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Econophysics and Levy Processes:
Statistical implications for financial economics

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Abstract

This paper deals with the use of Levy processes in financial economics. More precisely, I will emphasize theoretical features of Levy processes that could be in opposition with a plausible empirical approach. Indeed, despite these processes better fit to the financial data, some of their statistical properties can raise several theoretical problems in order to use them in empirical studies. Econophysicists developed more sophisticated Levy processes allowing to have a statistical tools in line with a plausible physically (empirical) analysis. This paper presented all these oppositions between mathematical characteristics of Levy processes and their empirical needs. This paper also deals with the future of financial economics since by (re)introducing, econophysicists provide a broader definition of financial uncertainty.

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I. Introduction

Financial economics is mainly characterized by a high level of mathematization in the modelling of stock market returns. Modelling stock market returns or stock market price variations is the first step in the development of financial models. Because these mathematical models require a statistical characterization of changes in price or returns, the work of determining the statistical distribution of returns is a key problem in financial economics and, more generally, in the work of modern financial theory. In this perspective, financial economists mainly use the Gaussian framework in order to characterize the evolution of financial prices. Three reasons can be mentioned to explain the success of this framework: the simplicity (only two parameters are needed to describe data), the notion of normality (that can refer to the key concept of economic equilibrium) and above all, the statistical justification which refers to the Central-Limit Theorem (CLT) with its notion of asymptotic convergence.

For a decade, a considerable number of physicists have started to apply concepts coming from physics to understand economics and financial phenomena in a more empiricist perspective. According to econophysicists real data are far away from the asymptote and from the Gaussian framework.

Econophysicists claim that the real world is not Gaussian. That is the reason why they have decided to use models that they consider more descriptive. All potential instabilities observed in the complex system must be taken into account in the econophysical approach. In this perspective, econophysicists try to explain the no-Gaussian distribution (i.e with leptokurticity) of financial distributions implying that extreme events have a significant probability of occurring.

Leptokurticity has generated a lot of debates in finance. Several authors tried to describe financial distributions in a non strictly Gaussian framework. In order to describe leptokurticity of financial distributions, financial economists used normal distribution whose large variations were simulated through "jumps processes". Such processes are a combination of two (or more) different kinds of distribution, usually a normal distribution combined with a Poisson law. Until recently, the finance literature focuses only on these two examples of Levy processes[2]: the Brownian (normal) motion developed by Black and Scholes (1973) and the compound Poisson

[2] See, for example, Beckers (1981) or Ball and Torous (1983)

process introduced by Merton (1976). Let us mention that the continuous process used by Black and Scholes (1973) implicitly assumed the dynamic completeness of market. This theoretical link between completeness of market and the continuous processes has been developed by Harrison and Kreps (1979) and Harrison and Pliska (1981). In this perspective, the compound Poisson process cannot describe complete markets in a perspective defined by Arrow and Debreu (1954)[3].

For the 1990s, financial economists have progressively began to study other Levy processes[4] like Variance Gamma Process (Madan, 1990), Generalized Hyperbolic Process (Eberlein, 1995) or CGMY process (Carr and al., 2000). However, all these processes do not present continuous properties in order to be applied in situations of complete market (Naik and Lee, 1990). The consequences of this property are very important since they refer to the possibility to have a unique price for each asset. If markets are not complete, prices can only be evaluated through an interval. In this perspective, the statistical properties of stable Levy processes are very interesting since they can describe the leptokurticity of financial markets and they are continuous processes (implying the uniqueness of prices).

In line with these works, econophycisists propose to use stable Levy processes in order to better describe leptokurticity of financial distributions. As I will show in this paper, the statistical features of Lévy processes are not really adapted for describing empirical systems. For example, Levy processes have an infinite variance. However the notion of variance often refers to the temperature in statistical physics (Gupta and Campanha, 1999). Therefore, if physicists want to use Levy processes to describe real systems, they have to find a way of computing variance. I will deepen this point in the paper and I will illustrate other statistical features of Levy processes that could be in opposition with a plausible physically approach. In other words, this paper analyses the possibility to use Levy processes in finance and it emphasizes all the opposition between their statistical characteristics of and the empirical needs of econophysicists. I will show then how econophysicists arrive to escape from the asymptotic argument of CLT by transforming Levy processes in a more empirical tool. All these responses given by econophysicists to apply Levy processes in a more physically analysis will be studied from a financial point of view. More precisely, I will present the consequences for financial economics of these statistical and all transformations proposed by econophysics.

[3] More precisely, Harrinson and Kreps (1979) and Harrison and Pliska (1981) showed how a process must be continuous in order to have the unicity of the martingal computing the financial price..

[4] See Miyahara (2005) for a detailed presentation of the application of these processes in finance.

I. Econophysics or complexity applied to economic phenomena

For a decade, a considerable number of physicists have started to apply concepts coming from physics to understand economics phenomena. The term "Econophysics" is now mainly used to describe these works. Econophysics, as a specific label and conceptual practice, was first coined by the physicist H. Eugene Stanley in 1996 in a paper published in *Physica A* (Stanley and al., 1996) As the name suggests, econophysics presents itself as a hybrid discipline which can be defined in methodological terms as "a quantitative approach using ideas, models, conceptual and computational methods of statistical physics" applied to economic and financial phenomena (Burda, Jurkiewicz and Nowak, 2003, p.1).

The influence of physics on economics is nothing new. A number of writers have studied the "physical attraction" exerted by economists on physics (Mirowski, 1989, Ingrao and Israel 1990 or Shabas 1990). But as McCauley points out (2004), in spite of these theoretical and historical links between physics and economics, econophysics represents a fundamentally new approach that differs from preceding influences (Schinckus, 2010b). Its practitioners are not economists taking their inspiration from the work of physicists to develop their discipline, as has been seen repeatedly in the history of economics. This time, it is physicists that are going beyond the boundaries of their discipline, studying various problems thrown up by social sciences in the light of their methods. Econophysicists are not attempting to integrate physics concepts into economics as it exists today, but are rather seeking to ignore, even to deny this discipline in an endeavour to replace the theoretical framework that currently dominates it with a new framework derived directly from statistical physics[5] (Gingras and Schinckus, 2010).

Econophysicists present their field of research as a new way of thinking about the economic and financial systems through the "lenses" of physics. Since econophysics is a very new field, it mainly focuses on financial economics even if some works exist about macroeconomics[6]. As

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[5] During past decades, a lot of physics models have been used in economics but these models were mainly used for their mathematical description of physical phenomena. Progressively, these imported models have been integrated in the mainstream (see Black & Scholes model, for example). This trend is not observed with econophysics in which economic phenomena are explained in terms of molecular mechanisms or complex interactions. In this perspective, econophysicists do not try to connect their works with the pre-existing economic theory. For an epistemological analysis of this attitude, see Rosser (2007).

[6] Some econophysicists work on the emergence of money (Shinohara, *et al.* 2001) or on the global demand (Donangelo, *et al.* 2000).

much as classical economics imported models from classical physics as formulated by Lagrange (Mirowski, 1989), and financial economics built on the model of Brownian motion also imported from physics, so too econophysics try to model economic phenomena using analogies taken from modern condensed matter physics and its associated mathematical tools and concepts.

Econophysics is directly in line with the development of the so-called "complexity science" during the 1990s (Rickles, 2007) for which economic systems are obvious candidates for a treatment in terms of "complexity" because they are composed of multiple components (agents) interacting in such a way as to generate the macro-properties of economic systems and subsystems (Rickles, 2008, p.4). These macro-properties can be characterized in terms of statistical regularities i.e. the fact that statistical properties appear and reappear in many diverse phenomena (McCauley, 2004). This statistical regularity is often characterized by econophysicists through power laws (and more generally Levy Processes) that are at the heart of econophysics[7].

II. Econophysics and the anti-asymptotic perspective

The empiricist dimension[8] is frequently mentioned in econophysics research (Bouchaud, 2002) and is often presented as the main difference with economics (Kitto, 2006). Rickles (2007, p.6) explained that real empirical data are certainly at the core of econophysics and the models are built around it, rather than some non-existent economic ideals (such as equilibrium or Gaussian framework for example). In other words, econophysicists claim to describe the world like it is and they consider that economists have "false a priori" (McCauley, 2006) about this world. Concerning models developed by financial economists, Mandelbrot also used severe words,

> "The standard theory, as taught in business school around the world, would estimate the odds of that final, August 31 [1998] collapse at one in 20 million – an event that, if you traded daily for nearly 100 000 years, you would not expect to see even once" (Mandelbrot, 2004, p.25).

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[7] As Farmer and Geanakoplos (2009, p.23) emphasized, power laws could be "consistent with the economic equilibrium but still need to be explained by other means". For further information, see Calvet and Fisher (2002a).

[8] According to an empiricist approach, all hypotheses and theories must be tested against observations of the natural world, rather than resting solely on *a priori* reasoning or intuition (even if they are confirmed by empirical data). In this perspective, empiricism can be seen as a radical empirical definition of science. About the empiricist dimension of econophysics, see Schinckus (2010b).

The generalized use of Gaussian framework in finance is considered as the first a priorism. According to econophysicists, this Gaussian framework does not describe financial data but economists continue to use it because they have an a priori opinion about the financial phenomena.

However, the use of the Gaussian framework is not a simple a priorism because it results from one of the most fundamental theorems[9] of probability: the famous Central Limit Theorem (CLT) which states that the sum

$$Z_n \equiv \sum_{i=1}^{n} x_i$$

of n stochastic variables x that are statistically independent, identically distributed[10] and with a finite variance converges when $n \to \infty$ to a Gaussian stochastic process (Feller, 1971).

This convergence is an asymptotic distribution and it provides then an approximation for a very large number of observations. More precisely, Bouchaud & Potters (2003) showed that the convergence toward a Gaussian distribution can be observed at the rate of $\sqrt{n\log(n)}$ standard deviation where n is the number of observations.

Therefore, the use of Gaussian framework in finance is not an a priori (as econophysicists claim it) because this use is based on one of the most fundamental theorem of probability theory (CLT). This CLT can be seen as the main result in the mainstream of statistics which essentially offers asymptotic reasoning, as Strasser (1985) emphasized it,

> "A considerable amount of statistical research during the last thirty years can be subsumed under the label "asymptotic justification of statistical methods". Already in the first half of this century asymptotic arguments became necessary, as it turned out that the optimization problems of statistics can be solved for finite sample size only in very few particular cases" (Strasser, 1985, p.2).

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[9] The first fundamental theorem is The Law of large numbers (Tijms 2004)

[10] A generalization of the theorem for variables which are not identically distributed (but always with a finite variance) was developed by Gnedenko and Kolmogorov (1954).

Econophysicists want to describe the world like it is and not like it would be according an idealized theoretical framework. For them, real data are far away from the asymptote and if the leptokurtic dimension must be described statistically, it must be done in a physically plausible process. In this perspective, the idea of Gaussian convergence is called into question by econophysicists on one hand, because, the Gaussian convergence will only happen at infinity and on the other hand, because, the distribution will become Gaussian in only its center but not in its tails. Of course financial economists have developed (improved) Gaussian distributions with finite moments. But, despite the fact that economists characterize fat tails through processes like diffusion-Poisson (Hanson & Westman, 2003), regime switching models (Calvet and Fisher, 2002b or Hamilton, 2005) or stochastic volatility methods (Broto & Ruiz, 2004), "all these distributions can be called pseudo-fat tails [because] the finiteness of all moments makes them collapse into thin tails" (Taleb, 2008, p.2).

In order to describe data with non-Gaussian but stable distributions, econophysicists had to develop some statistical models in order to avoid the asymptotical (theoretical) convergence evoked and demonstrated in the CLT. They did it by proposing more sophisticated statistical tools that we call Levy processes. However, although these processes better fit to the financial data, some of their statistical properties can raise several problems about their empirical applications. This paper emphasizes all these problems and presents the solution proposed by econophysicists in order to use Levy processes to describe financial phenomena in a physically plausible perspective. I will deal with this point in the following sections.

III. The Levy Processes or the post-Gaussian

Econophysicists reject the Gaussian framework usually used in finance to describe the evolution of financial prices. Two reasons can explain this rejection: first, the fact that financial data cannot be empirically described by a Gaussian form and second, the fact that the Gaussian framework is an asymptotic argument. In order to describe the complexity of financial data without using the Gaussian distribution, econophysicists used sophisticated processes called Levy processes in honour of Paul Lévy, the pioneer of this framework. These processes refer to a family of infinitely divisible distributions which can take into account skewness and exceed kurtosis (*i.e* leptokurticity). Examples of such distributions are numerous but we can mention, among the most used, the Variance Gamma (Madan and Seneta, 1987), the Normal Inverse Gaussian (Barndorff and Nielsen, 1995), the Generalized Hyperbolic Model (Eberlein and Keller, 1995)[11]. As mentioned before, all these processes do not present continuous properties and, therefore, they cannot describe complete markets with a unique price for each asset. In this perspective, Levy processes are very interesting because they can describe leptokurticity in a complete market context.

The fact that the statistical paradigm initiated by Lévy (and developed by econophysicists) was imported into finance is not without good reason as Belkacem (1996, 40) explained it: "from a practical point of view, stable distributions are able to explain the thick distribution tails observed in empirical distributions of asset profitability rates." A similar argument is found in the writings of Tankov (2004, p.13) who added that these processes are particularly interesting in financial economics because they allow discontinuities in the evolution of asset returns to be taken into account. This means that these processes are candidates for integration into "price jump" phenomena[12].

Of course, there are some papers disagreeing with Mandelbrot idea to use Levy process in order to characterize financial data but, as Taleb (2007) emphasized we can reject the Gaussian more easily than we can accept it while there is no evidence that we are not in a Lévy regime.

In this paper, I only focus on general forms of the Levy process that I will define in the second section of this part. After having reminded that Levy processes have already been used (but rejected) in finance in the 1960s, I will better define these processes (and their main statistical features) by emphasizing their main implications for financial economics.

III.1. History of Levy Processes in finance

In the 1960s Mandelbrot (1962, 1963, 1965) Samuelson (1965) and Fama (1965) proposed studying financial markets using a non-Gaussian statistical framework directly inspired by Lévy's

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[11] For a presentation of all Levy processes, see Schoutens (2003, p.44-71).

[12] For taking into account these price jumps or return jumps without rejecting the Gaussian framework, financial economists have developed another alternative that consists of modelling these variations using a combination of normal distribution and Poisson distribution. See for instance (Jorion 1988, Maheu and McCurdy 2004).

work (1924) on the stability of probability distributions[13] α-stable laws present Paretian distribution tails that allow them to take into account price variations that are very large in relation to average variations. This is an essential property of α-stable laws since it enables them to integrate the possibility of price "jumps." But this characteristic, together with the stability of the distribution, means that variance can vary considerably depending on the size of the sample and the observation scale. Consequently, this variance does not tend towards a limit value. The variation is said, therefore, to be *infinite* because it does not tend towards a fixed value[14]. This infinite variance appears then to be the cause of the abandonment of α-stable processes in financial economics.

Many researchers considered the infinite variance hypothesis unacceptable because it is meaningless in the financial economics framework. In the 1960s, the period in which financial economics was constituted as a scientific discipline, the relationship between risk and return was taken from Markowitz' work (1952, 1959). Markowitz associated risk with variance and return with the mean. Since then, variance and the expected mean have been become the two main variables for theoretical interpretations. In this perspective, if variance were infinite (as it is in a Lévy process), it became impossible to understand the notion of risk as Markowitz had defined it. Fama emphasized difficulities concerning the applicability of Paretian framework:

> "Although the model discussed in the previous sections provides a complete theoretical structure for a portfolio model in a stable Paretian market, there are several difficulties involved in applying the model in practical situations" Fama (1965, 414).

[13] In the 1920s, the famous French statistician Paul Lévy carried out work on the stability of probability distributions. This characteristic means that any linear combination performed on a probability distribution characterized by a particular statistical law will generate a new probability distribution in accordance with the same statistical law. Lévy demonstrated that this invariance of the distribution form in respect of the addition of independent variables was not specific to Gaussian distribution. It was precisely this characteristic of statistical invariance put forward by Lévy that enabled Mandelbrot to extend his fractal geometry (which itself was based on an invariance characteristic) to the study of statistical phenomena. Lévy's work leading to the identification of α-stable laws appeared, from that moment on, as a generalization of the Gaussian statistical framework. As Schoutens (2003) made clear, the family of α-stable distributions appear to be a general form of a number of known statistical laws such as normal distribution, Cauchy's law, Pareto's law and Lévy's law. For a statistical presentation of these specific laws, see Schoutens (2003).

[14] The adjective "indeterminate" would be more accurately employed, but the word "infinite" is used in the specialized literature.

These difficulties refer to on one hand, the indeterminacy of the variance and on the other hand the fact that no computational definition yet existed for evaluating parameters of stable Levy processes. Fama (1965) himself deplored this point by writing that "At the moment [1965] very little is known about the sampling behavior of procedures for estimating the parameters of these distributions", Fama (1965, p. 429) explained that the next step in the acceptability of Levy processes in economics would be "to develop more adequate statistical tools for dealing with stable Paretian distributions" (Fama, 1965, p.429). This statistical problem has been reminded in papers dedicated to the estimation of parameters of stable distributions (Fama and Roll, 1968, 1971). Moreover, an empirical work proposed by Officer (1972, p.811) contributed to the abandon of Levy process because it showed that financial distribution "have some but not all properties of a stable process". Therefore, Officer (1972) concluded that the evolution of financial markets cannot be described through a Levy process. Ten years after his 1965 article, this abandon seemed generalized since Fama (1976) himself confirmed he preferred to use normal distributions,

> « Nevertheless, as models for common stock returns, stable nonnormal distributions also have undesirable properties. Although Mandelbrot's 1963 paper led to much new work on the subject (see for example, Fama and Roll 1968, 1971 and Blattberg and Sargent 1971), statistical tools for handling data from nonnormal stable distributions are primitive relative to the tools that are available to handle data from normal distribution. Moreover, although most of the models of the theory of finance can be developed from the assumption of stable nonnormal return distributions, the exposition is simpler when models are based on the assumption that return distributions are normal. Thus, the cost of rejecting normality for securities returns in favor of stable nonnormal distributions are substantial and it behooves us to investigate the stable nonnormal hypothesis further" (Fama, 1976, p.26).

It is interesting to mention that Fama considered stable Levy processes as a generalized case of normal approach which, technically, correct but economically not so evident. Through their contributions, Harrinson, Kreps and Priska (1981) showed a stable Levy process cannot be so easy integrated to economics because they do not imply incomplete markets. Therefore, Fama did not have, in 1976, the axiomatic reference to really evaluate the economic problem generated by an integration of Levy processes in financial economics.

In the following section, I will give a definition of this stable processes and I will emphasize the main statistical characteristics and their consequences for financial data. I will then show how econophysicists can use these processes in their empirical description of financial phenomena. More precisely, I will emphasize how econophysicists have developed Levy processes with a finite variance. This necessity refers to two clear reasons that are, on one hand, a finite variance is more in line with a physical approach and, on the other hand, this notion of variance usually refers to the idea of risk in finance.

III.2 Definition of Levy-stable process

As mentioned in the previous section, Mandelbrot (1963) and Fama (1965) showed in the 1960s that evolution of financial returns can be described through power law distributions of the form: $p(x) \sim x^{-\alpha}$ (where p(x) is the probability of there being an event of magnitude x and the scaling exponent α is a constant which is set by the empirically observed behaviour of the system)[15].

The stability characteristic of a distribution refers to the distribution retain its shape (up to scale and shift) under addition. Stable laws appear like fixed points of convolution operation. In other terms, there are attractors for a distribution convoluted with itself a large number of times finally since it converges towards a stable law. More precisely, a random variable *X* is called stable if for *n* independent copies X_i of *X* there exists a constant *d* such that

$$X_1 + X_2 + \ldots + X_n \overset{d}{=} n^{1/\alpha} X + d_n$$

The symbol $\overset{d}{=}$ means equality in distribution i.e. the right and the left sides of the equations have the same distribution. The distribution is said to be *strictly stable* if this holds with $d_n = 0$ for all n (Nolan, 2009). The class of all laws that satisfy this equation is described by four parameters which we call (α, β, γ, δ). As Nolan (2005) explained there are a no closed formulas for stable densities *f* and cumulative distribution functions F since their characteristic function[16] can be summarized by

$$\psi_X(t) = \exp\left[\, i\delta\, t - \gamma^{\alpha} |t|^{\alpha} (1 - i\beta\, \mathrm{sign}(t)\; W(\alpha, t))\right]$$

[15] Let us mention that Pareto was the first to investigate, in his *Cours d'Economie Politique* (1897) the statistical character of the wealth of individuals by modelling them in using the power laws.
[16] For detailed demonstration, see Nolan (2005, 2009) or Schultz (2003).

where

- $i = 1,\ldots n$ and $t \in \Re$.
- $$W(\alpha, t) = \begin{cases} \tan \frac{\pi\alpha}{2} & \text{si } \alpha \neq 1 \\ -\frac{2}{\pi} \log|t| & \text{si } \alpha = 1 \end{cases}$$
- $$\text{sign}(t) = \begin{cases} 1 & \text{si } t > 0 \\ 0 & \text{si } t = 0 \\ -1 & \text{si } t < 0 \end{cases}$$

More precisely, a random variable X is said to be α-stable if we have specific value for parameters (α, β, γ, δ) such as $0 < \alpha \leq 2$, $\gamma \geq 0$, $-1 \leq \beta \leq 1$, $\gamma \in \Re$. All the usual stable distributions (Gaussian, Cauchy, Poisson etc) can then be found in function of the value of each parameters (α, β, γ, δ).

The *parameter* α is called the "characteristic exponent" and it shows the index of stability of the distribution. This value of this exponent refers to the shape of the distribution: the lower this exponent is, the fatter the tails are (extreme events have then a higher probability of occurring). In other words, the lower α is, the more often extreme events are observed. In financial terms, this parameter is then an indicator of risk since it describes how often important variations can occur. The *parameter* γ is the scale indicator that can be any positive number. It refers to the "random size" *i.e* the importance of variance whose the regularity is given by the exponent α. This parameter describes the importance of the variations (whose regularity is given by α). In a sense, γ can be seen as an indicator of statistical dispersion. α and γ allow to decompose the risk of a process into two kinds of risk: a "risk of shape" (α) and a "size risk" (γ) which is well known in neoclassical finance since it is supposed to be reduced thanks to diversification. However, financial economists neglect the "risk of shape" which is often reduced to a Gaussian distribution.

The *parameter* β is the skewness and it gives information about the symmetry of the distribution. If β = 0, the distribution is symmetric. If β < 0 it is skewed toward the left (totally asymmetric toward the left if β = -1) while a β > 0 refers to a skewed distribution toward the right (totally asymmetric toward the left if β = -1). Finally, the *parameter* δ a localisation factor, it shifts the distribution right if δ > 0 and left if δ < 0.

In order to parameterize these factors, there exist several methods based on different ways of generating random variables. I could mention, for example, the generator of Janicki and Weron (1994) or the method developed by Chambers, Mallows and Stucks (1976). According to some authors (Embrecht & *al*.,1997), the lack of uniformity in the parameterization methods can sometimes lead people to avoid this kind of distribution. For further information about these methods, see Embrecht and al. (1997) or Nolan (2009).

Let us mention that Levy processes are infinitely divisible random processes[17] and they have the property of scaling (self-similarity)[18]. The first means that pairwise independence implies mutual independence. The scaling property means that if we zoom in on each level of scale, we can find the same statistical features. This statistics characteristic is very important for financial economics because they imply the possibility to study the financial phenomena through different variables that would be identical at each time scales as for example, the price variations, the trading time or the number of transactions. In other words, variables (daily, weekly and monthly) will follow stable distributions of exactly the same form except for origin and scale.

Depending on the value of the four parameters evoked above, two categories of Levy processes can then be identified: Levy-stable and Levy non-stable. In the next sections, I will present these categories by connecting them with mainstream in financial economics and models developed in econophysics.

III.3. Econophysics and Levy-stable processes

As mentioned before, Levy processes refer to a class of distribution characterized by four parameters which determine different regimes in function of their value. After having presented the most used stable regime (Gaussian distribution), I will differentiate it from the other Levy

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[17] A random process y is infinitely divisible if, for every natural number k, it can be represented as the sum of k independent identically distributed random variables (x_i). For all value of x_i.

[18] Property of scaling suggests a fractal behaviour. A fractal is an object in which the parts are in some way related to the whole. Two kinds of relation can be evoked: self-similarity or multi-scaling. Self-similarity is an invariance with respect to scaling. It means that the object or process is similar at different scales. Each scale resembles the other scales, but is not identical. A self-similar object appears unchanged after increasing or shrinking its size. A stochastic process is said to be self-affine, or self-similar if

$$\{[rt1]\tau, \ldots, [rtk]\tau\} \; d{=} \{\tau H[rt1], \ldots, \tau H[rtk]\}$$

The exponent H, called self-affinity index, or scaling exponent, of $[rt]$, satisfies $0 < H < 1$. The operator, $d=$ indicates that the two probability distributions are equal. It means that samplings at different intervals yield the same distribution for the process $[rt]$ subject to a scale factor. Strictly speaking, self-affinity and self-similarity are not identical. A self-similar process is also called uniscaled or unifractal. Let us mention the notion of multifractality (or multiscaling) process that extends the idea of similarity to allow more general scaling functions. Multifractality is a form of generalized scaling that includes both extreme variations and long-memory (See, Mandelbrot, 1997 or Calvet and Fisher, 2001 for further information).

stable and no stable regimes used by econophysicists. Before dealing with processes that allow econophysicists to escape the Gaussian framework and its asymptotic argument, I will remind that this Gaussian process can be seen as a specific case of Levy framework.

III.3.a. The Gaussian framework

The Normal distribution N (μ, σ^2) is the most used stable regime and it refers to a specific case of Levy process in which $\alpha = 2$, the distribution is perfectly symmetric ($\beta = 0$), the scale parameter is proportional to the standard deviation ($\gamma = \frac{\sigma}{\sqrt{2}}$) and where the localisation factor is equal to the mean $(\delta = \mu)$[19].

The Gaussian distribution is the most used tool because this framework allows to give a statistical meaning to financial data since risk is associated with variance and return with the expected mean. In this perspective, information resulting from other Levy parameters are not necessary required.

The first statistical representations of variations in the price of financial assets were made on the basis of a Gaussian framework[20]. This Gaussian description of financial reality progressively crystallized and was reinforced when Samuelson (1965b) introduced geometric Brownian motion to describe the continuity of trajectories. Since then, the normal distribution of returns on assets has strongly contributed to the development of modern financial theory. From Markowitz' modern portfolio theory (MPT) to the Capital Asset Pricing Model (CAPM) and Black & Scholes' model, through to the recent development of Value at Risk (VaR), normal distribution of return on assets has played a central role in the construction of financial economics (German 2002). Indeed, all non-Gaussian observations and "white noise" were characterized through a Gaussian standardization. In this perspective, the leptokurticity of the observed distributions has often been characterized through an improved Gaussian framework based on composed processes like diffusion-Poisson (Hanson & Westman, 2003), regime switching models (Hamilton, 2005) or stochastic volatility methods (Broto & Ruiz, 2004). However, according to Taleb (2008, p.2), these processes can be called "pseudo-fat tails" because the finiteness of all moments makes them collapse into thin tails situation.

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[19] For all Levy stable processes with a $\alpha > 1$, the localisation factor is always equal to the mean (see Nolan, 2009) for a demonstration.
[20] Gaussian perspective is the most used framework in science to describe random phenomena (Stewart 1992).

Two arguments can explain this observation: the simplicity of Gaussian distribution (only two statistical momentums are needed in order to describe a random phenomenon) and the statistical foundations of this Gaussian framework that are directly rooted/embedded with the central-limit theorem (Belkacem 1996). Through only two keys parameters (mean and variance), the Gaussian distribution offers then a very useful and intuitive framework in order to describe financial data. However, as evoked before, Mandelbrot (1963) and Fama (1965) proposed, in the sixties, to describe the evolution of financial returns through a Levy stable framework. In an empiricist (econophysics) perspective, the process appear to be more adapted to the description of financial data.

III.3.b. Levy stable Processes

Some Levy stable processes refer to regimes having a characteristic exponent in the range of $0 < \alpha < 2$. All these distributions have heavy tails with a power lay Pareto decay. The term "Paretian" is often used to distinguish the $\alpha < 2$ cases from the Gaussian one. The Levy stable regimes with $\alpha < 2$ refer to a class of distribution in which we can find the Pareto distribution ($\alpha = 3/2$) or the Cauchy distribution ($\alpha = 1$).

A theoretical implication of the heavy tails is that only certain statistical moments exist (Nolan, 2005). It can be shown that the variance does not exist when $\alpha < 2$ and that the mean does not exist when $\alpha \leq 1$. More generally, the p^{th} moment exists if and only if $p < \alpha$ (Nolan, 2005). In contrast to this theoretical result, all statistical moments computed from sample exist because all samples are finite. However, as Nolan (2009) and Taleb (2007) emphasized it, the problem is that the sample does not tell you much about stable laws because the sample variance does not converge to a well-defined moment. We find here the gap evoked in the section (III.1) between the observational correspondence of Levy stable process with real data and the theoretical consequences that make the empirical use of these processes very difficult.

Mandelbrot was the first to attempt to emphasize this problem when he proposed an extended Gaussian framework in finance. Using two models which he called M1963 and M1965, he opened two new research themes in the statistical modelling of financial uncertainty: one calls into question the *independence* of observations (between themselves) while the other examines the *stationary* character of these observations[21]. The first makes it possible to take into account observable and apparent cycles on the markets, and the second the apparent discontinuity of the price of assets on the markets. Only the first model (M1963) called into question the normal distribution of returns on financial assets, that is why, one might consider this model (and not M1965[22], for example) as the first mathematical origins of what we call today econophysics[23]. In this model (M1963), Mandelbrot demonstrated how what Lévy called "α-stable" (α = 1.7) processes were entirely suitable for studying the discontinuity of price changes. To characterize this variability in respect of abrupt or discontinuous variations, Mandelbrot and Walis (1968) talked of a "Noah effect"[24]. Models that explicitly rejected the Gaussian framework and especially its continuity hypothesis needed to be integrated into a new probabilistic perception of uncertainty. Mandelbrot worked with Fama on applications such as these in finance. In these articles, Fama (1966, 1967) gave a mathematical reinterpretation of modern portfolio theory by Markowitz (together with Sharpe's diagonal model) in a Paretian statistical framework. However, in their attempt to provide a new way of thinking the notion uncertainty within a Lévy stable framework, these authors were unable to provide a theoretical interpretation of their work because the parameter of risk (variance) was infinite (Fama 1965, p.414). Finally, the logical consequence of these debates about the use of Levy regimes in the sixties, these processes have been rejected from the financial economics.

A statistical response to this indeterminate nature of variance was given by physicists in their attempt to characterize the turbulence phenomenon through the Levy stable processes. This

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[21] *Stationary* means that statistical properties do not depend on time. A time series may be stationary in respect to one characteristic, e.g. the mean, but not stationary in respect to another, e.g. the variance. *Independent* means that there is no link (no correlation) between variations in position
In his second model (M1965), Mandelbrot attempted to show how it was possible to study the long-term dependence of observations (in a solely Gaussian framework). This model, then, is chiefly concerned with studying correlations between observations. Mandelbrot and Walis (1968) used the term "Joseph effect"to characterize phenomena in which, for long periods, high (or low) values tend to be followed by high (or low) values. These studies focused essentially on the fractal dimension of the Gaussian framework without calling its normality into question. This type of model essentially re-examines the fractal nature and more specifically the self-similarity property of Brownian motion, proposing a generalization that challenges the statistical hypothesis of independence of observations. Mandelbrot's second model (M1965) did not reject the dominant framework, and proposed a generalization of the Brownian process.
[23] This origin is explicitly recognized in specialized econophysics literature (Mantegna and Stanley 1999, McCauley 2004) and claimed by Mandelbrot (2005) himself. It can be noted, however, that a (small) number of physicists have proposed work based on an α-stable analysis defended in Mandelbrot's first model – see, for example, Mantegna and Stanley (1999), Sornette and Johansen (1997).
[24] Mandelbrot and Walis (1968) referred indirectly to the biblical tale of Noah. When a "deluge" (stock market crash) is observed on financial markets, "even a big bank or brokerage house may resemble a small boat in a huge storm" (Mandelbrot 2004, p.222).

response, as I will show, also contributed to the emergence of econophysics since it was now possible to apply this standardization operation in such a way that the evolution of financial markets can be described using stable Lévy processes.

From the 1980s onwards, Lévy processes were increasingly used in physics[25], particularly in statistical physics. The latter discipline can be thought of as the continuation of thermodynamics, and the use of Lévy processes in this field allowed more accurate modelling of the phenomenon of turbulence. The first studies on the subject were those of Kolmogorov on the scale invariance of turbulence in the 1930s. This theme was subsequently addressed by many physicists and mathematicians, particularly by Mandelbrot in the 1960s when he defined fractal mathematics[26] and applied it to the phenomenon of turbulence[27].

Despite the extension of probability theory to thermodynamics, physicists did not seem disposed to integrate stable Lévy processes into physics (Gupta and Campanha 1999, p.382). This methodological position (like the abandonment of α-stable processes in financial economics) is explained by two reasons: The first reason, as we mentioned above, is the infinite characteristics of variance that is not physically plausible:

> "Stochastic processes with infinite variance, although well defined mathematically, are extremely difficult to use and, moreover, raise fundamental questions when applied to real systems. For example, in physical systems, the second moment is often related to the system temperature, so infinite variance implies an infinite temperature" (Mantegna and Stanley 1999, p.4).

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[25] See Frisch et al. (1994) for an analysis of the influence of Lévy processes in physics.
[26] Although modern probability theory was properly created in the 1930s, in particular through the works of Kolmogorov, it was not until the 1950s that the Kolmogorov's axioms became the dominant paradigm in this discipline.
[27] Fractal mathematics was essentially developed by Mandelbrot, who attempted to develop a new geometrization to describe a large number of complex, "irregular" phenomena encountered in nature (Mandelbrot 2005, 147). The main idea of fractal geometry is that certain aspects of reality have the same structure seen from afar or close up, at any scale, and that only the "details having no effect change when they are enlarged for a close-up view" (Mandelbrot 1995, p.36). Mandelbrot uses the term "principle of scale" to illustrate this constancy of structure between two levels of enlargement and he adds that "a phenomenon satisfies the principle of scale if all the quantities relating to this phenomenon are linked together by a law of scale" (Mandelbrot 1995, p.53). Lévy processes allowed a statistical reformulation of fractal geometry by means of the notion of invariance. It then becomes possible to link two variables (X and Y) each characterizing the level of the zoom operated on the phenomenon under study. In this way, two levels, to each of which a variable is associated and which comply with the principle of scale, can be linked by a scaling law (or a power law). Regarding the history of statistical physics and importance of fractal mathematics in this discipline, see Ruelle (1991) or Barberousse (2002).

As Gupta and Campanha (2002, p.232) pointed out, "Lévy fights have mathematical properties that discourage a physical approach because they have infinite variance." In their view, this property of physical systems is the direct result of the thermodynamic hypotheses set out by Boltzmann and Gibbs in 1884 when they laid the foundations of contemporary statistical mechanics. Physicists, then, seem to be facing the same theoretical impasse as Fama and Mandelbrot in the 1960s: the infinite character of variance[28].

The second reason to the reject of Levy process in Physics refers to the fact that Levy stable processes are also based on an asymptotical argument. Indeed, according to a generalized version of the central limit theorem proposed by Gnedenko and Kolmogorov (1954), if many independent random variables are added whose probability distributions have power-laws tails such as $p_i(x_i) \sim |x_i|^{-(1+\alpha)}$ with an index $0 < \alpha < 2$, their sum will be distributed according to a stable Lévy distribution.

As reminded Bouchaud (2002, p.28), there is a very important difference between the Gaussian CLT and the generalized Levy CLT: in the first case ($\alpha = 2$), the order of magnitude of the sum of N terms is $\sqrt{N}$ and is much larger (in the large N limit) than the largest of these N. In other words, no single term contributes to a finite fraction of the whole sum (i.e there is no important deviation but a lot of small deviations). In opposition with the case $\alpha < 2$, the whole sum and the largest term have the same order of magnitude. The largest term in the sum contributes now to a finite fraction of the sum, even if $N \rightarrow \infty$.

All physical systems refer to real phenomena that have neither infinite parameters nor an asymptotical behaviour. By choosing to characterize turbulence phenomena using Lévy processes, physicists have to solve two theoretical problems in order to fit the statistical tools to the reality they study: on one hand, they look for a finite variance and on the other hand, they want to work in a (locally)[29] no-asymptotical framework. In this perspective, a number of writers have proposed new statistical methods in order to "standardize" of "α-stable" distribution so that variance is no longer infinite. I will deal with this method in the following section.

[28] Note that a number of studies have been carried out on a new data dependency structure to replace the concept of variance with the notion of "covariation" (Föllmer, *et al.* 1995). However, these studies have not been unanimously accepted by physicists.

[29] I will explain and illustrate what I mean by "locally" in the following sections.

I.V. Econophysics and Levy non-stable processes

IV.1) Truncated Levy processes

The most widespread solution to solve the infinite variance problem consists in truncating Lévy distributions. The truncation of a Lévy distribution consists in normalizing it using a particular function so that its variance is finite. In this perspective, Lévy stable distribution are used with the specific condition that there is a cut-off length for the price variations above which the distribution function is set to zero in the simplest case (Mariani and al., 2007) or decreases exponentially (Gupta and Campanha, 2002; Gunaratne and McCauley, 2002). These function are chosen in order to obtain the best fit with the empirical data (Mariani and al., 2007) or they are derived from models like percolation theory (Gupta and Campanha, 2002 or Richmond, 2001) or the generalised Focker Plank equation (Gunaratne and McCauley, 2002). Generally speaking, the probability distribution of a truncated levy flight can be defined as followed,

$$P(\mathrm{x}) = \begin{cases} N(0,\sigma) & |x| > l \\ k\,\mathrm{L}(\mathrm{x})_{(\alpha,\beta,\gamma,\delta)} & |\mathrm{x}| \leq l \end{cases}$$

Where $\mathrm{L}(\mathrm{x})_{(\alpha,\beta,\gamma,\delta)}$ is a symmetrical Levy stable distribution with an index α ($0 < \alpha \leq 2$), a scale factor $\gamma > 0$ and $\beta = 0$ since the distribution is symmetrical. δ is a scale factor which is > 0, *k* is to a normalizing constant and *l* is the cut-off length. Usually, the standard Lévy distribution is abruptly cut to zero at a cutoff point (Mantegna and Stanley, 1994). In other words, the probability of taking a step is abruptly cut to zero at a certain critical step size. *K* and *l* are then chosen in order to better fit with the financial data but also to have a crossover value $N^*(l)$ which increases rapidly with the cut-off length *l* ($N^*(l) \sim l^{\alpha}$). For small value of x, *P (x)* takes a value very close to the one expected for a Lévy stable process. But for large values of n, *P (x)* assumes the value predicted for a Gaussian process. In other words, if x is not too large, a truncated Lévy distribution behaves very much like a pure Lévy distribution since most of value expected for x fall in the Lévy-like region. When x is beyond the crossover value N^*, the variable x progressively converges towards a Normal distribution.

Although Levy stable processes have an infinite variance, physicists have developed a statistical method allowing them to apply these processes in experimental studies. Thanks to the truncated Levy, physicists can have a finite variance and then a more realistic analysis. However, these truncated processes are still based on an asymptotic argument coming from the convergence towards a Gaussian distribution even if this process of convergence is very slow. In this perspective, the asymptotic problem have been partially solved by physicists since they are able to escape locally (only for < N^*) to this asymptotic behavior.

A more important drawback of truncated Levy distributions refers to the fact they are not stable[30] and often not infinitely divisible because the truncation of the distribution is abrupt. This implies that its shape is changing at different time horizons and that distribution at different time horizons do not obey scaling relations[31]. More precisely, scaling turns out to be approximate and valid for a finite time interval only. For longer time intervals, scaling must break down. However, scaling can be very useful in the analysis of finance data since this concept allows us to describe the statistic features of financial distributions independently of time horizon. Moreover, some physicists claimed that an abrupt truncation is only useful for very specific cases but that this methodology would not be physically plausible enough.

Physicist have two reasons to go beyond truncated Levy process and to use another statistical framework, on one hand, in a more realistic perspective, physicists would use statistical processes that have neither an infinite variance nor an asymptotical behavior and, on the other hand, they want to use an infinitely divisible process avoiding to have an abruptly cutoff to zero which is a very special case and which have not physical basis (Gupta and Campanha, 1999). I will deal with this problem in the next section.

IV.2) Gradually Truncated Levy processes

In real systems, the variance of a stationary process is necessarily finite. In this perspective, when we try to describe a system with Levy processes, an unavoidable cut off is needed. Mantegna and Stanley (1994) introduced truncated Levy process to solve this problem. As presented in the previous section, this truncated Levy distributions allow to have a finite variance but, in this case,

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[30] Only non-truncated Lévy processes are stable (Mantegna and Stanley, 1999).

[31] Indeed, the variable x progressively converges towards a Levy distribution for $x < N^*$ while it converges toward to a Normal distribution when x is beyond the crossover value N^*

the probability of taking a step is abruptly cut to zero at a certain critical step. However, as Gupta and Campanha (1999) or Matsushita and al. (2004) emphasized, the truncated distribution proposed by Mantegna and Stanley (1994) is a very special (only related to specific problems). Moreover, these authors also stressed that abrupt truncated levy distributions are not based on a physical basis. Indeed, as Gupta and Campanha wrote,

> "The dynamical properties of a complex physical or social system depend on the dynamical evolution of a large number of non-linear coupled subsystems. Thus, it is expected that in general, the probability of taking a step [a variation] should decrease gradually and not abruptly, in a complex way with step size due to limited physical capacity".

In this perspective, truncated Levy distributions have been introduced by Koponen (1986) who considered the truncation with an exponential cutoff. Koponen (1986) considered a TLF in which the cut off is a decreasing exponential function. Gupta and Campanha (1999) generalized this approach in econophysics by the following probability distribution,

$$P(\mathrm{x}) = \begin{cases} \mathrm{L(x)}_{(\alpha,\beta,\gamma,\delta)} & |x| > l \\ e^{-f(t,l)}\ \mathrm{L(x)}_{(\alpha,\beta,\gamma,\delta)} & |\mathrm{x}| \leq l \end{cases}$$

where l is the cut off at which the distribution begins to deviate from Lévy distribution. $e^{-f(t,l)}$ is a decreasing function depending on time (t) and the cut off parameter(s) (l) . Let us mention that there are different ways of evaluating the cut off parameter (t) which is often related to the limited physical capacity of systems to take larger steps[32]. In general, authors compute parameters in order to fit to the data[33]. In line with abruptly truncated levy distributions, gradually truncated Levy distributions have finite variance, they allow to avoid locally the asymptotic behaviour, they are infinitely divisible and they have scaling property. However, this kind of distribution has an advantage from an empiricist perspective: it would be more adapted to describe reality[34] than

[32] We deal with this parameter in more details in the following sections.
[33] See Matushita and al. (2004) for further information about this point.
[34] This kind of distribution has been applied to the currency data. See Figueiredo and al (2003) or Gupta and Campanha (1999) for empirical tests of this kind of distribution.

abruptly truncated levy distribution because it is more physically justified since real systems rarely evolve abruptly (Gupta and Campanha, 1999).

Despite all these statistical features, gradually levy truncated distributions have an empirical drawback: they under-estimate the fat tails (Matsushita and al. 2004) and moreover, they only describe negative variations. In order to improve the gradually truncated levy distributions by keeping its statistical feature, physicists developed a more sophisticated process which better fit the data. I will introduce this process in the next section.

IV.3) Exponentially Damped truncated Levy processes

It has been pointed out (Gutpa and al., 1999 or Gupta and al. 2000) that the gradually truncated Levy distributions break down in the presence of positive variations. That also means that tails are fatter than those estimated by gradually truncated levy distributions. This drawback is important since in real system, we also observe increasing deviations for which a more realistic process must necessary be developed. Matsushita and al. (2004) introduced what they called exponentially damped truncated distributions which distributions encompass the previous cases. These new truncated levy distributions are more justified from a physical point of view and they better fit the data.

$$P(\mathrm{x}) = \begin{cases} \mathrm{L(x)}_{(\alpha,\beta,\gamma,\delta)} & |x| > l_{\max} \\ e^{-f(t,l)}\,\mathrm{L(x)}_{(\alpha,\beta,\gamma,\delta)} & l \le |\mathrm{x}| \le l_{max} \\ k\,\mathrm{L(x)}_{(\alpha,\beta,\gamma,\delta)} & |\mathrm{x}| < l \end{cases}$$

The truncation parameters (l and l_{max}) are estimated to provide an optimized fit for the tails. l is the step size at which variations begin to deviate from the Levy following a decreasing exponentially processes and l_{max} is the step size at which variations follow a process in line with an abruptly truncated distribution. These truncated Levy processes keep all statistical features observed for previous cases. Therefore, they appear to be the best appropriated process in order to describe financial data since they encompass statistical features[35] of all previous cases but they also take into account negative and positive variations.

[35] These processes have a multiscaling property implying a complex scaling relations between different level of scales can be identified (Matsushita and al. 2003).

IV.4) Other non-stable Levy processes

Although Lévy processes have enjoyed considerable popularity in physical sciences, the interest has been generally confined to stable processes (Cont and Tankov, 2004). This appeal to the stable processes refers to their properties of scaling and self-similarity. However, as we mentioned before, the Lévy stable processes have features (infinite variance and an asymptotical behaviour) which are not adequate for an empirical application (Petroni, 2007). The need to go beyond the Levy stable processes leads some physicist to use non stable Levy processes.

New applications for non stable processes appear in different fields of physics: quantum physics (Vivoli and al, 2006), models for spinning particles (De Angelis and Jona-Lasinio, 1982) and relativistic quantum mechanics (De Angelis, 1990). In this perspective, several statistical processes can be mentioned in the family of non-stable (non-truncated) Levy distributions: the compound Poisson process (Vivoli and al., 2006), the VG process (Petroni, 2007) or the Student process (Gopi and al, 1998)[36]. Some of these processes can also be used to avoid the infinite variance of Levy stable process. For example, one might use Levy sections theorem which extends the central limit theorem to encompass dependent variables. Paul Lévy employed his notion of “sections” to outline a proof for a variant of the central limit theorem that considers the sums of correlated random variables[37]. Student’s t-distribution can also be used to avoid a infinite variance since this kind of distribution may have both finite or infinite variance (depending on parameters chose to have a finite moment of order 2)[38]. As emphasized above, this paper only focuses on the truncated solutions and their statistical implications for financial economics. For an introduction to these non stable (and non-truncated) Levy processes, see Petroni (2007).

[36] According to Gopi and al (2002) and Gabaix (2003), the stylized fact can be described with a t-student process with power law exponent of 3.

[37] The classical central limit theorem does not take chains of random variables that are dependent into account. Levy section theorem can then be seen as a less restrictive argument. More precisely, this theorem states that the sum S_t converges to a Gaussian distribution even if the usual sum $S_n = x_1 + \ldots + x_n$ does not.

[38] See Mantegna and Stanley (1999, p.62) for further information about the use of t-Student’s distribution

Conclusion

This paper dealt with the application of Levy processes in finance. After having reminded the reasons for why the Gaussian perspective is the most used framework in financial economics, I emphasized why econophysicists reject this frame claiming that Levy processes appear to be a more descriptive approach of financial phenomena. This new statistical description of financial distribution refers to the emergence of new parameters describing the risk. In this paper, we evoked the four parameters of risk computed in a Levy framework (while Gaussian framework is mainly based on two parameters). As mentioned in this paper, stable Levy processes are very interesting for financial economics because, in opposition to Jump processes or to Pure Jump diffusion models, stable Levy processes imply a scaling property and they can describe the leptokurticity of financial markets through a continuous process (implying the uniqueness of prices).

However, although these processes better fit to the financial data, some of their statistical properties can raise several theoretical problems in order to use them in empirical studies. In this perspective, econophysicists had to developed more sophisticated Levy processes in order to have a statistical tools in line with a plausible physically analysis. This paper presented all these oppositions between statistical characteristics of Levy processes and the empirical needs of econophysicists.

This paper implicitly dealt with the future of financial economics. Indeed, by (re) introducing Levy processes in finance, econophysicists challenge neoclassical financial economics (Jovanovic and Schinckus, 2010) because they offer a more generalized statistical framework and therefore they provide a broader definition of financial uncertainty (Schinckus, 2009).

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