij}^{\beta}}{\beta}$. In a Nash equilibrium, as equilibrium effort cannot be negative, the first order condition implies that the optimal effort of team member i is given by:

$$\max \left\{ \left[V \sum_{k=1}^{n} \frac{\partial E}{\partial e_{ij}} \frac{E_i}{(E_1+E_2)^2} \lambda_{ik} C_k^n \left[k P_j^{k-1} (1-P_j)^{n-k} - (n-k) P_j^k (1-P_j)^{n-k-1} \right] \right]^{\frac{1}{\beta-1}}, 0 \right\}. \quad (12)$$

At a symmetric Nash equilibrium, the above boils down to:

$$e_i = \max\left\{\left[V\sum_{k=1}^{n}\frac{\partial E}{\partial e_i}\frac{1}{4E}\lambda_{ik}C_k^n\left(\frac{1}{2}\right)^{n-1}(2k-n)\right]^{\frac{1}{\beta-1}},0\right\}. \tag{13}$$

Simplifying and forgetting for now that effort cannot be negative, we get:

$$E = \left\{\sum_i\left[\frac{V}{2^{n+1}}\sum_{k=1}^{n}\lambda_{ik}C_k^n(2k-n)\right]^{\frac{1}{\beta-1}}\right\}^{\frac{\beta-1}{\beta}}. \tag{14}$$

Denoting $\Delta_{ik} = \lambda_{i(n-k)} - \lambda_{ik}$ and exploiting the fact that $\lambda_{i0} = 0$, we can rewrite team output as:

$$E = \left\{\sum_i\left[\frac{V}{2^{n+1}}\sum_{k=0}^{\lfloor n/2\rfloor}\Delta_{ik}C_k^n(n-2k)\right]^{\frac{1}{\beta-1}}\right\}^{\frac{\beta-1}{\beta}}. \tag{15}$$

Thus, team output is maximized when $\sum_i\left[\sum_{k=0}^{\lfloor n/2\rfloor}\Delta_{ik}C_k^n(n-2k)\right]^{\frac{1}{\beta-1}}$ is maximized.

The constraints take two forms. First, there is the prize budget constraint: $0 \leq \sum_i \Delta_{ik} \leq 1$. Second, individual probabilities must be between 0 and 1, implying that: $-1 \leq \Delta_{ik} \leq 1$.

5.1 Optimal monotonic rules

We first derive the optimal mechanism under a monotonicity constraint, which requires that $\lambda_{ik} \geq \lambda_{i,k-1}$, that is, that $\Delta_{ik} \geq 0$ for all k and i. This constraint imposes that the probability that an individual wins a prize is (weakly) increasing in the number of prizes won by the team. Solving for the optimal allocation rule yields the following:

Proposition 5: When $\beta \geq 2$, the egalitarian allocation rule is the optimal monotonic allocation rule. When $\beta \leq 2$, the list allocation rule is the optimal monotonic allocation rule.

Proof. See appendix. ■

The intuition behind the result is simple. The intra-team prize allocation rule determines individual incentives and effort choices. When $\beta \geq 2$, it is optimal to equalize incentives across team members, while when $\beta \leq 2$, it is optimal to make incentives as strong as possible for some individuals. The maximization problem is similar to that of the optimal allocation of risk. With risk-averse individuals, the allocation is as egalitarian as possible, while with risk-loving agents, it is optimal to make the allocation as unequal as possible.

Thus, as already anticipated, the analysis above proves that the use of closed-lists under proportional representation in multiple-seat electoral districts is the optimal incentive mechanism to use (provided condition $\beta < 2 - 2\sigma$ is met). This finding also offers a novel interpretation to two important and related puzzles in the empirical literature in political economy: the lack of empirically significant incentive costs associated with the use of closed-lists under proportional representation —see for example Persson, Tabellini and Trebbi (2003)— and the fact that some of the countries that are most praised for their electoral system's capacity to hold their politicians on their toes have as their electoral rule proportional representation with closed-lists. The above findings suggest that this electoral rule can actually be the optimal one to use in elections in multiple-seat districts if the goal is to maximize the party list's total electoral output.

Applying our findings to the optimal organization of firms, one testable prediction is that departments in which members are highly complementary for team production should treat their members in a more egalitarian fashion than departments in which individual efforts are more substitutable. One casual observation about the organization of firms in the technologically most advanced sectors, such as that of software applications for mobile technologies, is that at least some of these firms are much more horizontal, much less hierarchical than the firms in older, more mature sectors. Whereas this observation falls very short of being an empirical test of our findings, it is certainly consistent with them: as the efforts of coders working on the same project in a software company are very complementary, we should expect the way their department is organised to be such that all workers face equal treatment. Taking this observation to the data would certainly be a very interesting exercise.

5.2 Optimal non-monotonic rules

A non-monotonic allocation rule can give negative incentives to some team members. Negative incentives appear when an individual faces a higher probability of getting a prize when his team gets fewer prizes. Yet, effort cannot be negative. Also, negative incentives free up incentive tokens that can be redistributed to other team members. The combination of the zero lower bound on effort and the possibility of redistributing incentives within the team may then generate a higher team output than what was possible when the monotonicity constraint was binding.

We illustrate how this redistribution of incentives can indeed generate higher team output with the help of two examples. As the optimal rule can be non-monotonic, the member labels should not be given any ranking interpretation anymore. A team member with a lower number is not necessarily treated more favorably by the intra-team allocation rule.

Example 1 *Optimal rules with four prizes*

In that case, the optimal allocation rule maximizes $\sum_{i=1}^{4} \max\left(1+2\left(\lambda_{i3}-\lambda_{i1}\right)^{\frac{1}{\beta-1}}, 0\right)$. When $\beta=2$, the optimal allocation rule maximizes $\sum_{i=1}^{4} \max\left(1+2\left(\lambda_{i3}-\lambda_{i1}\right), 0\right)$. Under the list allocation rule and the egalitarian allocation rule, $\sum_{i=1}^{4}(\lambda_{i3}-\lambda_{i1})=2$. This means that $\sum_{i=1}^{4}\left[(1+2(\lambda_{i3}-\lambda_{i1})\right]=8$.

Once we remove the monotonicity constraint, we can set $(\lambda_{i3}-\lambda_{i1})$ to be negative, generating negative incentives for team member i who, as a consequence, chooses zero effort. Yet, this also frees incentive tokens that can be strategically redistributed to the most responsive team member(s). Exploiting this redistribution, it is easy to check that an optimal rule[10] is $\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.

The optimal allocation rule is non-monotonic in the sense that team member 1 wins a prize if the team wins one prize but does not win a prize if the team wins three prizes. With that rule, $\sum_{i=1}^{4} \max\left(1+2\left(\lambda_{i3}-\lambda_{i1}\right), 0\right)=9>8$. Also, observe that members 2 to 4 are all treated equally (as the weights in columns 2 and 4 do not matter for incentives): the entries in columns 1 and 3 are the same for these players.

This rule clearly remains optimal when $\beta \leq 2$ given that it creates even more unequal incentives than the list allocation rule. For $\beta>2$, there is a trade-off between relying on the egalitarian allocation rule and an optimal non-monotonic rule. To pin down the value of β below which it is optimal to depart from the egalitarian rule, one need only compare $\sum_{i=1}^{4}(1+2*1/2)^{\frac{1}{\beta-1}} = 4*2^{\frac{1}{\beta-1}}$ and $3*(1+2*1)^{\frac{1}{\beta-1}}=3*3^{\frac{1}{\beta-1}}$, where the effort of only three team members matter for the second team's output as, under the non-monotonic rule above, member 1 exerts zero effort. Then, simple algebra implies that the egalitarian rule is suboptimal whenever $4*2^{\frac{1}{\beta-1}}<3*3^{\frac{1}{\beta-1}} \iff \beta<\frac{\ln(3/2)+\ln(4/3)}{\ln(4/3)} \simeq 2.4$.

Example 2 *Optimal rule with five prizes*

With more prizes, the optimal non-monotonic rule can take different forms depending on the value β takes on.

The optimal allocation rule maximizes $\sum_{i=1}^{4} \max\left[5+15\left(\lambda_{i4}-\lambda_{i1}\right)+10\left(\lambda_{i3}-\lambda_{i2}\right), 0\right]^{\frac{1}{\beta-1}}$.

[10] This is not the only optimal rule, as optimality puts no constraints on the value the different λ_{i2} can take on.

As before, when $\beta \leq 2$ the function above is a convex function of $\lambda_{i4} - \lambda_{i1}$ and $\lambda_{i3} - \lambda_{i2}$, implying that it is optimal to make incentives as heterogeneous as possible. Also, as all columns of the rule matrix but the first and last enter in the equation of team effort, the optimal non-monotonic rule is this time unique. It is easy to check that it is given by $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

Thus, when $\beta \leq 2$, it is optimal to treat differently members 1 and 2 and treat equally members 3 to 5.

When $\beta > 2$ identifying which intra-team allocation rule is optimal is a bit more involved. Using numerical analysis to pin down the critical values for β, it appears that there are two cases to consider: $\beta \geq 3$ and $\beta \in (2, 3)$.

When $\beta \geq 3$, the egalitarian rule is optimal: the convexity of the individual cost of effort is too strong for the benefits of generating negative incentives to compensate for their costs. When $\beta \in (2, 3)$, the optimal rule gives equal incentives to four team members and negative incentives to one: $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1/4 & 3/4 & 1 & 1 \\ 0 & 0 & 1/4 & 3/4 & 1 & 1 \\ 0 & 0 & 1/4 & 3/4 & 1 & 1 \\ 0 & 0 & 1/4 & 3/4 & 1 & 1 \end{bmatrix}$.

The last example shows that, under non-monotonicity and with a sufficient number of prizes, the space of parameter values for β can be partitioned more finely to fine tune incentives than when the allocation rules are constrained to be monotonic.

6 Extensions

6.1 Contractible efforts

Assume efforts are now fully observable or contractible. What are the optimal allocation rules in this case? Because efforts can be contracted upon, any optimal allocation rule will depend not only on the number of prizes won by the team, but also on the actual effort decision of each team member. This means that the participation constraint of team members now becomes central.

The egalitarian rule then specifies that each team member who exerts effort $e \geq e^*$ has an equal chance of getting one of the prizes won by the team. The list rule still assigns to each team member a rank in the list, but now also specifies a minimal effort level associated with each rank. If a team member exerts the specified effort (or more), they get a prize if the team wins more prizes than their rank. If they exert less effort or if the team wins less prizes than their rank, they get no prize. Then, we have the following proposition:

Proposition 6: When efforts are contractible, the egalitarian allocation rule leads to higher team output than the list allocation rule for all values of the parameters β and σ.

Proof. See appendix. ■

When efforts are observable, the level of effort is determined by the participation constraint of team members. The contract can impose an effort level that drives each team member's utility to his outside option. Then, the cost of effort enters directly in the participation constraint. When effort is not observable, it is the marginal cost of effort that is central to any member's effort choice. Comparing team outputs across the two intra-team allocation rules, we find that the egalitarian rules generates higher team output than the list rule if and only if $\beta \geq 1 - 2\sigma$, which is always satisfied. Thus, when effort is contractible, it is always optimal to treat every team member in the same way. Unfair allocation rules are inefficient allocation rules. This last finding suggests that the imperfect contractibility of effort is a necessary condition for the optimality of discriminatory rules regarding incentives.

6.2 More than two teams

How are our findings modified if we let $K > 2$ teams compete? This extension is natural for the political economy application as it is common to see more than two parties competing in elections under a proportional system. Surprisingly perhaps, the main result of Proposition 3 extends to this set-up. We still get that closed-lists may be optimal under proportional representation when there are more than two parties in the election.

With more than two teams, the distribution of efforts within each team becomes asymmetric and right-skewed. Indeed, in any symmetric equilibrium, each team member expects his team to win n/K prizes. Thus, depending on how many teams enter the competition, the relevant 'intermediate' list members, those who exert highest effort, are around position n/K on their team list. Figure 2 below illustrates this for $K = 2$ (the blue bars) and $K = 4$ (the red bars) (with $V = 1$, $\beta = 2$, $\sigma = 0$ and $n = 30$).

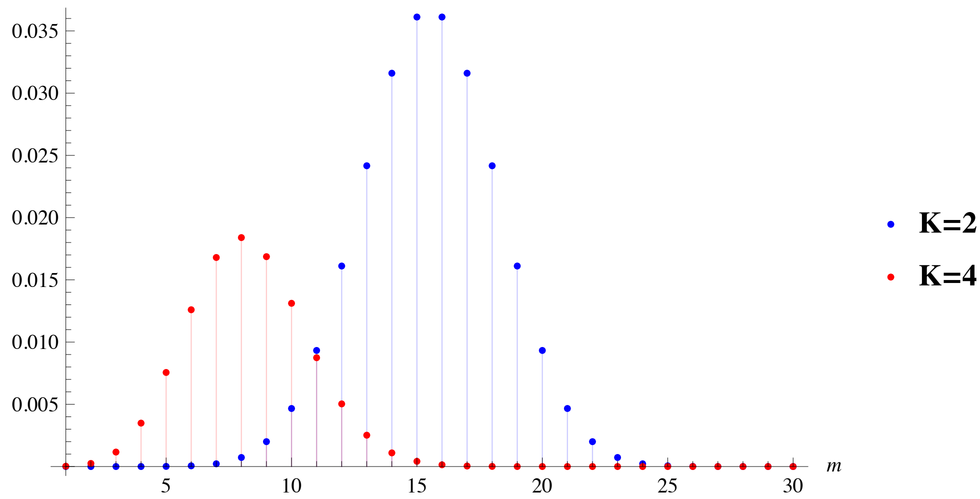

Figure 2: Effort distributions with two and four parties

Despite the fact that efforts are now distributed asymmetrically, we still have that:

Proposition 7: With $K > 2$ teams, the list allocation rule leads to higher team output than the egalitarian rule if and only if $\beta < 2 - 2\sigma$. The two allocation rules yield the same team output if and only if $\beta = 2 - 2\sigma$.

Proof. See appendix. ■

6.3 Asymmetric contests (one team with an advantage)

Often, inter-team competitions are biased in favour of some team. In electoral contests, for example, some parties 'enjoy' an ex-ante ideological advantage over some of their competitors. In contests for the allocation of local public goods such as schools, hospitals or military bases, the socioeconomic conditions of the different regions may make some regions (dis-)advantaged in the competition. In firms, the CEO may favour one department over the others.

Again somewhat surprisingly, condition 1 still determines the ranking of effort between allocation rules. Indeed, consider a contest between two teams that is biased in favor of team 1, say. Let the probability that team 1 wins a prize given efforts E_1 and E_2 be given by $\lambda E_1 / (\lambda E_1 + E_2)$ with $\lambda > 1$. The distribution of the number of prizes is now:

$$P_1(k) = C_k^n \left(\tfrac{\lambda E_1}{\lambda E_1 + E_2}\right)^k \left(1 - \tfrac{\lambda E_1}{\lambda E_1 + E_2}\right)^{n-k},$$

and we have:

Proposition 8: In a biased contest between two teams $(\lambda > 1)$, the list allocation rule leads to higher team output than the egalitarian allocation rule if and only if $\beta < 2 - 2\sigma$. The two allocation rules yield the same team output if and only if $\beta = 2 - 2\sigma$.

Proof. See appendix. ■

6.4 Some other possible extensions

We discuss quickly some other extensions.

Increasing the size of the teams. We have assumed in the model that the number of members in each team is equal to the number of prizes that a team can win. We can easily adapt the model to analyze how the number of team members influence total effort under the two allocation rules of interest. When the team uses the egalitarian allocation rule, it is easy to show that increasing the number of team members leads to higher team output when condition 1 is satisfied, whereas the opposite is true when condition 1 is not satisfied. Of course, with a list allocation rule, increasing the number of team members has no benefit. Additional team members have no chance of getting a prize and would thus exert zero effort.

Our findings also allow us to extend the literature on the group size paradox (see for example Esteban and Ray (2001) and the contributions that follow). If that literature focuses on the case of a single prize, we offer some thoughts about the optimal way to organize a team when prizes are multiple and indivisible and the number of team members is higher than the number of potential prizes. Our results prove that the optimal intra-team prize allocation rule is crucial to understand the conditions such that larger groups are more efficient at generating team output. Indeed, when the list allocation rule is optimal, increasing the size of the team has no effect on individual effort decisions and thus team output. It thus makes sense to think about the group size paradox in our setup only when the egalitarian allocation rule is optimal.

Broader objective function for team members. We can also extend the model to analyze the consequences of letting team members care about, for example, the total number of prizes their team wins. The presence of this type of preferences seems especially relevant for political applications: the typical political candidate does not care only about themselves getting elected, but also about the overall performance of their party. There are several ways to model this type of preferences. For instance, we could add a component to the utility function of the individual that depends on the number of prizes won by the team. We could also add a benefit if the team wins a majority of the prizes (as winning a majority of legislative seats implies control of government and thus a discrete jump in utility for the political party that achieves such a result). The algebra then becomes less transparent, but the same condition on β and σ is still driving the comparison of team outputs.

Allowing for aggregate noise. We can also modify the contest technology to allow for aggregate noise in the inter-team distribution of prizes. For example, we could make use of $\frac{E_i^r}{E_1^r+E_2^r}$ or $\frac{\gamma}{2}+(1-\gamma)\frac{E_i}{E_1+E_2}$ where r and γ parametrize the responsiveness of success to effort. It is easy to show that it is still true that the ranking between the egalitarian allocation rule and the list allocation rules follows condition 1. This finding mirrors Balart, Chowdhury and Troumpounis (2015) which proves that the above two ways of modelling aggregate noise yield the same incentives for contest participants.

7 Conclusion

We developed a model to study the impact of intra-team prize allocation rules in team contests when prizes are indivisible. We showed that the convexity of the marginal cost function and the degree of effort complementarity drive which type of allocation rule is best for incentives. When the marginal cost of effort is convex or efforts are complements, the egalitarian allocation rule, that treat all team members the same way, dominates. When the marginal cost of effort is concave or efforts are substitutes, it is optimal to treat team members asymmetrically. In particular, the list allocation rule is optimal. These results hold in a context in which individual efforts are not contractible and in which the allocation of prizes only depends on the total number of prizes won by the team. When efforts are contractible, the optimal mechanism is one that links directly individual effort provision to the prospect of obtaining a prize, and in that case the egalitarian rule dominates, always. We also considered several extensions, such as having more than two teams, having an asymmetric contest or introducing a luck component in the contest between teams, to show that our results are robust to such extensions.

We believe our model is very amenable to further extensions and applications. For example, in a companion paper, we analyze a similar game but allow members to differ in ability, which impacts their cost of effort function, and productivity, which impacts team output via the CES production function. These additional sources of heterogeneity allow us to answer several additional interesting questions, such as understanding how to treat the most and least productive team members and pinning down the environments in which heterogeneous teams are most useful.[11] In political economy, Crutzen and Sahuguet (2017) adapt the present model to compare incentives under different electoral rules, and in particular British-style first-past-the-post and Israeli-style proportional representation. Finally, the stark predictions of our model should be empirically testable, as some economic sectors exhibit much stronger complementarity across workers than others. Carrying out such empirical tests is thus another interesting avenue for further research.

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[11] A very interesting recent contribution that has a simlar flavour is Cubel and Sanchez-Pages (2017), in which intra-group income inequality is mapped into the groups' capacity to defend themselves from external threats.

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Appendix: proofs

Proof of Proposition 1

We start by proving a useful lemma.

Lemma 1: $\frac{\partial E_j}{\partial e_{ij}} = \left(\frac{E_j}{e_{ij}}\right)^{\sigma}$.

Proof of Lemma 1

Using the definition of E_j, $E_j = \left[\sum_{i=1}^{n} (e_{ij})^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$, we get:

$$\begin{aligned} \frac{\partial E_j}{\partial e_{ij}} &= \frac{1}{1-\sigma}(1-\sigma)(e_{ij})^{-\sigma}\left[\sum_{i=1}^{n}(e_{ij})^{1-\sigma}\right]^{\frac{1}{1-\sigma}-1} \\ &= (e_{ij})^{-\sigma}\left[\sum_{i=1}^{n}(e_{ij})^{1-\sigma}\right]^{\frac{\sigma}{1-\sigma}} \\ &= \left(\frac{E_j}{e_{ij}}\right)^{\sigma}. \end{aligned}$$

■

The first-order condition associated to maximising (6) is:

$$V\frac{\partial E_1}{\partial e_{i1}}\frac{E_2}{(E_1+E_2)^2} - e_{i1}^{\beta-1} = 0.$$

At a symmetric Nash equilibrium, $e_{ij} = e_{kj}$ for any i and k and j, thus $\left(\frac{E_j}{e_{ij}}\right)^{\sigma} = n^{\frac{\sigma}{1-\sigma}}$ and $\frac{E_2}{(E_1+E_2)^2} = 1/4E$. We thus get:

$$\frac{V}{4E}n^{\frac{\sigma}{1-\sigma}} - e^{\beta-1} = 0.$$

Thus:

$$e = \left(\frac{V}{4E}n^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1}{\beta-1}}$$

Therefore:

$$
\begin{aligned}
E &= \left\{ \sum_{k=1}^{n} \left[\left(\frac{V}{4E} n^{\frac{\sigma}{1-\sigma}} \right)^{\frac{1}{\beta-1}} \right]^{1-\sigma} \right\}^{\frac{1}{1-\sigma}} \\
E &= n^{\frac{1}{1-\sigma}} \left(\frac{V}{4E} \right)^{\frac{1}{\beta-1}} n^{\frac{\sigma}{(1-\sigma)(\beta-1)}}
\end{aligned}
$$

which implies

$$
\begin{aligned}
E &= \left[n^{\frac{1}{1-\sigma}} \left(\frac{V}{4} \right)^{\frac{1}{\beta-1}} n^{\frac{\sigma}{(1-\sigma)(\beta-1)}} \right]^{\frac{\beta-1}{\beta}} \\
E &= \left(\frac{V}{4} \right)^{\frac{1}{\beta}} \left[n^{\frac{1}{1-\sigma}} n^{\frac{\sigma}{(1-\sigma)(\beta-1)}} \right]^{\frac{\beta-1}{\beta}} \\
E &= \left(\frac{V}{4} \right)^{\frac{1}{\beta}} n^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}.
\end{aligned}
$$

■

Proof of Proposition 2

We start by proving a useful lemma.

Lemma 2

$\sum_{k=m}^{n} C_k^n \left[kp^{k-1} (1-p)^{n-k} - (n-k)(1-p)^{n-k-1} p^k \right] = mC_m^n p^{m-1} (1-p)^{n-m}$.

Proof:

We show that terms in the sum cancel. Consider the second term within the summation sign: $(n-k) C_k^n (1-p)^{n-k-1} p^k$. Using the identity $(n-k) C_k^n = (k+1) C_{k+1}^n$ we can write it as $(n-k) C_k^n (1-p)^{n-k-1} p^k = (k+1) C_{k+1}^n (1-p)^{n-k-1} p^k$, which corresponds exactly to the first term within the summation sign for the index $k+1$. These two terms cancel leaving only the first and last term of the sum. The first term is $mC_m^n p^{m-1} (1-p)^{n-m}$. The last term is equal to zero.

■

Under the list system, the individual on the mth position on the list maximizes:

$$
\sum_{k=m}^{n} C_k^n \left(\frac{E_j}{E_1+E_2} \right)^k \left(1 - \frac{E_j}{E_1+E_2} \right)^{n-k} V - \frac{e_{mj}^{\beta}}{\beta}
$$

Denoting $P_j = \frac{E_j}{E_1+E_2}$, the first-order condition is:

$$
V \sum_{k=m}^{n} \frac{\partial E_j}{\partial e_{ij}} \frac{E_i}{(E_1+E_2)^2} C_k^n \left[k (P_j)^{k-1} (1-P_j)^{n-k} - (n-k)(P_j)^k (1-P_j)^{n-k-1} \right] - (e_{mj})^{\beta-1} = 0
$$

At a symmetric equilibrium, using Lemma 1 and Lemma 2, the first-order condition simplifies to:

$$\left(\frac{E}{e_m}\right)^{\sigma} \frac{m}{E} C_m^n \left(\frac{1}{2}\right)^{n+1} V - e^{\beta-1} = 0$$

Thus

$$\begin{aligned}
(e_m)^{\beta+\sigma-1} &= E^{\sigma-1} m C_m^n \left(\frac{1}{2}\right)^{n+1} V \\
&\Longleftrightarrow \\
e_m &= \left[E^{\sigma-1} m C_m^n \left(\frac{1}{2}\right)^{n+1} V\right]^{\frac{1}{\beta+\sigma-1}}
\end{aligned}$$

Total team effort is thus:

$$\begin{aligned}
E &= \left\{\sum_{k=1}^{n} \left[\left[E^{\sigma-1} k C_k^n \left(\frac{1}{2}\right)^{n+1} V\right]^{\frac{1}{\beta+\sigma-1}}\right]^{1-\sigma}\right\}^{\frac{1}{1-\sigma}} \\
&\Longleftrightarrow \\
E^{\frac{\beta}{\beta+\sigma-1}} &= \left\{\sum_{k=1}^{n} \left[k C_k^n \left(\frac{1}{2}\right)^{n+1} V\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{1}{1-\sigma}} \\
&\Longleftrightarrow \\
E &= \left\{\sum_{k=1}^{n} \left[k C_k^n \left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}} \left(\frac{V}{4}\right)^{\frac{1}{\beta}}
\end{aligned}$$

■

Proof of Proposition 3:

Comparing efforts under both allocation rules, we see that the egalitarian rule dominates the list rule when:

$$n^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}} > \left\{\sum_{k=1}^{n} \left[k C_k^n \left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}$$

We can rewrite this inequality as:

$$1 > \left\{\frac{\sum_{k=1}^{n} \left[k C_k^n \left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}}{n}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}$$

which simplifies as:

$$\frac{\sum_{k=1}^{n}\left[kC_k^n\left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}}{n}<1.$$

Note that $\sum_{k=1}^{n}kC_k^n\left(\frac{1}{2}\right)^{n-1}=n$. We can now use Jensen's inequality to show that whether the inequality is satisfied or not depends on the concavity or convexity of the function $g\left(x\right)=x^{\frac{1-\sigma}{\beta+\sigma-1}}$.

We have that $\sum_{k=1}^{n}\left[kC_k^n\left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}/n\leq 1$ when $\frac{1-\sigma}{\beta+\sigma-1}\leq 1$. This last inequality simplifies to $\sigma\geq\frac{2-\beta}{2}$, which completes the proof.

■

Proof of Proposition 4

Given the choice of allocation rule by one team, the best response of the other team is the choice of the intra-team allocation rule the allocation rule that maximizes effort and thus the number of prizes won. We then need to prove that the condition $\beta\geq 2-2\sigma$ also determines the ranking of effort between the egalitarian rule and the list rule..

We first consider the egalitarian allocation rule. The first order condition of team 1's ith member under the egalitarian allocation rule is:

$$V\left(\frac{E_1}{e_{i1}}\right)^{\sigma}\frac{E_2}{\left(E_1+E_2\right)^2}-\left(e_{i1}\right)^{\beta-1}=0.$$

This yields, denoting $p_1=\frac{E_1}{E_1+E_2}$:

$$Vn^{\frac{\sigma}{1-\sigma}}\frac{p_1\left(1-p_1\right)}{E_1}=\left(e_{i1}\right)^{\beta-1}.$$

Thus:

$$e_1=\left[Vn^{\frac{\sigma}{1-\sigma}}\frac{p_1\left(1-p_1\right)}{E_1}\right]^{\frac{1}{\beta-1}}$$

Therefore

$$\begin{aligned}E_1 &= \left\{\sum_{k=1}^{n}\left[\left[Vn^{\frac{\sigma}{1-\sigma}}\frac{p_1\left(1-p_1\right)}{E_1}\right]^{\frac{1}{\beta-1}}\right]^{1-\sigma}\right\}^{\frac{1}{1-\sigma}}\\ E_1 &= n^{\frac{1}{1-\sigma}}\left[\frac{p_1\left(1-p_1\right)}{E_1}V\right]^{\frac{1}{\beta-1}}n^{\frac{\sigma}{(1-\sigma)(\beta-1)}}\end{aligned}$$

which implies

$$E_1=\left[p_1\left(1-p_1\right)V\right]^{\frac{1}{\beta}}n^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}.$$

We now turn to the list allocation rule. Under this rule, the first order condition for the mth member on the list of team 1 is:

$$V\left(\frac{E_1}{e_{m1}}\right)^{\sigma}\frac{E_2}{(E_1+E_2)^2}\sum_{k=m}^{n}C_k^n\left[kp_1^{k-1}\left(1-p_1\right)^{n-k}-(n-k)\,p_1^k\left(1-p_1\right)^{n-k-1}\right]=\left(e_{m1}\right)^{\beta-1}.$$

Using Lemma 2 and exploiting the fact that $\frac{E_2}{(E_1+E_2)^2}=\frac{p_1(1-p_1)}{E_1}$, this simplifies to:

$$e_{m1}=\left[VE_1^{\sigma}C_m^n\frac{m}{E_1}p_1^m\left(1-p_1\right)^{n-m+1}\right]^{\frac{1}{\beta+\sigma-1}}$$

And thus

$$\begin{aligned}
E_1 &= \left\{\sum_{k=1}^{n}\left[\left[VE_1^{\sigma}C_k^n\frac{k}{E_1}p_1^k\left(1-p_1\right)^{n-k+1}\right]^{\frac{1}{\beta+\sigma-1}}\right]^{1-\sigma}\right\}^{\frac{1}{1-\sigma}}\\
&\Longleftrightarrow\\
E_1^{\frac{\beta}{\beta+\sigma-1}} &= \left\{\sum_{k=1}^{n}\left[kC_k^np_1^k\left(1-p_1\right)^{n-k+1}V\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{1}{1-\sigma}}\\
&\Longleftrightarrow\\
E_1 &= \left\{\sum_{k=1}^{n}\left[kC_k^np_1^{k-1}\left(1-p_1\right)^{n-k}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}\left[p_1\left(1-p_1\right)V\right]^{\frac{1}{\beta}}.
\end{aligned}$$

We need to compare the expected number of prizes won under both allocation rules, given the other team's allocation rule. We know that if both teams choose the same allocation rule, individual efforts are symmetric and thus team outputs are equal and both teams win on average $n/2$ prizes. We therefore need to compare team outputs when teams choose different allocation rules.

From the calculation above, if team 1 uses the egalitarian rule and team 2 uses the list rule, denoting $p_j=E_1/(E_j+E_2)$, we get:

$$\begin{aligned}
E_1 &= \left[p_1\left(1-p_1\right)V\right]^{\frac{1}{\beta}}n^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}},\\
E_2 &= \left\{\sum_{k=1}^{n}\left[kC_k^np_2^{k-1}\left(1-p_2\right)^{n-k}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}\left[p_2\left(1-p_2\right)V\right]^{\frac{1}{\beta}}.
\end{aligned}$$

Dividing the above two team outputs and exploiting the fact that $p_1=1-p_2$ which implies that $\left[p_1\left(1-p_1\right)V\right]^{\frac{1}{\beta}}=\left[p_2\left(1-p_2\right)V\right]^{\frac{1}{\beta}}$, we get:

$$\frac{E_2}{E_1}=\left\{\frac{\sum_{k=1}^{n}\left[kC_k^np_2^{k-1}\left(1-p_2\right)^{n-k}\right]^{\frac{1-\sigma}{\beta+\sigma-1}}}{n}\right\}^{\frac{\beta+\sigma-1}{\beta(1-\sigma)}}.$$

We need to pin down the conditions such that $\frac{E_2}{E_1}$ is greater or smaller than 1. Remark now that $\sum_{k=1}^{n}\left[kC_k^n p_2^{k-1}(1-p_2)^{n-k}\right]=n$. Then we use the argument based on Jensen's inequality used in the proof of Proposition 3 to show that the condition $\beta > 2-2\sigma$ determines which team exerts the most effort. This implies in turn that it is a dominate strategy for both teams to choose the egalitarian allocation rule if and only if $\beta > 2-2\sigma$. If $\beta = 2-2\sigma$ both rules yield the same payoff. ■

Proof of Proposition 5

We first note that the monotonicity constraint imposes that $\Delta_{ik} \geq 0$ for any i and k.

There are two cases to consider depending on the value of β.

Case 1: $\beta \geq 2$

At a symmetric equilibrium, team output is given by

$$E^{\frac{\beta}{\beta-1}} = \sum_i \left[\frac{V}{2^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \Delta_{ik} C_k^n (n-2k)\right]^{\frac{1}{\beta-1}}.$$

When $\beta \geq 2$, we have that $\frac{1}{\beta-1} \leq 1$ and team output is the sum over team members i of a concave function of their individual efforts. Any mean preserving spread of individual effort reduces team output and a rule that equalizes $\sum_{k=1}^{\lfloor n/2 \rfloor} \Delta_{ik} C_k^n (n-2k)$ among team members is optimal. The egalitarian rule is thus the optimal rule.

Case 2: $\beta \leq 2$

In that case, any mean-preserving spread of the incentives would increase team output. We need to show that the list rule can not be improved upon —that no mean-preserving spread is possible.

The list rule gives the highest incentives (and thus highest effort) to the median team member(s). For this team member, all the Δ_{ik} are equal to 1, and $\sum_{k=0}^{\lfloor n/2 \rfloor} \Delta_{ik} C_k^n (n-2k)$ is maximized. For team members who are one position away from the median on the list, $\Delta_{i\lfloor n/2 \rfloor} = 0$ and all other $\Delta_{ik} = 1$. For team members who are two positions away from the median on the list, $\Delta_{i\lfloor n/2 \rfloor} = 0$ and $\Delta_{i\lfloor (n-1)/2 \rfloor} = 0$ and all other $\Delta_{ik} = 1$. Proceeding similarly for all other team members, we can conclude that the list rule maximizes the heterogeneity of efforts across team members and is thus optimal.

Proof of Proposition 6

In a symmetric equilibrium, when both teams use the egalitarian rule, individual efforts are given by the participation constraint:

$$e^{\beta}/\beta = V/2.$$

Individual effort is given by:

$$e = (\beta V/2)^{1/\beta}.$$

This leads to team output being equal to

$$\begin{aligned} E &= \left(\sum e^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \\ &= n^{\frac{1}{1-\sigma}} e \\ &= n^{\frac{1}{1-\sigma}} \left(\frac{\beta V}{2}\right)^{1/\beta}. \end{aligned}$$

Turning to the case where both teams use the list allocation rule, at a symmetric equilibrium, individual effort of the team member in mth position is given by the participation constraint:

$$\begin{aligned} (e_m)^{\beta}/\beta &= \sum_{k=m}^{n} C_k^n \left(\frac{1}{2}\right)^n V \\ &\Longleftrightarrow \\ e_m &= \left[\sum_{k=m}^{n} C_k^n \left(\frac{1}{2}\right)^n \beta V\right]^{1/\beta}. \end{aligned}$$

Team output is thus equal to:

$$\begin{aligned} E &= \left(\sum_{m=1}^{n} e_m^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \\ &= \left\{\sum_{m=1}^{n} \left[\sum_{k=m}^{n} C_k^n \left(\frac{1}{2}\right)^n \beta V\right]^{\frac{1-\sigma}{\beta}}\right\}^{\frac{1}{1-\sigma}}. \\ &= \left\{\sum_{m=1}^{n} \left[\sum_{k=m}^{n} C_k^n \left(\frac{1}{2}\right)^{n-1}\right]^{\frac{1-\sigma}{\beta}}\right\}^{\frac{1}{1-\sigma}} \left(\frac{\beta V}{2}\right)^{1/\beta} \end{aligned}$$

The argument of Proposition 3 applies but now the condition for the egalitarian rule to lead to higher team output is that $(1-\sigma)/\beta < 1$ that is $\beta > 1-\sigma$ which is always the case. ■

Proof of Proposition 7

The proof of Proposition 3 applies directly with $p = E_1/\sum_{j=1}^{K} E_j$.

■

Proof of Proposition 8

The proof of Proposition 3 applies directly with $p = \lambda E_1/(\lambda E_1 + E_2)$ given that:

$$\frac{\partial \frac{\lambda E_1}{\lambda E_1 + E_2}}{\partial e_{i1}} = \frac{\lambda E_2}{(\lambda E_1 + E_2)^2} = \frac{p(1-p)}{E_1}.$$

■