Aggregate Consequences of Firm-Level Financing Constraints*

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Abstract

Recent empirical studies have documented important differences in the financial behavior of small and large firms. These empirical regularities are widely interpreted as indirect evidence of frictions in financial markets. Frictions in financial markets have been cited both as a possible source of, and as an amplifier of business cycle fluctuations. In order to quantitatively investigate the aggregate effects of financial frictions, I compare two general equilibrium model economies. In the two model economies, firm-level borrowing constraints arise because of contracts that are constrained efficient under private information or under limited enforcement. The results show that an economy subject to private information problems is less volatile than a frictionless economy, while an economy with contract enforcement problem is substantially more volatile than a frictionless economy when the punishment for defaulting entrepreneurs is low.

JEL Code: E10; E23; E22; D82; G32; L14

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1 Introduction

Recent empirical studies in industrial organization have documented important differences in the financial behavior of small and large firms. In particular, these studies have rejected Gibrat's Law, which states that firm size and growth are independent. Furthermore, a vast body of empirical literature has rejected the prediction of the standard model of investment with convex adjustment costs that movements in the investment rate should be determined by changes in Tobin's q. These results have been widely interpreted as evidence of frictions in financial markets. Models such as the one of Clementi and Hopenhayn [2006] and Albuquerque and Hopenhayn [2004] have shown that long-term contractual arrangements that are constrained efficient under private information or limited enforceability can help account for some of the growth characteristics of small and young firms.

To what extent do frictions in financial markets due to private information or limited enforcement amplify business cycle fluctuations? Starting with Bernanke and Gertler [1989] and Kiyotaki and Moore [1997], financial frictions have become an important focal point of the business cycle literature. However, most of this literature abstracts from the effect of financing frictions on individual firm dynamics. A notable exception is Cooley, Marimon, and Quadrini [2004] who show that long-term contracts that are constrained efficient when contracts are not fully enforceable amplifies business cycle fluctuations. This paper complements Cooley et al. [2004] by studying a general equilibrium model economy in which entrepreneurs and investors enter into a long-term contractual relationship that is constrained optimal under private information; and compares the predictions of the model to an economy in which entrepreneurs and investors enter into a long-term contractual relationship that is optimal, subject to enforceability constraints.

The main results of this paper can be summarized as follows: In the absence of aggregate shocks, the aggregate quantities of the two economies are constant over time, although individual firms are continually in motion by expanding, contracting, starting up, and shutting down. I show that while the firm dynamics implied by the two models are broadly consistent qualitatively with microeconomic empirical regularities, their quantitative implications at the micro and macroeconomic level differ substantially. For instance, I show that conditional on age, the growth rate of firms in an economy with private information problems is about four times more volatile than in an economy with enforcement problems. This, in turn, is associated with important differences in job relocation, firms' entry and exit, the adoption of new technology, and the equilibrium allocation.

With aggregate shocks, the aggregate quantities of the two economies fluctuate over time. I devise two algorithms based on the method of Krusell and Smith [1998] to approximate the equilibrium allocations. The key result of this paper is that an economy with private information problems is less volatile than a frictionless economy, while an economy with limited enforcement is substantially more volatile than a frictionless economy when defaulting entrepreneurs cannot be excluded from financial markets. When defaulting entrepreneurs can be excluded from financial markets, an economy in which financial contracts cannot be fully enforced is not substantially more volatile than a frictionless economy.

The mechanics behind the key results are as follows: In an economy in which financial contracts cannot be enforced, the outside value option is – a fraction of – the expected value of diverting and consuming the current revenue and starting a new project. When high-productivity projects are abundant, the outside value option increases, which optimally tightens the financing constraints for small – young – firms in order to prevent default. This slows down the growth of small firms disproportionally to that of larger firms. Furthermore, new low-productivity firms face a substantially higher hazard rate of exit than high-productivity firms during their first year of operation. This implies the economy retains relatively more high productivity firms. Last, the size of a firm does not decrease after receiving a low revenue shock. These three effects yields a mass of high-productivity firms operating at full scale that are not immediately replaced when exiting a the exogenous rate. This creates substantial fluctuations in the distribution of firms and thus of aggregate output.

When defaulting entrepreneurs can be excluded from the market, the value of the outside option is only the value of the diverted resources in one period, and the financing constraint faced by small firms is less binding. New firms start larger and take fewer years on average to become financially unconstrained. This reduces the first effect and decreases the spread of firms in the economy yielding smaller fluctuations of the equilibrium distribution of firms and thus of aggregate output.

Finally, in an economy in which revenue realizations are private information, firm growth characteristics are more dynamics than in an economy with limited contract enforcement because size decreases with bad revenue shocks and increases with high ones. However, the economy does not penalize low-productivity as in the case of limited enforcement, and firms are more evenly replaced when shrinking or exiting. This in turns does not lead to large fluctuations in the equilibrium distribution of firms and aggregate output, and moderately dampens the effect of the arrival of new technology.

The rest of the paper is organized as follows: section 2 presents the key empirical regularities on firm dynamics, and the recent developments in the industrial organization to address them. section 3 describes the economic environment, and section 4 discusses the source of the financial frictions in the two models. Section 5 presents the recursive formulation and section 6 defines the stationary equilibria. Section 7 describes the parametrization and the properties of the stationary and the stochastic steady state for the economies, and section 8 concludes. Proofs of propositions and numerical strategies are relegated to the Appendix.

2 Empirical regularities and related literature

Firm dynamics broadly include firm entry and exit, growth and volatility of growth, and job creation and job destruction. Hall [1987] and Evans [1987] finds that the growth and volatility of growth of manufacturing firms are negatively related to their size and age. Evans [1987] show this relationship holds when conditioning on age or on size. Dunne, Roberts, and Samuelson [1989] and Dunne and Hughes [1994] show that the output of new firms is substantially smaller than the output of an average incumbent. Davis, Haltiwanger, and Schuh [1998] shows that the rates of job creation and job destruction in U.S. manufacturing firms are negatively associated with the age and the size of firms; and that, conditional on the initial size, small firms grow faster than larger ones. Cooley and Quadrini [2001] summarize the aforementioned empirical regularities on firm dynamics as follows:

1. conditional on age, the dynamics of firms are negatively related to the size of firms;
2. conditional on size, the dynamics of firms are negatively related to the age of firms;
3. smaller and younger firms pay fewer dividends, take on more debt and invest more; and
4. the investments of small firms are more sensitive to cash flows, even after controlling for their future profitability.

Firm dynamics are typically attributed to selection or to financing constraints. Two canonical models of selection are Jovanovic [1982] and Hopenhayn [1992]. In Jovanovic [1982], entrepreneurs learn about their ability over time, and decide to stay or exit depending on the signals they receive each period. In Hopenhayn [1992], persistent productivity shocks affect firm growth, and a long sequence of bad shocks may force a firm to exit the market. A key assumption in these two models is that firms are uncertain about their productivity ex-ante. In their original presentation, these two models cannot simultaneously account for the dependence on size conditional on age, and conversely, the dependence on age conditional on size. However, recent models of selection such as Klette and Kortum [2004], and Luttmer [2007] are able to account for the age-size dependency. Lee and Mukoyama [2008] extend Hopenhayn and Rogerson [1993], which is a general equilibrium extension of Hopenhayn [1992], to include aggregate shocks to study firm entry and exit over the business cycles. Clementi and Palazzo [2010] study the contribution of firm entry and exit to aggregate dynamics by characterizing the partial equilibrium allocation in a model based on Hopenhayn [1992] and augmented with investment in physical capital and aggregate shocks.

Starting with Fazzari, Hubbard, and Petersen [1988], a large body of literature has developed to assess empirically the effect of frictions in financial markets on firms investment and growth – see Hubbard [1998] for a survey of the literature. Recent empirical studies such as Cabral and Mata [2003], Oliveira and Fortunato [2006], Fagiolo and Luzzi [2006], and Lu and Wang [2010] suggest that financing constraints are an important determinant of the age-size dependence of firm dynamics. Albuquerque and Hopenhayn [2004] and Clementi and Hopenhayn [2006] show that long-term contracts that are constrained efficient under limited enforcement and private information respectively, can account for the empirical regularities discussed above. In Albuquerque and Hopenhayn [2004], borrowing constraints arise because borrowers face limited liability and debt repayments cannot be perfectly enforced. In Clementi and Hopenhayn [2006], borrowing constraints arise because borrowers face limited liability and the revenue realizations of a firm are private information.

In the business cycle literature, the work of Bernanke and Gertler [1989], Carlstrom and Fuerst [1997], Kiyotaki and Moore [1997], and Bernanke, Gertler, and Gilchrist [1999] sparked a large body of literature that developed dynamic general equilibrium models of investment with financial constraints to study the aggregate implications of financial frictions. Gertler and Gilchrist [1994] argue that small firms with lesser access to credit are more negatively affected by a contractionary monetary shock than larger firms, thus amplifying the response of shocks. Sharpe [1994] argues that highly leveraged firms are more sensitive to aggregate demand shocks. However, Lee and Mukoyama [2008] document patterns of firm dynamics of U.S. manufacturing plants over the business cycle between 1972 and 1997, and find that the entry rate is substantially more cyclical than the exit rate, and entrants' average size and productivity vary significantly over the business cycle. Their findings suggest incumbent small and large firms are not disproportionally affected by business cycle contractions, in the sense that a recession does not lead to a "cleansing" of low productivity firms.

Cooley et al. [2004] are the first to introduce micro-foundations for firm-level financing constraints and firm dynamics into a dynamic general equilibrium models to study the implications of financial frictions on aggregate fluctuations. They propose a general equilibrium model in which entrepreneurs finance investment with long-term financial contracts that are constrained efficient because of enforceability problems. Cooley et al. [2004] is a general equilibrium extension of Marcet and Marimon [1992] with aggregate shocks, and is related to Albuquerque and Hopenhayn [2004]. In this paper, I use long-term contractual arrangements of Albuquerque and Hopenhayn [2004] and Clementi and Hopenhayn [2006] to construct two general equilibrium models with heterogeneous firms and aggregate shocks in which firm dynamics are qualitatively consistent with the aforementioned empirical regularities. A key difference between the model with limited enforcement of this paper and the model of Cooley et al. [2004] is firms revenue is subject to idiosyncratic shock and young firms face a an endogenous non-zero probability of exit as in Albuquerque and Hopenhayn [2004]. I show below that this has important implications for the mechanism driving the amplification of business cycle fluctuations.

Lastly, Beck, Demirg-Kunt, and Maksimovic [2008] use a firm-level survey database of small and medium firms in 48 countries, and show that while small firms are not able to take on as much debt as large firms, they benefit disproportionally from higher levels of property rights protection by using significantly more external debt financing from banks. This is consistent with the results of this paper. The result that private information does not amplify the business cycle thereby suggesting that the level of contract enforcement, and not financing frictions per-se, is the source of an important amplification mechanism of aggregate shocks.

3 Environment

Time is discrete and infinite. The economy is populated by a mass of overlapping generations of atomistic workers and entrepreneurs with uncertain lifetimes. Workers and entrepreneurs survive into the next period with a fixed probability. There is one all-purpose good used for consumption and investment. Workers are risk averse and are endowed with one unit of time each period, which they allocate between labor and leisure. Workers supply labor in a competitive labor market. Entrepreneurs are risk neutral and have access to a project with diminishing returns to scale technology requiring a fixed initial investment, and period resources to be used as working capital and to hire labor. Entrepreneurs do not make labor-leisure decisions. Once started, the project generates a stream of uncertain revenues subject to independent and identically distributed revenue shocks. Entrepreneurs finance their project by borrowing the initial capital, and the period working resources from financial intermediaries. Death of an entrepreneur terminates the firm, and is analogous to a permanent zero-productivity shocks.

3.1 Workers

Workers are born with zero non-human wealth, survive into the next period with exogenous probability $(1 - \gamma_w)$, and are instantly replaced by new ones when deceased. Workers discount the future at rate $(1 - \gamma_w)\hat{\beta}$, are endowed with one unit of time each period, and receive capital income d_t and labor income $w_t h_t$, where d_t are the workers' assets at time t, h_t is the fraction of time spent working, and w_t is the wage rate. Following Blanchard [1985] and Rios-Rull [1996], workers use their income to either buy numeraire consumption good c_t, or to purchase contingent claims d_{t+1} at price p_t^a that pay one unit of consumption in the next period if the agent is alive, and zero otherwise. Agents do not value bequests and thus place all their savings in these claims. Workers assess their consumption-leisure decision according to

$$E_0 \sum_{t=0}^{\infty} [(1 - \gamma_w)\hat{\beta}]^t u(c_t, 1 - h_t) \tag{1}$$

which is subject to the following constraint each period:

$$c_t + p_t^a d_{t+1} \leq d_t(1 + r_t) + w_t h_t. \tag{2}$$

3.2 Entrepreneurs

Entrepreneurs, like workers, are born with zero wealth, survive into the next period with probability $(1 - \gamma_e)$, and are instantly replaced by new ones when deceased. Entrepreneurs are risk neutral and discount the future at rate $\beta = 1/(1 + r_t)$. I assume entrepreneurs do not make a leisure-labor decision – an interpretation of this is entrepreneurs devote a fixed fraction of their time to supervise the operation of their project. Entrepreneurs assess their consumption decision according to

$$E_0 \sum_{t=0}^{\infty} [(1 - \gamma_e)\beta]^t c_t. \tag{3}$$

Risk neutrality implies entrepreneurs consume all their income every period, and do not take part in the annuity market introduced above.

3.3 Financial intermediation

Financial intermediaries are not modeled explicitly. I assume perfect competition in the financial markets and focus on a representative risk-neutral financial intermediary who discounts the future at $1/(1+r_t)$. The financial intermediary receives deposits from the workers in period t, which are then invested into the entrepreneurs' projects in period $t+1$. Repayments from the entrepreneurs to the intermediary are used to repay the deposits with interest. The assumptions on the workers imply a stationary demographics for workers so that annuities can be offered without risk by the financial intermediaries. Given an interest rate r_t, zero profits in the annuity market drive the price of annuities to the survival rate $p_t^a = (1-\gamma_w)$.

3.4 Technology and information

3.4.1 Idiosyncratic shocks

New entrepreneurs are born with a blue print for a project. There are at most two types of project with a fixed productivity levels $z \in \{z_L, z_H\}$ with $z_L < z_H$, decreasing return to scale technology $f_z(k,n)$ such that $f_z(k,0) = f_z(0,n) = 0$, and which require a sunk fixed initial investment I_0 – i.e., using firms to organize production is meaningful. Once started, a project requires period resources R_t to be used as capital k_t and to hire labor n_t at wage rate w so that $R_t = k_t + wn_t$. I assume period capital k_t is fully depreciated at the end of a period.

Project returns are subject to a sequence of independent and identically distributed idiosyncratic revenue shocks $(\nu_t)_{t\geq 0}$, where $Pr(\nu_t = 1) = 1 - Pr(\nu_t = 0) = p$. Project returns are only observed by the entrepreneur in an economy with private information, while there is no informational asymmetry in an economy with enforcement problem. A firm is terminated if the entrepreneur dies, which is analogous to receiving a permanent zero-productivity shock. This assumption is convenient to capture other source of exit not modeled explicitly, and allows me to pin down the distribution of firms without keeping track on individual firms.[1]

3.4.2 Aggregate shocks

I follow Cooley et al. [2004] and assume that high productivity projects are available in limited supply each period. Let $(N_t)_{t\geq 0}$ be the sequence of random measures of high-productivity projects available in an economy each period, and let Γ_t be the measure of new-born entrepreneurs. It follows that If, in period t, $\Gamma_t > N_t$, a fraction $1 - N_t/\Gamma_t$ of new-born entrepreneurs do not have access to a high productivity project. In the parametrization, I set the productivity differential between the project and the sunk cost so that it is never optimal for an entrepreneur to wait for a high productivity project, or to give up a low-productivity project once it is started to switch to a new high-productivity project. Finally, the probability for a new-born entrepreneur to start a high-productivity project in period t is $q_t = \min\{1, N_t/\Gamma_t\}$. The realizations of the aggregate shock are observed by all agents in the economy.

4 Financial frictions and optimal contracts

I consider endogenous financing constraints induced by a long-term contract between an entrepreneur and the financial intermediary in an economy with private information and in an economy with limited enforcement. In an economy with private information, the sequence of idiosyncratic revenue shocks is only observed by the entrepreneurs who then make ex-post reports to the financial intermediary. The financial intermediary

[1]This assumption is common in the related literature. See for instance Cooley and Quadrini [2001], Cooley et al. [2004], and Smith and Wang [2006].

is able to make a new entrepreneur sign a long-term contract that is optimal given a limited-liability constraint, and an incentive compatibility constraint. In an economy with limited enforcement, project returns are public information, but entrepreneurs may default on their debt obligation, consume the current period revenue, and search for a new project.

4.1 Private information

The financial intermediary make new entrepreneurs sign a long-term contract, and fund the project by lending the initial sunk cost I_0, and some initial working resources R_0.[2] A reporting strategy for an entrepreneur is a sequence of reports $\underline{\hat{\nu}} = \{\hat{\nu}_t(\nu^t)\}_{t\geq 0}$, where $\nu^t = (\nu_1, \ldots, \nu_t)$ is the true history of the revenue shocks received by an entrepreneur. The history of reports is denoted by $h^t = (\hat{\nu}_1, \ldots, \hat{\nu}_t)$. A contract is a quadruple $\kappa = \{\ell_t(h^{t-1}), Q_t(h^{t-1}), R_t(h^{t-1}), \tau(h^{t-1}, \nu_t)\}$ that specifies a liquidation rule $\ell_t(h^{t-1})$, the payment $Q_t(h^{t-1}) \geq 0$ from the financial intermediary to the entrepreneur in case of liquidation, the period resource advancement $R_t(h^{t-1})$ to be allocated between capital k_t and labor n_t at cost w, and the period repayments τ_t to the financial intermediaries contingent on the entrepreneur's report on the revenue realization. Let the revenue function for a loan of size R be[3]

$$\begin{aligned} F_z(R) = \ & \max_{k,n} f_z(k, n) \\ \text{s.t.} \ & k + wn \leq R \end{aligned} \tag{4}$$

and write current revenue as a function of resources R_t as $\nu_t F_z(R_t(h^{t-1}))$.

Conditional on surviving, the project is either liquidated, in which case the entrepreneur receives $Q_t(h^{t-1})$, and the financial intermediary receives $S - Q(h^{t-1})$, where

[2] The optimal contract with private information I consider is based on Clementi and Hopenhayn [2006]. The results established in Clementi and Hopenhayn [2006] are stated without proof.

[3] It will be clear later that any other allocations of R id not optimal as it would decrease the value of the firm forever.

S is the project's salvage value; or the project remains in operation, in which case the entrepreneur receives additional working resources $R_t(h^{t-1})$, observes the revenue realization, and makes repayment $\tau(h^{t-1}, \hat{\nu}_t)$ to the financial intermediary, conditional on the ex-post report $\hat{\nu}_t$. After every history h^{t-1}, the pair of contract and report strategy $(\kappa, \underline{\hat{\nu}})$ implies an expected discounted cash flow $V_{z,t}(\kappa, \underline{\hat{\nu}}, h^{t-1})$ and $B_{z,t}(\kappa, \underline{\hat{\nu}}, h^{t-1})$ for the entrepreneur and the financial intermediary respectively. A feasible and incentive compatible contract is optimal if it maximizes $B_{z,t}(V_t)$ for every possible V_t. Clementi and Hopenhayn [2006] refer to V and B_z as equity and debt respectively. It follows that an optimal contract maximizes the value of the firm $W_z(V) = B_z(V) + V$. The value of the firm is generally not independent of the combination of debt and equity, and of repayment, and as in other models with financial frictions, the Modigliani-Miller theorem does not hold.

Let $\tau_t = \tau_t(h^{t-1}, \hat{\nu}_t = 1)$ since repayments are 0 when the entrepreneur reports a bad shock. Clementi and Hopenhayn [2006] show that the contract can be expressed recursively by using V as a state variable, and V' as the continuation value awarded to an entrepreneur conditional on his report to the intermediary. A feasible contract κ is optimal if it maximizes the value of joint surplus $W_z(V) = V + B_z(V)$ subject to a participation constraint

$$V = p(F_z(R) - \tau) + \beta \left(pV^H + (1-p)V^L \right) \,, \tag{5}$$

where V^L and V^H are the next period levels of equity awarded to the entrepreneur conditional on the report of a low and high revenue shock respectively. Incentive compatibility requires that truthful reporting is optimal in every period. Since the entrepreneur has no incentive to report a good shock when receiving a bad shock, the relevant incentive compatibility constraint is

$$F_z(R) - \tau + \beta V^H \geq F_z(R) + \beta V^L \Rightarrow \tau \leq \beta(V^H - V^L). \tag{6}$$

Limited liability requires the period repayment to never exceed the reported income, so that

$$\tau \leq F_z(R). \tag{7}$$

Let $\mathbf{s}_t$ denote the vector of aggregate state variables. Conditional on surviving, the value of firm is given by

$$\begin{aligned} \widehat{W}_z(V, \mathbf{s}) = \max_{\tau, R, V^H, V^L} \quad & pF_z(R) - (1 + r(\mathbf{s}))R + \beta \mathbb{E} W_z(V', \mathbf{s}') \\ \text{s.t.} \quad & (5),\ (6),\ \text{and}\ (7) \\ & V^H, V^L \geq 0 \end{aligned} \tag{8}$$

Clementi and Hopenhayn [2006] show that for low values of equity it is optimal to put a lottery on the liquidation decision such that

$$\begin{aligned} W_z(V, \mathbf{s}) = \max_{\alpha \in [0,1], Q, V_r} \quad & \alpha S + (1 - \alpha)\widehat{W}_z(V_r, \mathbf{s}) \\ \text{s.t.} \quad & \alpha Q + (1 - \alpha) V_r = V \\ & Q, V_r \geq 0 \end{aligned} \tag{9}$$

where $\alpha(\mathbf{s})$ is the probability of liquidation, and $V_r(\mathbf{s})$ is the continuation value utility when the firm is not liquidated. Thus, whenever V falls below $V_r(\mathbf{s})$, the financial intermediary offers the entrepreneur a lottery where the project is either liquidated with probability $\alpha(\mathbf{s}) = (V_r - V)/V_r$ in which case the entrepreneur receives Q from the intermediaries, or kept in operation with probability $1 - \alpha(\mathbf{s})$ and is awarded $V_r(\mathbf{s})$. It follows that $V_r(\mathbf{s}) = \sup_V \{W_z'(V, \mathbf{s}) - S - (W_z(V, \mathbf{s}) - S)/V\}$ and $Q = 0$.

Conditional on the aggregate state $\mathbf{s}$, there is a natural upper bound on equity $\widetilde{V}_z(\mathbf{s}) = pF_z(\widetilde{R}_z(\mathbf{s}))/(1 - \beta)$ corresponding to the unconstrained size of the firm. It is optimal to set the repayment to $\tau = F_z(R)$ whenever $V^H \leq \widetilde{V}_z(\mathbf{s})$ as it allows for the fastest accumulation of equity toward the unconstrained level. However, repayments in Clementi and Hopenhayn [2006] are indeterminate when $V^H > \widetilde{V}_z(\mathbf{s})$. For the purpose of this paper, it is convenient to assume the optimization problem takes place on the convex set $[0, \widetilde{V}_z(\mathbf{s})]$ so that $V^H = \widetilde{V}_z(\mathbf{s})$ for V such that $(V + (1-p)F_z(R))/\beta > \widetilde{V}_z(\mathbf{s})$ and pins down the unique optimal repayment for all V. Constraints (5) and (6) imply

$$\tau = \begin{cases} F_z(R) & \text{if } V^H < \widetilde{V}_z(\mathbf{s}) \\ \beta(\widetilde{V}_z(\mathbf{s}) - V^L) & \text{if } V^H = \widetilde{V}_z(\mathbf{s}) \end{cases} \tag{10}$$

That is, conditional on a good shock, repayments increase with the firm's equity until $V^H = \widetilde{V}_z(\mathbf{s})$, and then start declining until they eventually reach 0 when the firm is unconstrained.

Conditional on surviving, the level $\widetilde{V}_z(\mathbf{s})$ can be reached with a sufficiently long, but finite, sequence of good shocks at which point, conditional on $\mathbf{s}$, the firm's equity no longer changes, and the borrowing constraints ceases. The value of the firm then is

$$W_z(\widetilde{V}, \mathbf{s}) = \underbrace{\frac{pF_z(\widetilde{R}_z(\mathbf{s}))}{1-\beta}}_{\widetilde{V}(\mathbf{s})} - \underbrace{\frac{(1+r(\mathbf{s}))\widetilde{R}_z}{1-\beta}}_{B_z(\widetilde{V}(\mathbf{s}))}, \tag{11}$$

where

$$\widetilde{R}_z(\mathbf{s}) = \underset{R}{\operatorname{argmax}}\{pF_z(R) - (1+r(\mathbf{s}))R\} \,. \tag{12}$$

Thus, a firm becomes unconstrained when an entrepreneur has repaid his long term debt I_0, and has accumulated enough capital via the repayments to finance the firm operation at full scale in every period and under all contingencies. Figure 1 summarizes the sequence of events within one period in an economy with private information.

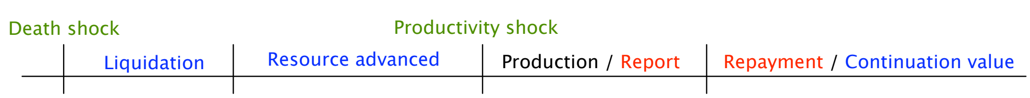

Figure 1: Timing within a period (private information)

4.2 Limited enforcement

Consider now an economy in which the sequence of revenue shocks $(\nu_t)_{t\geq 0}$ is public knowledge, but there is limited enforcement and the intermediaries cannot prevent entrepreneurs from defaulting on his debt obligation. Assume that, after R_t is advanced and after ν_t is realized, the entrepreneur can divert $\nu_t F_z(R_t)$ for his own consumption, and search for a new project in the next period. In case the entrepreneur chooses not to default, he makes repayment $\tau(\nu^{t-1}, \nu_t)$ to the financial intermediary. This optimal dynamic contract is a special case of Albuquerque and Hopenhayn [2004], and is also related to Marcet and Marimon [1992] of which Cooley et al. [2004] is the general equilibrium extension. As before, a contract is a quadruple $\kappa = \{\ell_t(\nu^{t-1}), Q_t(\nu^{t-1}), R_t(\nu^{t-1}), \tau(\nu^{t-1}, \nu_t)\}$, and Albuquerque and Hopenhayn [2004] show it can be written recursively by using V as a state variable. The participation constraint is the same as in the case of private information, and the relevant no-default constraint is

$$\mathcal{O}(\mathbf{s}) \leq F_z(R) - \tau + \beta V^H \tag{13}$$

where $\mathcal{O}(\mathbf{s})$ is the value of the outside opportunity, and is endogenous in general equilibrium.

Assumptions about $\mathcal{O}(\mathbf{s})$ are critical and, as I discuss below, play an key role in amplifying the response of an aggregate shock to productivity. If a defaulting entrepreneur is able to find a new project and a financial intermediary to fund it at no cost, the value of the outside value is equal to the value of the diverted in the current period $\nu_t F_z(R_t)$ plus the expected present discounted value of starting a new project next period $\bar{V}(\mathbf{s})$. This is the case where $\mathcal{O}(\mathbf{s}) = F_z(R) + \beta\bar{V}(\mathbf{s})$, and default is 'cheap.' This assumption may capture economies where contracts are not well enforced. A more developed legal system may be modelled by introducing a fixed cost $\mathcal{K}$ that reduces the value of the outside option such that $\mathcal{O}(\mathbf{s}) = F_z(R) - \mathcal{K} + \beta\bar{V}(\mathbf{s})$. An important limiting case is $\mathcal{K} = \beta\bar{V}(\mathbf{s})$ so that a defaulting entrepreneur is excluded from the market, and may represent OECD countries.[4]

In what follow, I consider the case where $\mathcal{O}(\mathbf{s}) = F_z(R) - \beta\bar{V}(\mathbf{s})$ as a benchmark. The no-default constraint becomes

$$\tau \leq \beta(V^H - \bar{V}(\mathbf{s})) \tag{14}$$

As for the model with private information, the value of firm conditional on surviving is given by

$$\begin{aligned} \widehat{W}_z(V, \mathbf{s}) = \max_{\tau, R, V^H, V^L} \quad & pF_z(R) - (1 + r(\mathbf{s}))R + \beta\mathbb{E}W_z(V', \mathbf{s}') \\ \text{s.t.} \quad & (5), (14), \text{ and } (7) \\ & V^H, V^L \geq 0 \end{aligned} \tag{15}$$

and the highest value of the firm is found by allowing for a lottery on the liquidation decision as in Problem 9. A notable feature of this model is for a non-zero salvage value S, young firms face a positive, and possibly large, probability of liquidation, while firms in Cooley et al. [2004] are not subject to endogenous liquidation. Once a firm reaches the unconstrained level of equity $\widetilde{V}_z = pF_z(\widetilde{R}_z(\mathbf{s}))/(1 - \beta)$, the entrepreneur is able to self-finance the operation of the firm at full scale conditional on the aggregate state $\mathbf{s}$. As before, it is optimal in this contract to set repayment to $\nu F_z(R)$ as long as $V < \widetilde{V}_z(\mathbf{s})$ as it allows for the fastest accumulation of equity toward the unconstrained level. Figure 2 summarizes the sequence of events within one period in an economy with limited enforcement.

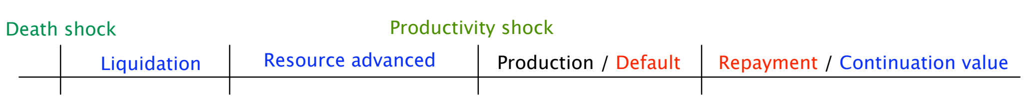

Figure 2: Timing within a period (limited enforcement)

[4]See Albuquerque and Hopenhayn [2004] for a more general formulation of the contract with limited enforcement.

4.3 Financing constraints and firm dynamics

Private information and limited enforcement have different implications for the evolution of firm's equity and resource advancements, and I refer the reader to Albuquerque and Hopenhayn [2004] and Clementi and Hopenhayn [2006] for a complete treatment. In this section, I concentrate on the key differences in the growth characteristics of firms implied by the two optimal contracts.

First, the size of a constrained firm decreases after being hit by a low revenue shock in the private information case, while firm size never decreases in the limited enforcement case. This implies that firms in the limited enforcement model that are outside of the liquidation region only exit when receiving a death shock, while all constrained firms in the moral hazard model may receive a finite sequence of low revenue shocks long enough to force them to exit. More formally, conditional on surviving $V_z^L < V < V_z^H$ in the economy with private information, and $V = V_z^L < V_z^H$ in the economy with limited enforcement for all $V < \widetilde{V}_z(\mathbf{s})$. Figure 3 summarizes the implication of the optimal contracts on firm dynamics.

Second, and as explained in the previous section, firms in an economy with private information begin to pay dividends to the entrepreneur before reaching their unconstrained size – V such that $V^H(V) = \widetilde{V}(\sim)$ – and the dividend payment from this point increases with the size of the firm. In contrast, firms in an economy with limited enforcement only make dividend payments once they reach the unconstrained level.

Third, depending on the value of the parameters and the nature of the outside opportunity in the limited enforcement model, the entrepreneur may receive the unconstrained level of resources $\widetilde{R}_z$ with equity less than $\widetilde{V}_z(\sim)$. To see this, note that setting $\tau_z = \nu F_z(R)$ implies the period resource advancement is given by

$$F_z(R) \leq \beta(V_z^H - \bar{V}(\mathbf{s})). \tag{16}$$

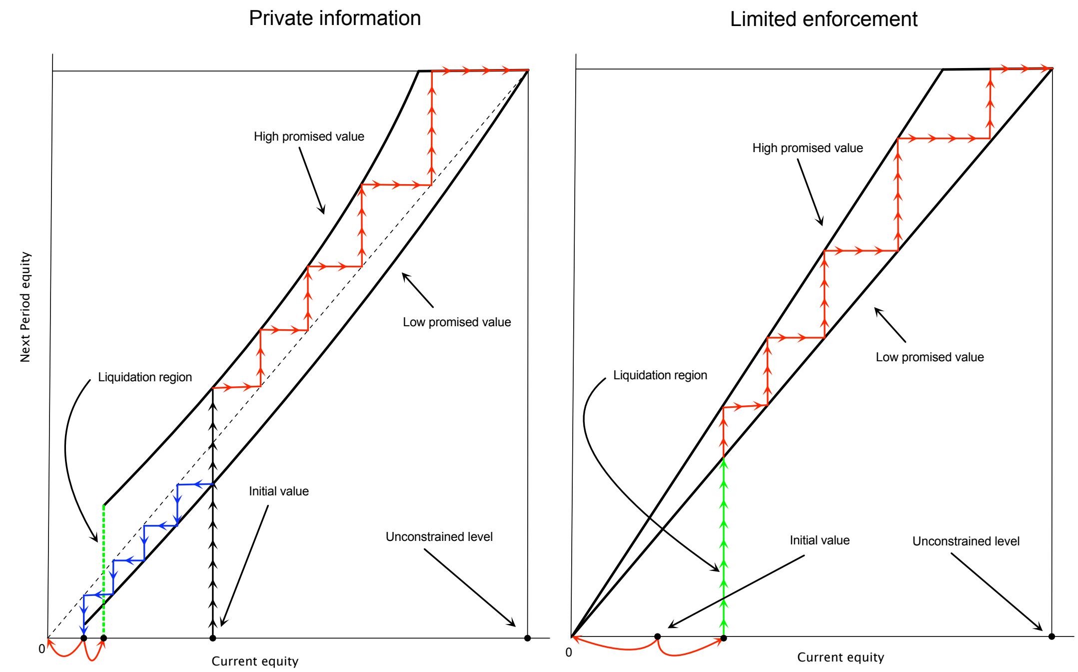

Figure 3: Firm dynamics

The constraint stops binding whenever $F_z(\widetilde{R}) < \beta(V_z^H - \bar{V}(\mathbf{s}))$. For instance, when $\bar{V}(\mathbf{s}) = 0$ the maximum value of the right hand side is $\beta\widetilde{V}_z = aF_z(\widetilde{R})$ where $a = p(1-\gamma_e)/(r+\gamma_e) >> 1$ for reasonable values of the parameters. This is in contrast to the model with private information for which Clementi and Hopenhayn [2006] show that $R < \widetilde{R}_z(\sim)$ as long as $V < \widetilde{V}_z(\sim)$, which implies firms do not operate at full capacity before reaching the (conditional) maximum level of equity $\widetilde{V}_z(\sim)$. This implies there is a greater variability in firm size in an economy with private information.

Last, constrained firms in an economy with private information always face a non-zero hazard rate of liquidation, while firms in an economy with limited enforcement face a zero hazard rate of liquidation once they are outside the randomization region $[0, V_r)$. I discuss the quantitative implication of the above differences in Section 7 below.

5 Recursive formulation

Let $\mathbf{M}$ the state space for firm state so that $V \in \mathbf{M}$. Let $\mathcal{M}(V)$ the Borel $\sigma-$algebra generated by $\mathbf{M}$, and μ the measure defined over $\mathcal{M}$. The worker's problem can be written recursively as

Problem 1 (Workers)

$$\begin{aligned} U(d, \mathbf{s}) = \max_{d', c, h} \quad & u(c_t, h_t) + (1-\gamma_w)\hat{\beta}\mathbb{E}U(d', \mathbf{s}') \\ s.t. \quad & c + (1-\gamma_w)d' = (1 + r(\mathbf{s}))d + w(\mathbf{s})h \\ & d' \geq 0 \\ & \\ & \mathbf{s}' \sim \mathcal{H}(\mathbf{s}) \end{aligned} \tag{P0}$$

where $\mathbf{s} = (N, \mu)$ is the set of aggregate state variables, and $\mathcal{H}(\mathbf{s})$ is the law of motion for $\mathbf{s}$. The last constraint $d' \geq 0$ implies workers are not allowed to borrow, which simplifies the analysis and is never binding given my parametrization.

The optimal contract with private information can be written recursively as

Problem 2 (Firms: private information)

$$\widehat{W}_z(V,\mathbf{s}) = \max_{\tau,R,V^H,V^L} pF_z(R) - (1+r(\mathbf{s}))R + \beta \mathbb{E} W_z(V',\mathbf{s}')$$

$$\begin{aligned} s.t.\quad & V \geq \beta[pV^H + (1-p)V^L] \\ & \tau \leq \beta(V^H - V^L) \\ & \tau \leq F_z(R) \\ & V^H, V^L \geq 0 \end{aligned} \tag{P1}$$

$$W_z(V,\mathbf{s}) = \max_{\alpha\in[0,1],Q,V_c} \alpha S + (1-\alpha)\widehat{W}_z(V_c,\mathbf{s})$$

$$\begin{aligned} s.t.\quad & \alpha Q + (1-\alpha)V_c = V \\ & Q, V_c \geq 0 \end{aligned}$$

$$\mathbf{s}' \sim \mathcal{H}(\mathbf{s})$$

Similarly the optimal contract with limited enforcement can be written recursively as follows

Problem 3 (Firms: limited enforcement)

$$
\begin{aligned}
\widehat{W}_z(V,\mathbf{s}) = \quad & \max_{\tau,R,V^H,V^L} pF_z(R) - (1+r(\mathbf{s}))R + \beta \mathbb{E} W_z(V',\mathbf{s}') \\
s.t. \quad & V \geq \beta[pV^H + (1-p)V^L] \\
& \tau \leq \beta(V^H - \bar{V}(\mathbf{s})) \\
& \tau \leq F_z(R) \\
& V^H, V^L \geq 0
\end{aligned}
\tag{P2}
$$

$$
\begin{aligned}
W_z(V,\mathbf{s}) = \quad & \max_{\alpha\in[0,1],Q,V_c} \alpha S + (1-\alpha)\widehat{W}_z(V_c,\mathbf{s}) \\
s.t. \quad & \alpha Q + (1-\alpha)V_c = V \\
& Q, V_c \geq 0 \\
& \mathbf{s}' \sim \mathcal{H}(\mathbf{s})
\end{aligned}
$$

The value of searching for a new project in Problem 3 is given by

$$
\bar{V}(\mathbf{s}) = q(\mathbf{s})V_0(z_H;\mathbf{s}) + (1-q(\mathbf{s}))V_0(z_L;\mathbf{s}) \tag{17}
$$

where $q(\mathbf{s})$ is the probability for a new-born entrepreneur to start a high-productivity project, and $V_{0,z}(\mathbf{s})$ is the initial state of a contract for a project of productivity z.

Proposition 1 *There exists a stationary distribution of firms in the two economies without aggregate shocks.*

6 Stationary equilibrium

The assumption on the workers implies a stationary distribution of workers since each period $(1-\gamma_w)$ survive and γ_w are replaced by new ones. Setting the mass of workers to $m=1$, let $d_j(\mathbf{s})$ and $h_j(\mathbf{s})$ the deposits and hours worked of a j-years old worker. In every period t, γ_w new workers are born with zero wealth and therefore contribute $\gamma_w d_0 = 0$ to aggregate deposits, and j-years old workers contribute $\gamma_w(1-\gamma_w)^j d_j(\mathbf{s})$ to aggregate deposits. It follows the aggregate deposits each period is

$$D(\mathbf{s}) = \gamma_w \sum_{j=1}^{\infty} (1-\gamma_w)^j d_j(\mathbf{s}), \tag{18}$$

and the aggregate labor supply in each period is

$$H(\mathbf{s}) = \gamma_w \sum_{j=0}^{\infty} (1-\gamma_w)^j h_j(\mathbf{s}). \tag{19}$$

It follows aggregate consumption by workers is defined by

$$C_w(\mathbf{s}) = \gamma_w \sum_{j=0}^{\infty} (1-\gamma_w)^j c_j(\mathbf{s}). \tag{20}$$

Perfect competition in the financial sector implies the intermediaries break even so that the initial value of a contract in Problem 2 and Problem 3 is such that

$$V_{0,z}(\mathbf{s}) = \sup_V \{W_z(V,\mathbf{s}) - V = I_0\} \tag{21}$$

where $B_z(V,\mathbf{s}) = W_z(V,\mathbf{s}) - V$ is the value to the intermediary, and I_0 is the set up cost. The labor market clears when labor supply equal labor demand so that

$$H(\mathbf{s}) = \int_{\mathcal{M}} n(\mathbf{s}) d\mu. \tag{22}$$

Solving for the equilibrium of Problem 3 requires solving for the outside value opportunity which is the fixed point of the following mapping

$$\bar{V}^{j+1}(\mathbf{s}) = T(\bar{V}^j)(\mathbf{s}). \tag{23}$$

Aggregate consumption by entrepreneurs $C_e(\mathbf{s})$ is equal to total revenues to entrepreneurs minus repayments made to the financial intermediary.

$$C_e(\mathbf{s}) = p \int F(R(\mathbf{s}))d\mu - p \int \tau(\mathbf{s})d\mu. \tag{24}$$

The capital market clears when the total resources advanced to firms is equal to workers' deposits in the previous period, so that

$$D(\mathbf{s}) = \int R(\mathbf{s}')d\mu + \Gamma(\mathbf{s})I_0, \tag{25}$$

where $\Gamma(\mathbf{s}) = \Gamma_b(\mathbf{s}) + \gamma_e$, where $\Gamma_b(\mathbf{s})$ is the measure of liquidated (b for bankrupt) firms. Financial intermediaries must service their debt to workers at the interest rate, so that

$$D(\mathbf{s})(1 + r(\mathbf{s}')) = p \int \tau(\mathbf{s}')d\mu + \Gamma_b(\mathbf{s}')S. \tag{26}$$

Walras's Law implies that if the above two equilibrium conditions hold, the goods market also clears, so that

$$Y(\mathbf{s}) = C_w(\mathbf{s}) + C_e(\mathbf{s}) + K(\mathbf{s}) \ , \tag{27}$$

where $K(\mathbf{s}) = \int k(\mathbf{s})d\mu + \Gamma(\mathbf{s})I_0 - \Gamma_b(\mathbf{s})S$, which is total capital expenditure. To show this, note that

$$\begin{aligned} Y(\mathbf{s}) &= p \int F(R(\mathbf{s}))d\mu = C_e(\mathbf{s}) + p \int \tau(\mathbf{s})d\mu \\ &= C_e(\mathbf{s}) + D(\mathbf{s})(1 + r(\mathbf{s})) - \Gamma_b(\mathbf{s})S \ . \end{aligned} \tag{28}$$

Capital market clearing (25) and using the definition of the use of resources yields

$$\begin{aligned} Y(\mathbf{s}) &= C_e(\mathbf{s}) + D(\mathbf{s})(1+r(\mathbf{s})) - \Gamma_b(\mathbf{s})S \\ &= C_e(\mathbf{s}) + \int R(\mathbf{s})d\mu + \Gamma(\mathbf{s})I_0 - \Gamma_b(\mathbf{s})S + D(\mathbf{s})r(\mathbf{s}) \\ &= C_e(\mathbf{s}) + \int n(\mathbf{s})d\mu + K(\mathbf{s}) + D(\mathbf{s})r(\mathbf{s}) \ . \end{aligned} \tag{29}$$

Finally, labour market clearing (19) and the aggregate budget constraint for workers, which is given by $C_w + D = wH + D(1+r)$ yields

$$\begin{aligned} Y(\mathbf{s}) &= C_e(\mathbf{s}) + \int n(\mathbf{s})d\mu + K(\mathbf{s}) + D(\mathbf{s})r(\mathbf{s}) \\ &= C_e(\mathbf{s}) + w(\mathbf{s})H(\mathbf{s}) + K(\mathbf{s}) + D(\mathbf{s})r(\mathbf{s}) \\ &= C_e(\mathbf{s}) + C_w(\mathbf{s}) + K(\mathbf{s}) \ . \end{aligned} \tag{30}$$

The definition of a recursive competitive equilibrium in an economy with private information is as follow

Definition 1 (Recursive competitive equilibrium: private information) *A recursive competitive equilibrium consists of*

1. *labor supply* $h(\mathbf{s})$, *consumption function* $c(\mathbf{s})$ *for worker*
2. *contract policy: promised value* $V^H(z;\mathbf{s})$, $V^L(z;\mathbf{s})$, *resources advancement* $R(z;\mathbf{s})$, *probability of liquidation* $\alpha(\mathbf{s})$, *repayment* $\tau(z;\mathbf{s})$
3. *initial contract state* $V_0(z;\mathbf{s})$
4. *wage* $w(\mathbf{s})$ *and interest rate* $r(\mathbf{s})$
5. *a law of motion for the state variables* $\mathbf{s}' \sim H(\mathbf{s})$

such that

1. *the labor and consumption function are optimal*

2. *the contract policy solve the contract*

3. *the initial state* $V_0(z; \mathbf{s})$ *is such that the intermediary breaks even*

4. *the wage and interest rate clear the labor and capital market*

5. $H(\mathbf{s})$ *is consistent with individual decision and the shocks*

and similarly for an economy with limited contract enforcement

Definition 2 (Recursive competitive equilibrium: limited enforcement) *A recursive competitive equilibrium consists of*

1. *labor supply* $h(\mathbf{s})$, *consumption function* $c(\mathbf{s})$ *for worker*

2. *contract policy: promised value* $V^H(z; \mathbf{s})$, *resources advancement* $R(z; \mathbf{s})$, *probability of liquidation* $\alpha(\mathbf{s})$, *repayment* $\tau(z; \mathbf{s})$

3. *initial contract state* $V_0(z; \mathbf{s})$

4. *value of searching for new project* $\bar{V}(\mathbf{s})$

5. *wage* $w(\mathbf{s})$ *and interest rate* $r(\mathbf{s})$

6. *a law of motion for the state variables* $\mathbf{s}' \sim H(\mathbf{s})$

7. *a mapping* T

such that

1. *the labor and consumption function are optimal*

2. *the contract policy solve the contract*

3. *the initial state* $V_0(z; \mathbf{s})$ *is such that the intermediary breaks even*

4. *the wage and interest rate clear the labor and capital market*

5. *$H(\mathbf{s})$ is consistent with individual decision and the shocks*

6. *the value of searching $\bar{V}(\mathbf{s})$ is the fixed point of T*

While I do not prove existence and uniqueness of the stochastic steady state, existence of the stationary equilibrium with no aggregate shocks follows if for each V_0 there exists a stationary distribution of firms that is continuous in the pair of prices (r, w).

Proposition 2 *There exists a stationary equilibrium for the two economies without aggregate shocks.*

7 Contrasting economies with private information and limited enforcement

In this section, I parametrize the instantaneous utility function for workers, the production function for firms, the stochastic process for the idiosyncratic and aggregate shocks, calibrate the parameters of the frictionless economy to be broadly consistent with the U.S. economy, and use the calibration to compare the two economy with friction. I would like to stress that this calibration is only useful to compare the two economies with frictions and is not used to assess to what extent each friction accounts for aggregate fluctuations observed in the data. I then contrast numerically the models prediction for two economies with and without aggregate shocks. Throughout the section, firm size refers to the equity level.

7.1 Parametrization

Let the instantaneous utility function for the workers be[5]

$$u(c, l) = \ln(c) + \eta \ln(l) \ , \tag{31}$$

and let the production function be

$$f_z(k, n) = z\zeta(k^{\xi} l^{1-\xi})^{\theta} \ . \tag{32}$$

Given the above parametrization, it remains to assign values on the workers' inter-temporal discount rate β, the elasticity of leisure η, the probability of death for workers γ_w, the probability of high revenue shocks p, the production parameters z, ζ, ν, and θ, the exogenous exit rate for firms γ_e, the salvage value S, and the setup investment I_0.

A period in this model is 1 year. The death rate for a worker is set to $\gamma_w = 0.02$ which implies an average working life of 50 years. The exogenous probability of firm exit is set to $\gamma_e = 0.05$ which is consistent with studies on industry dynamics such as Evans [1987]. I set the workers' discount rate to $\beta = 0.962$, which roughly implies $r = 0.04$ at steady state in the two economies.

I follow Cooley et al. [2004] and set $\theta = 0.85$. Labor income share of unconstrained firm is around 0.6, and hour worked are typically one third. Hence, $\theta(1 - \xi) = 0.6$ implies $\xi = 0.294$. The mass m coincides with the average employment size of firm, which is also determined by η. I set $\eta = 2$ and $m = 1$ so that hours worked are roughly around one-third of period time endowment. However, the presence of constrained firms at steady state implies the labor income share is less than 0.6 in economies with frictions. The parameter ζ is set to normalize resources used by the unconstrained firms

[5] Smith and Wang [2006] use this functional form and show it implies a close-form solutions for the aggregate supply of labor and aggregate deposits given the workers' demographic assumption. This simplifies the numerical implementation and reduces the computational burden. See Smith and Wang [2006] for more details.

Table 1: Parameter values

Worker's discount rate	$\hat{\beta}$	0.962
Elasticity of leisure	η	2
	ξ	0.294
	θ	0.85
Production technology	ζ	5.516
	z_h	1.2
	z_l	0.8
Salvage value	S	0.5
Setup investment	I_0	0.2
Workers' death shock	γ_w	0.02
Entrepreneurs' death shock	γ_e	0.05
Firm-level revenue shock	p	0.5
Aggregate technological shock	$q \in$	$\{0.5, 1\}$

to 1 when there is only one type of firm – i.e., no aggregate shocks and $z_H = z_L = 1$. I assume the difference in productivity between high and low productivity firms is 1/3 so that $z \in \{z_L, z_H\}$ where $z_L = 0.8$ and $z_L = 1.2$ for the economies with aggregate shocks.

The salvage value, setup investment, and probability of high revenue shock affect the value of the contract and the distribution of firms in the economy and could be set to study different experiments. A higher S can be interpreted as a higher collateral to back the debt obligation, and yield a higher level of resources in every period. A higher I_0 implies a lower starting value V_0 which implies a higher probability of liquidation across unconstrained firms as it take a longer sequence of good shocks to reach the unconstrained level. In what follows, I set $S = 0.5$, $I_0 = 0.2$, and $p = 0.5$. This implies that firms revenue has a volatility 25

I assume there are two aggregate states: a bad state (or recession) for which 50% of the new-born entrepreneurs have access to a high productivity project, and a good state (or boom) for which all new born entrepreneurs have access to a high productivity project. That is, I assume $(N_t)_{t\geq 0}$ is such that the state space for $(q_t)_{t\geq 0}$ is restricted to $\{0.5, 1\}$. Section 7.3 considers different specifications for the evolution of q_t. Table 1 summarizes the parametrization.

7.2 Stationary steady state

After solving the models, I simulate the two economies for a large number of periods and with large number of firms. I then build a panel with observations on approximately $60,000$ firms from which I estimate the results discussed in this section. Table 2 reports the values of the key endogenous variables for the two economies with frictions and the frictionless economy for reference.

The wage rate is about 11% higher in an economy with limited enforcement. This is due to two factors. First, from Section 4 firms in an economy with limited enforcement start receiving the efficient level of resources $\widetilde{R}$ after reaching a size that is less than the unconstrained level. Second, there is a greater fraction of unconstrained firms. This implies a relatively larger number of firms demand the unconstrained level of labor, pushing the wage up in order to clear the market.

Aggregate output is 16% lower in an economy with moral hazard problems, and 9% lower in an economy with enforcement problems than aggregate output in the frictionless economy. This means that lending contracts that are optimal under limited enforcement are able to achieve an allocation of resources that is significantly closer to first best as compared to an optimal lending contract in an economy with private information – i.e., private information puts more burden on the economy than limited enforcement does.

Entrepreneurial consumption is about the same in the two economy subject to financial frictions. Recall that entrepreneurs in an economy with limited enforcement only start receiving dividends (and consuming) once the firm has reached the unconstrained level, while entrepreneurs in an economy with private information start receiving divi-

Table 2: Key endogenous variables

	Private information	**Limited enforcement**	**No friction**
Wage rate	1.394	1.568	2.159
Interest rate	0.040	0.040	0.040
Initial size (Equity)	1.379	1.369	13.584
Liquidation region	0.137	1.509	–
Average firm size (Equity)	5.646	6.422	13.584
Unconstrained firm size (Equity)	13.585	13.585	13.584
Share of unconstrained firms	0.243	0.257	1.000
Aggregate consumption (workers)	0.482	0.542	0.747
Aggregate consumption (entrepreneurs)	0.299	0.303	0.116
Aggregate output	0.979	1.064	1.167
Aggregate hours worked	0.327	0.327	0.327

dends sooner. This implies that while aggregate consumption from entrepreneurs is the same in the two economies, consumption inequality among entrepreneurs is higher in an economy with limited enforcement. Workers' consumption is about 11% higher in an economy with limited enforcement, which follows form a higher wage paid for the same number of hour worked. Using aggregate consumption as a measure of welfare, the above suggests that welfare is about 7.5% higher in an economy limited enforcement than in an economy with repeated moral hazard problems. Furthermore, aggregate consumption of workers is lower and aggregate consumption of entrepreneur is higher in economies with frictions than in an frictionless economy. This follows form the fact the extra resources are required in economies with frictions to prevent entrepreneurs from miss-reporting or defaulting.

[FIGURE 4 ABOUT HERE]

Initial firm size is about the same in the two economies. However, new firms in an economy with private information start outside the liquidation region, while firms in an economy with limited enforcement face a 10% chance of being liquidated during their first year of operation.[6] Furthermore, the liquidation regions reflect the fact that a long but finite sequence of low revenue shocks may drive a constrained firm to exit in an economy with private information, but not with limited enforcement for firms that are two years old or older. Given the parametrization, the unconstrained level of equity is roughly the same in the three economies . However, a greater fraction of firms are unconstrained in an economy with limited enforcement, and the average firm size is lower in an economy with private information.

[FIGURE 5 AND FIGURE 6 ABOUT HERE]

Figure 5 plots the mean growth rates of firms conditional on age. Young firms grow on average faster than older firms, which is consistent with the empirical regularities; and the growth rate is broadly the same in the two economies. Figure 6 plots the mean standard deviation of firms' growth rates of firms conditional on age, and shows the volatility of growth, however, is about four times higher in an economy with private information. Inspection of the age-size density of firms in Figure 9 and Figure 10 reveals younger firms tend to be smaller in an economy with private information, which contribute to the excess volatility as the size of constrained firms is subject to wider fluctuation in this economy.

[FIGURE 10 AND FIGURE 9 ABOUT HERE]

Figure 9 and Figure 10 plot the age-size density of firms in the economies without aggregate shock. Most firms are unconstrained after 15 years of operation in an economy with private information, while it takes about five more years with limited enforcement. Clementi and Hopenhayn [2006] shows the spread of continuation values in the contract with the moral hazard is asymmetric. That is, $V^H - V > V - V^L$ for large firms, and vice-versa for small firms. This implies firms that receive a sequence of high revenue shocks in an economy with private information are disproportionally

[6] Recall that the probability of liquidation is $\alpha = V_0/V_r$. Furthermore, note that Figure 4 is a histogram with 4-year break points and only one-year old firms face positive probability of liquidation.

pushed toward the unconstrained region, while firms receiving a sequence of low revenue shocks are disproportionally pushed toward the liquidation region. This results in a thinner population of medium-sized firms. Indeed, Figure 9 shows the density is close to 0 for size between 5 and 10 irrespective of age.

Firms in the limited enforcement economy are not penalized by continuation values that are lower than their current equity, which explain the larger population of medium-sized firms. Furthermore, recall firms in an economy with limited enforcement start receiving the efficient level of resources $\widetilde{R}$ after reaching a size that is less than the unconstrained level. This is the main channel through which volatility of growth is determined. That is, the outside opportunity affects how binding the resource advancement constraint is, and in turn affects the volatility of growth.

[FIGURE 8 AND FIGURE 7 ABOUT HERE]

Figure 7 and Figure 8 plot the mean job creation rand destruction rate conditional on age respectively. Figure 8 show that the mean job creation rate is initially higher in an economy with private information as firm size cannot decrease, but drops to zero as firms reach the unconstrained level. Figure 7 shows job destruction is above the exogenous 5% probability of receiving a permanent productivity shock, even for 50 years and older firms. Job destruction is exactly 5% for firm one year and older firm in an economy with limited enforcement. Recall firms in this economy only face a positive probability of liquidation during their first year of operation.

7.3 Stochastic steady state

In this section, I study to what extent private information and limited enforcement are amplifier of business cycle fluctuations in economies subject to aggregate technological shocks. Recall that $(N_t)_{t\geq 0}$ is the sequence of random measure (number) of high-productivity projects available in an economy each period, Γ_t is the measure (number) of new entrepreneurs, and $q_t = \min\{1, N_t/\Gamma_t\}$ is the probability that a new entrepreneur finds a high-productivity project in period t.

I assume there are two states of the world: one in which high-productivity projects are available in an unlimited supply; and one in which half of the new entrepreneurs are able to start high-productivity projects, and the other half start low-productivity projects. I assume that the arrival of technology follows a five year cycle on average so that $(q_t)_{t\geq 0}$ is a two-state time-homogenous Markov process with state space $\{0.5, 1\}$ that evolves according to the transition matrix

$$P_q = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}. \tag{33}$$

Since $\mathbf{s} = (N, \mathcal{M})$ is an infinite dimensional state variable and prices are determined implicitly by the market clearing conditions, I devise two algorithms based on the approximate aggregation method of Krusell and Smith [1998].[7] Each economy is populated by 10,000 firms every period, which yields a satisfactory estimate of the law of motion for the distribution of firms.

Figure 11 plots the series of log-aggregate output with mean removed, and Figure 12 and Figure 13 plot the cycles and trends of the HP-filtered log-aggregate output for a 100-year window. Given one period is calibrated to represent 1 year, I follow Ravn and Uhlig [2002] and use $\lambda = 6.5$ for the HP-Filter. Table 3 and Table 4 report the standard deviations for, and the correlation between the four series respectively.

[FIGURE 11, FIGURE 13, AND FIGURE 12 ABOUT HERE]

Table 3 shows that aggregate output in an economy with private information is about 35% less volatile than in the frictionless economy, while it is about four times more volatile in an economy with limited enforcement when defaulting entrepreneurs cannot be excluded from financial markets. When defaulting entrepreneur can be excluded, aggregate output is no more volatile than in the frictionless economy. Figure 12 shows

[7] See Appendix for details.

that the average length of a business cycle is about 5 years, which is consistent with the parametrization of the stochastic process for the arrival of new technology q_t. Table 3 shows that the volatility of the business cycle is about twice as high in an economy with limited enforcement than in the frictionless economy, and than an economy with private information is moderately less volatile. Furthermore, Figure 13 show that an economy in which defaulting entrepreneurs cannot be excluded from the financial market feature large low frequency cycles.

Table 3: Standard deviations of output growth rates ($\times 100$)

	Frictionless	**Private info.**	**Limited enfo.**	**Market exclusion**
Log-output	1.2	0.8	5.3	1.2
High frequency	0.4	0.3	0.9	0.5
Low frequency	1.0	0.7	4.9	1.0

The intuition behind the above results is as follow. In the frictionless economy, new entrepreneurs are able to set up and operate their firm at the unconstrained level from their first year of operation. The mass of the new cohort subsequently decreases geometrically at the exogenous exit rate. Since by assumption, each type of firm faces the same exogenous exit rate, the overall productivity of the economy fluctuates with the arrival of new technology at the entry/exit rate.

In an economy in which financial contracts cannot be enforced, the outside value option is – a fraction of – the expected value of diverting and consuming the current revenue, and starting a new project. Three effects are at play. First, when high-productivity projects are abundant, the outside value option increases which optimally tightens the financing constraints for small or young firms in order to prevent default.

This slows down the growth of small firms disproportionally more than of larger firms. Second, new low-productivity firms face a substantially higher hazard rate of exit than high-productivity firms during their first year of operation. High-productivity firms start inside but close to the boundary of the liquidation region, so that the probability of exiting during the first year is less than 10%. Low-productivity firms start relatively smaller than high-productivity firm, and hence face a greater probability of liquidation during their first year of operation which is about 50%. This implies the economy retains relatively more high productivity firms. Third, firm size does not decrease after receiving a low revenue shock.

These three effect combined yields a mass of high-productivity firms operating at full scale that is not immediately replaced when exiting a the exogenous rate. This creates substantial fluctuations in the distribution of firms and thus of aggregate output. Table 4 shows that these combined effects also yield fluctuations in aggregate output that are not correlated with the ones of the frictionless economy.

When defaulting entrepreneurs can be excluded from financial markets, the value of the outside option is only the value of the diverted resources in one period, and the financing constraint faced by small firms is less binding. New firms start larger and take fewer years on average to become financially unconstrained. To see this, recall that the no-default constraint is

$$\mathcal{O}(\mathbf{s}) \leq F_z(R) - \tau + \beta V^H \tag{34}$$

Market exclusion implies a defaulting entrepreneur is able to divert $F_z(R)$ for his own and immediate consumption and is not able to find a new project. That is, $\mathcal{O}(\mathbf{s}) = F_z(R)$ and adding the limited liability constraint yield:

$$F_z(R) \leq \beta V^H \tag{35}$$

Table 4: Correlations of output growth rates

	Private info.	**Limited enfo.**	**Market exclusion**
		Log-output	
Frictionless	0.79	0.05	-0.09
Private information		0.13	0.16
Limited enforcement			0.36
		High frequency	
Frictionless	0.75	0.04	-0.03
Private information		0.12	0.26
Limited enforcement			0.39
		Low frequency	
Frictionless	0.90	0.42	-0.12
Private information		0.48	0.08
Limited enforcement			0.19

so that resources advancement is tied up to the promised continuation value in case of high revenue shock, which implies a greater loan size for all firms and a faster growth toward the unconstrained level. This reduces the first effect and also decreases the spread of firms in the economy yielding smaller fluctuations of the equilibrium distribution of firms, and thus of aggregate output. This results is consistent with Cooley et al. [2004] who consider the same enforcement problem in a model in which firms are not subject to idiosyncratic revenue shocks. Furthermore, inspection of Figure 13 Table 4 shows that the variations in the distribution of firms in economies with enforcement problems imply fluctuations in output that are asynchronous with the one from frictionless economy, and that regardless wether or not defaulting entrepreneurs can be excluded from financial markets.[8]

In an economy with private information, high-productivity and low-productivity firms start at a size that is less than the unconstrained level, and low-productivity firms start relatively smaller than high-productivity firms. Firm growth characteristics are more dynamics than in an economy with limited contract enforcement because size decreases with bad revenue shocks and increases with high ones. However, the economy does not penalize low-productivity firm as in the case of limited enforcement, and firms are more evenly replaced when shrinking or exiting since half of the firm decrease shrink and half expand every period. This in turns does not lead to large fluctuations in the equilibrium distribution of firms, and in fact mitigates the effect of the aggregate shocks on aggregate output. Table 4 shows that contrary to the economies with limited enforcement, the economy with private information is very highly correlated with the frictionless economy.

8 Concluding comments

Two general equilibrium models with heterogenous firms, endogenous firm-level borrowing constraints, and aggregate shocks were proposed to study to what extent private information and limited enforcement amplify business cycles fluctuations. I used the contractual arrangements proposed by Albuquerque and Hopenhayn [2004] and Clementi and Hopenhayn [2006] to construct model economies in which firm dynamics are qualitatively consistent with recent empirical evidence on firms. The key result of this paper is frictions in financial markets due to private information moderately

[8]The precise determinant of the periodicity of output implied by the different types of financial friction is work in progress.

dampens the effect of the arrival of new technology in the economy, while an economy in which contracts cannot be enforced is substantially more volatile than a frictionless economy when the punishment for defaulting entrepreneur is low. When defaulting entrepreneur can be excluded from financial markets, an economy with limited contract enforcement is no more volatile than a frictionless economy. In the light of the recent findings by Beck et al. [2008] on the effect of institutional developments on firm financing, the results of this paper suggest that the extent to which financial contracts can be enforced in an economy is an important determinant of business fluctuations.

The framework introduced in this paper can be used to answer a wide range of questions. For example, the models offer a new mechanism through which one can study the effect of financial reforms on aggregate volatility and welfare under different types of financing frictions. Financial reforms such as an increase in banks' capital reserve requirement can be easily added to the models by forcing the intermediaries to operate at a fixed surplus. This will change the growth characteristics of all firms and may have substantial effect – positive or negative – on the amplification of business cycle fluctuations and welfare. I leave these questions to future research.

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A Stationary equilibrium

In the stationary steady state (one type of firm and no aggregate shocks), given interest rate r and wage w, perfect competition in the financial market implies financial intermediaries earn zero-profit on a new contract. This implies new entrepreneurs receive an initial equity $V_0 = \sup_V \{V | W(V) - V - I_0 = 0\}$, for which the lender earns just enough to break even. The initial equity V_0 is endogenous in the general equilibrium.

Consider the sequence $(X_t)_{t \geq 0}$ of equity levels from a single firm indefinitely replaced by a new one when it dies. It is clear $(X_t)_{t \geq 0}$ is a sequence of random variables, and its evolution depends on the properties of the contracts and on the sequence of shocks – productivity, death, and liquidation. In what follows, I show that for the two models $\mathbf{X} = (X_t)_{t \geq 0}$ is a time-homogeneous Markov chain such that

$$X_{t+1} = T_\omega(X_t, \epsilon_t), (\epsilon_t)_{t \geq 0} \sim \phi_\omega \in \mathscr{P}(Z), X_0 = V_0 \in S \tag{36}$$

where $T_\omega : S \times Z \to S$ is a collection of measurable functions indexed by $(r, w) = \omega \in \Omega$ the parameter space, $(\epsilon_t)_{t=1}^{\infty}$ is a sequence of independent random shocks with (joint) distribution ϕ_ω, and S and Z are the state space and the probability space respectively. The state space S is infinite for the model with moral hazard, and finite for the model with limited enforcement. The existence of an invariant distribution of firms follows if $\mathbf{X}$ has a unique and ergodic invariant distribution. Existence of stationary equilibrium follows if the stationary distribution is continuous in prices.

Proposition 3 (Stationary distribution of firms: private information) $\mathbf{X}$ *is a time-homogeneous Markov chain on a general state space and is globally stable.*

Equip the state space S with a boundedly compact, separable, metrizable topology $\mathscr{B}(S)$.[9] Let $(Z, \mathscr{Z})$ be the measure space for the shocks. Let A be any subset of $\mathscr{B}(S)$. It follows for any $x \in [V_r, \widetilde{V})$

$$P(x, A) = \begin{cases} (1-\gamma_e)(1-p)\alpha(V^L(x)) + \gamma_e & \text{if } A = \{V_0\} \text{ and } V^L(x) < V_r \\ (1-\gamma_e)(1-p)(1-\alpha(V^L(x)) & \text{if } A = \{V_r\} \\ (1-\gamma_e)(1-p) & \text{if } A = \{V^L(x)\} \text{ and } V^L(x) \geq V_r \\ (1-\gamma_e)p & \text{if } A = \{V^H(x)\} \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

And for $x = \{\widetilde{V}\}$

$$P(x, A) = \begin{cases} \gamma_e & \text{if } A = \{V_0\} \\ (1-\gamma_e) & \text{if } A = \{\widetilde{V}\} \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

[9]The notation used in the proof is consistent throughout the appendix but currently conflicts with the notation used in the main text.

For each $A \in \mathscr{B}(S)$, $P(\cdot, A)$ is a non-negative function on $\mathscr{B}(S)$, and for each $x \in S$, $P(x, \cdot)$ is a probability measure on $\mathscr{B}(S)$. Therefore, for any initial distribution ψ, the stochastic process $\mathbf{X}$ defined on S^∞ is a time-homogeneous Markov chain. Let $\mathbf{M}$ denote the corresponding Markov operator, and let $\mathscr{P}(S)$ denote the collection of firms distribution generated by $\mathbf{M}$ for a given initial distribution.[10]

Write the stochastic kernel P with the density representation p so that $P(x, dy) = p(x, y)dy$ for all $x \in S$. The Dobrushin coefficient $\alpha(p)$ of a stochastic kernel p is defined by[11]

$$\alpha(p) := \min\left\{\int p(x, y) \wedge p(x', y)dy : (x, x') \in S \times S\right\}$$

$(\mathscr{P}(S), \mathbf{M})$ is globally stable if $(\psi\mathbf{M}^t)_{t\geq 0} \to \psi^*\mathbf{M}$ where $\psi^* \in \mathscr{P}(S)$ is the unique fixed point of $(\mathscr{P}(S), \mathbf{M})$. This occurs if the Markov operator is a uniform contraction of modulus $1 - \alpha(p)$ on $\mathscr{P}(S)$ whenever $\alpha(p) > 0$. A firm dies with a fixed, exogenous and independent probability γ_e each period, and is instantaneously replaced by a new one of size V_0. Therefore,

$$P(x, \{V_0\}) \geq 0 \; \forall \, x \in S. \tag{39}$$

Equation (11.15) and Exercise (11.2.24) in Stachurski [2009] yield $\alpha(p) > \gamma_e$. By Stachurski [2009, Th. 11.2.21], this implies

$$||\psi\mathbf{M} - \psi'\mathbf{M}||_{TV} \leq (1 - \gamma_e)||\psi - \psi'|| \tag{40}$$

for every pair ψ, ψ' in $\mathscr{P}(Z)$, and where $_{TV}$ indicate the total variation norm. ∎

Proposition 4 (Existence of stationary equilibrium: private information) *The unique and ergodic invariant distribution of* $\mathbf{X}$ *is continuous in prices.*

The result follows if the conditions of LeVan and Stachurski [2007, Proposition 2] are satisfied.[12] Consider again the state space S equipped with a boundedly compact,

[10] Note that Stokey, Lucas, and Prescott [1989, Theorem 12.12] fails to apply in this case because the stochastic kernel is not monotone on $[V_r, a]$ where a is such that $V^L(a) = V_r$. For instance, consider the function $f(x) = x$. From the above,

$$\begin{aligned}&\int_{[V_r,a]} P(x, dy)y\\ &= (1-\gamma_e)\{(1-p)[\alpha(V^L(x))V_0 + (1-\alpha(V^L(x)))V_r] + pV^H(x)\} + \gamma_e V_0\\ &= (1-\gamma_e)[(1-p)V_0(1 - V^L(x)/V_r) + (1-p)V^L(x) + pV^H(x)] + \gamma_e V_0\\ &= (1-\gamma_e)[(1-p)V_0\alpha(V^L(x)) + x/\beta] + \gamma_e V_0\end{aligned}$$

Which is generally not increasing. The (non-economic) intuition is when x falls below a, the probability of liquidation α becomes non-zero in case of a bad shock. Liquidation sends x to V_0 which can be larger than V_r and $V^H(x)$ so that the lower the x, the higher the expected value of next period x.

[11] This is the first abuse of notation. It is customary to denote the Dobrushin coefficient by α.

[12] LeVan and Stachurski [2007, Proposition 2] is an application of LeVan and Stachurski [2007, Theorem 1] of which Stokey et al. [1989, Theorem 12.13] is a special case.

separable, metrizable topology, $(Z, \mathscr{Z})$ a measure space, and $\mathscr{P}(Z)$ the collection of probabilities on $(Z, \mathscr{Z})$. From the above, the model can be written as

$$X_{t+1} = T_\omega(X_t, \epsilon_t), \text{ where } \epsilon_t \sim \psi_\omega \in \mathscr{P}(Z), \quad \forall t \in \mathbb{N} \tag{41}$$

where $(\epsilon_t)_{t=1}^\infty = (\{D_{1,t}, D_{2,t}, D_{2,t}\})_{t=1}^\infty$ independently distributed, and $T_\omega : S \times Z \to S$ is measurable. Given prices ω, the stochastic kernel can be written as $P_\omega(x, B) := \psi_\omega\{z \in Z : T_\omega(x, z) \in B\}$, and given the parameter space Ω, the family of stochastic kernel is $\{P_\omega : \omega \in \Omega\}$. Let N be any subspace of Ω, and define $\Lambda(\omega) := \{\mu \in \mathscr{P}(S) : \mu = \mu P_\omega\}$ the collection of invariant distribution corresponding indexed by ω.

Proposition 5 (LeVan and Stachurski [2007]) *If $\Lambda(\omega) = \{\mu_\omega\}$, then $\omega \mapsto \mu_\omega$ is continuous on N if the following four conditions are satisfied:*[13]

1. *the map $N \ni \omega \mapsto T_\omega(x, z) \in S$ is continuous for each pair $(x, z) \in S \times Z$*

2. *for each compact $C \subset S$, there is a $K < \infty$ with*

$$\int d(T_\omega(x, z), T_\omega(x', z))\psi_\omega(dz) \leq K d(x, x'), \forall x, x' \in C, \forall \omega \in N \tag{42}$$

3. *$\exists$ a Lyapunov function $V \in \mathscr{L}(S)$, $\lambda \in (0, 1)$, and $L \in [0, \infty)$ s.t. $\forall \omega \in N$*

$$P_\omega V(x) := \int V(T_\omega(x, z))\psi_\omega(dz) \leq \lambda V(x) + L \quad \forall x \in S \tag{43}$$

4. *$\omega \mapsto \psi_\omega$ is continuous in total variation norm.*

That $\Lambda(\omega)$ is nonempty, and $\Lambda(\omega) = \{\mu_\omega\}$ for each $\omega \in N$ follows from Proposition 3. Condition (1) requires the optimal value function $W(x)$ to be continuous in ω which follows from Berge's theorem (see for example Ausubel and Deneckere [1993]). Condition (4) holds as the shocks are independent and the probability of liquidation is $\alpha = (x - V_r)/V_r$, which is continuous in ω since V_r is continuous in ω from condition (1). To show condition (2) holds, define again $a \ni V^L(a) = V_r$, and $b \ni V^H(b) \geq \widetilde{V}$. Pick any $x, x' \in C \subset [V_r, a)$. Without loss of generality assume $x > x'$ so that $\alpha(V^L(x')) > \alpha(V^L(x))$. By noting that $\alpha(V^L(x')) - \alpha(V^L(x)) = (V^L(x') - V^L(x))/V_r$, and $x = \beta[pV^H(x) + (1-p)V^L(x)]$ at optimum, it follows easily that

$$\begin{aligned} & \int d(T_\omega(x, z), T_\omega(x', z))\psi_\omega(dz) \\ < \; & |pV^H(x) + (1-p)V^L(x) - pV^H(x') - (1-p)V^L(x')| \\ = \; & \frac{1}{\beta}|x - x'| = \frac{1}{\beta}d(x, x'). \end{aligned}$$

[13] This is another abuse of notation. It is customary to use $V(x)$ to denote a Lyapunov function.

The above inequality also holds for any $x, x' \in C \subset [a, b)$. Last, recall that $V^L(x) = (x - pF(R(x)))/\beta$. It follows for any $x, x' \in C \subset [b, \widetilde{V})$

$$\begin{aligned} & \int d(T_\omega(x,z), T_\omega(x',z))\psi_\omega(dz) \\ < \ & (1-p)|V^L(x) - V^L(x')| \\ = \ & \frac{(1-p)}{\beta}|x - x'| = \frac{(1-p)}{\beta} d(x, x'). \end{aligned}$$

It remains to show condition (3) holds. Pick $V(x) = x$ which is a Lyapunov function (since S is boundedly compact). Then,

$$P_\omega x = \begin{cases} (1-\gamma_e)\{(1-p)[\alpha(V^L(x))V_0 + (1-\alpha(V^L(x)))V_r] + pV^H(x)\}\mathbf{1}_{[V_r,a)}(x) \\ + [pV^H(x) + (1-p)V^L(x)]\mathbf{1}_{[a,b)}(x) \\ + \widetilde{V}\mathbf{1}_{(x=\widetilde{V})}(x)\} + \gamma_e V_0 \end{cases} \tag{44}$$

Pick any $x \in [V_r, a)$ so that $V_r \leq x \leq V^H(x)$. Then,

$$\begin{aligned} P_\omega x \ & < \ (1-p)(1-\alpha(V^L(x)))V_r + pV^H(x) + [(1-p)\alpha(V^L(x)) + \gamma_e]V_0 \\ & \leq \ (1-p)V_r + pV^H(x) + V_0 \\ & \leq \ \lambda x + \sup_{\omega \in N} V_0 = \lambda V(x) + L \end{aligned}$$

The same inequality holds for any $x \in [a, b)$, since $V^L(x) < x < V^H(x)$. Last, when $x = \widetilde{V}$,

$$\begin{aligned} P_\omega x \ & = \ (1-\gamma_e)\widetilde{V} + \gamma_e V_0 \\ & \leq \ \lambda x + \sup_{\omega \in N} V_0 \end{aligned}$$

∎

Proposition 6 (Stationary distribution of firms: limited enforcement) **X** *is a time-homogeneous Markov chain on a finite state space and is globally stable.*

In an economy with limited enforcement, a firm's equity cannot decrease. This implies $\widetilde{V}$ can be reached in a minimum of T steps corresponding to the T level of equity $S = \{V_0, V_1, \ldots, V_{T-2}, \widetilde{V}\}$. It follows **X** is a finite state Markov-chain with Markov operator M and stochastic kernel P such that given a distribution $\psi \in \mathscr{P}(S)$ $M : \psi \mapsto \psi M$ where $\psi M(y) = \sum_{x \in S} P(x, y)\psi(x)$ for each $y \in \psi$. $(\mathscr{P}(S), M)$ is globally stable if $(\psi M^t)_{t \geq 0} \to \psi^* M$ where $\psi^* \in \mathscr{P}(S)$ is the unique fixed point of $(\mathscr{P}(S), M)$. The

Dobrushin coefficient in this case is given by

$$\alpha(P) := \min \left\{ \sum_{y \in S} P(x,y) \wedge P(x',y) : (x,x') \in S \times S \right\}$$

Given firms face the exogenous death probability γ_e, and are replaced by a new firm of size V_0, the same arguments as in Proposition 3 yield

$$||\psi \mathbf{M} - \psi' \mathbf{M}||_{TV} \leq (1 - \gamma_e)||\psi - \psi'|| \tag{45}$$

for every pair ψ, ψ' in $\mathscr{P}(Z)$ $\blacksquare$

Proposition 7 (Existence of stationary equilibrium: limited enforcement)) *The unique and ergodic invariant distribution of* $\mathbf{X}$ *is continuous in prices.*

The proof is another application of LeVan and Stachurski [2007, Proposition 2], and is omitted $\blacksquare$

B Numerical implementation

All computations were done in **R** version 2.11.1 on a Mac Pro 8-core 3.2GHz.[14]

B.1 Stationary steady state

B.1.1 Private information

(1) guess a value for r,w from a interval

(2) solve the contract (value iteration on a grid) and get the initial value from $V_0 = \sup_V \{W(V) - V = I_0\}$

(3) estimate the invariant distribution using the Look-Ahead Estimator as described in Stachurski and Martin [2008]

  (1') check labor market clearing: if does not clear, guess a new w using bisection and go to (2'), if clears go to (1")

  (2') solve the contract (value iteration on a grid), and get the initial values. Note the program is now the 'old' distribution as an approximation since it should not be too different around a particular value of w. This increases the efficiency of the algorithm substantially.

  (3') go to (1')

  (1") check capital market clearing: if does not clear, guess a new r using bisection go to (2"), if clears go to (3)

  (2") solve the contract (value iteration on a grid) and get the initial values from $V_0 = \sup_V \{W(V) - V = I_0\}$

  (3") go to (1")

(4) stop when the maximum absolute difference between supply and demand in two market falls below the desired tolerance level.

B.1.2 Limited enforcement

(1) guess a value for r,w from a interval

(2) guess a value for $\bar{V}$ from a interval

(3) solve the contract (value iteration on a grid) and get the initial value from $V_0 = \sup_V \{W(V) - V = I_0\}$. If $V_0 \neq \bar{V}$ go to (2), if equal go to (4)

[14] See http://www.r-project.org.

(4) solve for the invariant distribution. Stachurski [2009, Excercise 4.3.7] shows that for a finite Markov chain with N state and a stochastic kernel P, the invariant distribution ψ is such that $\mathbf{1}_{1\times N} = \psi(\mathbf{I}_N - P + \mathbf{1}_{N\times N})$, where $\mathbf{1}_{N\times M}$ is a $N \times M$ matrix of ones, and $\mathbf{I}_N$ is an identity matrix of size N.

   (1') check labor market clearing: if does not clear, guess a new w using bisection and go to (2'), if clears go to (1")

   (2') guess a new value for $\bar{V}$ using bisection

   (3') solve the contract (value iteration on a grid) and get the initial values from $V_0 = \sup_V\{W(V) - V = I_0\}$. If $V_0 \neq \bar{V}$ go to (2'), if equal go to (4')

   (4') solve for the invariant distribution as in (4)

   (5') go to (1')

   (1") check capital market clearing: if does not clear, guess a new r using bisection and go to (2"), if clears go to (1')

   (2") guess a new value for $\bar{V}$ using bisection

   (3") solve the contract (value iteration on a grid) and get the initial values from $V_0 = \sup_V\{W(V) - V = I_0\}$. If $V_0 \neq \bar{V}$ go to (2"), if equal go to (4")

   (4") solve for the invariant distribution as in (4)

   (5") go to (1")

(5) stop when the maximum absolute difference between supply and demand in two market falls below the desired tolerance level.

B.2 Stochastic steady state

The algorithms below are based on the approximate aggregation method of Krusell and Smith [1998]. The algorithm for the economy with private information finds the exact sequence of prices that clear the markets, and the set of consistent decision rules for a given realization of shocks. This can be interpreted as a non-parametric estimation of the law of motion for prices. While it works well, it is very computationally intensive, especially since the results are conditional on a sequence of shocks.

The algorithm for the economy with limited enforcement is similar to Rios-Rull [1997], this assumes agents are bounded rational and forecast prices, and upgrade the state space with the sequence of prices that would clear the markets each period. With the forecasts, I can solve the contracts on a grid and interpolate the initial values and outside opportunity value which would otherwise require finding the fixed point of the mapping T in every period.

The first algorithm yields the rational expectation equilibrium in a small number of (long) iterations on the sequence of prices. The success of second algorithm depends crucially on the quality of the forecasting functions, and the thinness of the grid (which in turn affects the computational burden).[15]

For the two algorithms, I find that a minimum of 10,000 trajectories over 1150 is necessary to have a satisfactory convergence. I let the economy burn in for 1050 periods. I assess the accuracy of the law of motions and the forecasting functions by calculating the maximum point-wise error from the market-clearing generated by the law of motion without updating. This type of stopping criterion is similar to the one proposed by Den Haan [2010]. The algorithm stop when the discrepancy fall below 0.01%.

B.2.1 Private information: stochastic steady state

(1) Generate the sequences of idiosyncratic shocks, and the sequence of aggregate shock (always use the same seed)

(2) Set an initial firm distribution (it is useful to start from an arbitrary sequence of prices, let the algorithm do one iteration, and then use the last period distribution of firms and the updated sequence of prices as starting values).

(3) Guess a sequence for prices (from (2)), and in each period

  (1') solve the contract for each type of firm using current and next period prices

  (2') update the distribution of firms given the realization of the shocks (idiosyncratic and aggregate)

  (3') estimate the marginal distribution of firms (Stachurski and Martin [2008]), and use the market clearing conditions to find the prices that would have cleared the markets. This requires solving the system of non-linear equations from the market clearing conditions.

  (4') update the sequence of prices, and go to (2)

(4) iterate until the maximum error from the markets clearing falls below the desired threshold (allowing for a burn-in period)

B.2.2 Limited enforcement: stochastic steady state

(1) Guess two sets of forecasting functions for prices (one for each price)

(2) Guess a sequence for prices (see above), and in each period

[15] I am currently working on an algorithm based on Den Haan and Rendahl [2010] that should be less computationally intensive.

(3) Create two two-dimensional grids (one for each state) for r and w, solve the contract for each point on the two grid using the forecasting function to determine next period prices, solve for the outside value option $\bar{V}$, and the initial values $V_{0,z}$

(4) Generate the sequences of idiosyncratic shocks, and the sequence of aggregate shocks (always use the same seed)

(5) Set an initial firm distribution (see above)

  (1') find the decision rules using the current and next period prices from the sequence of prices, and by interpolating (use bi-linear interpolation) the outside opportunity value from the value on the grids

  (2') update the distribution of firms given the shock realization

  (3') estimate the marginal distribution of firms (Stachurski and Martin [2008]), and use the market clearing conditions to find the prices that would have cleared the markets. This requires solving the system of non-linear equations from the market clearing conditions.

  (4') update the sequence of prices and the forecasting functions by regressing prices on their lag values, and go to (3)

(5) iterate until the maximum error from the markets clearing falls below the desired threshold (allowing for a burn-in period)

C Figures

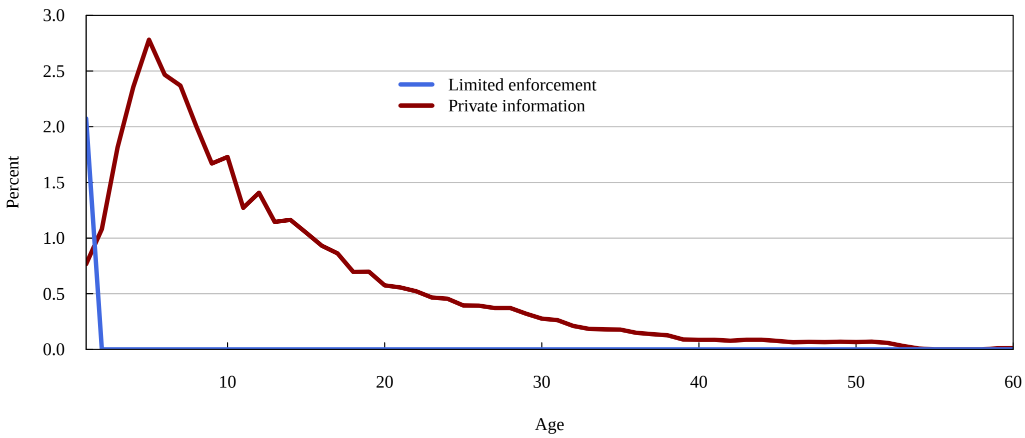

Figure 4: Conditional probability of liquidation (Age)

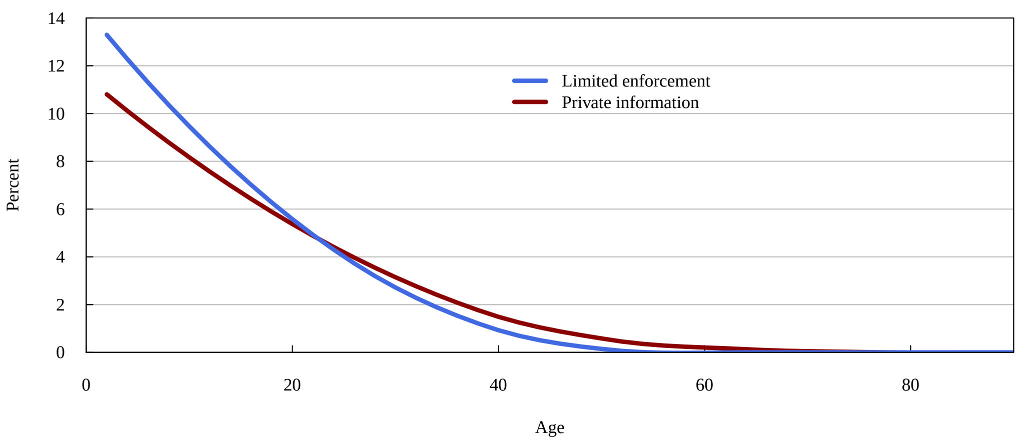

Figure 5: Conditional growth rates (Age)

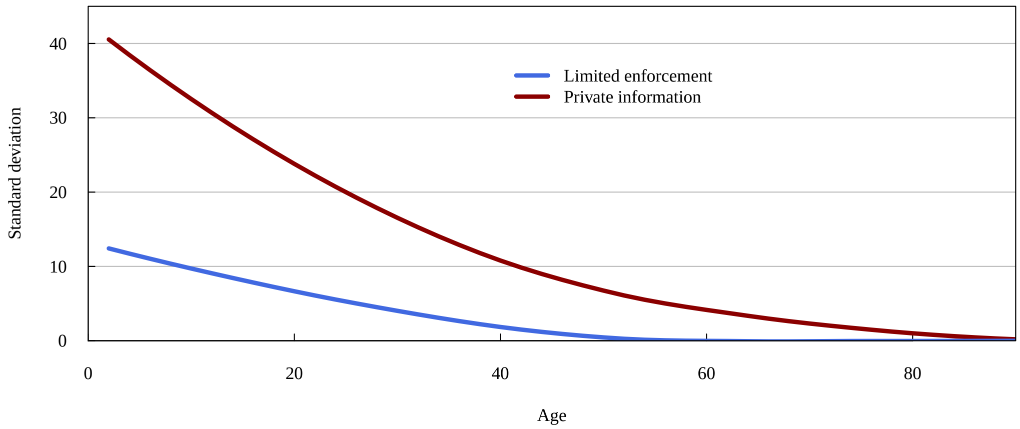

Figure 6: Conditional volatility of growth (Age)

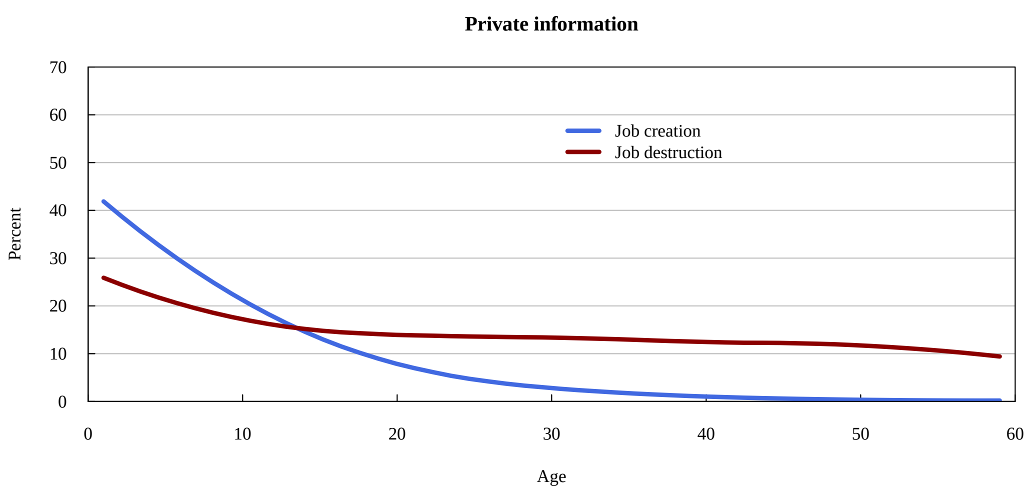

Figure 7: Job creation and job destruction (private information)

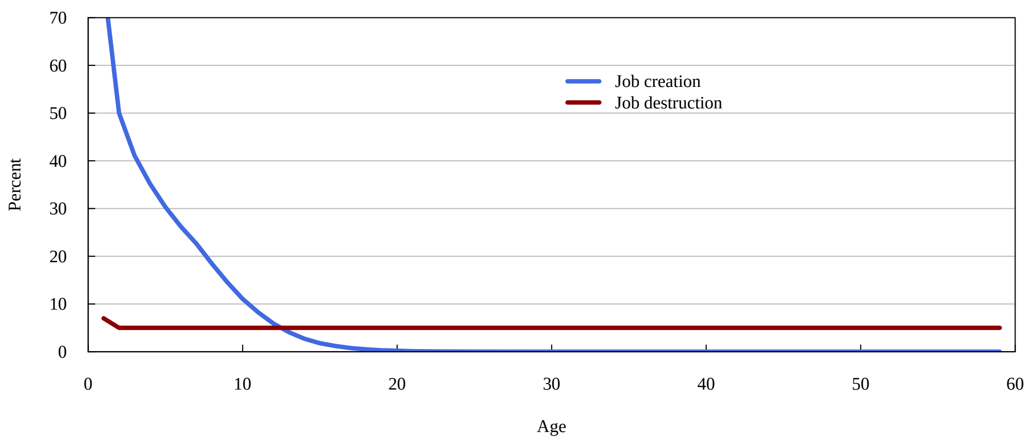

Figure 8: Job creation and job destruction (limited enforcement)

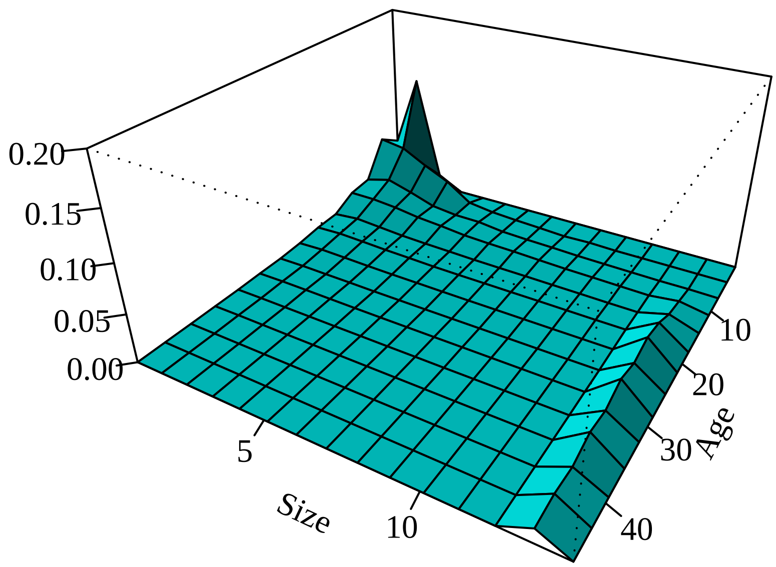

Figure 9: Age-Size density (private information)

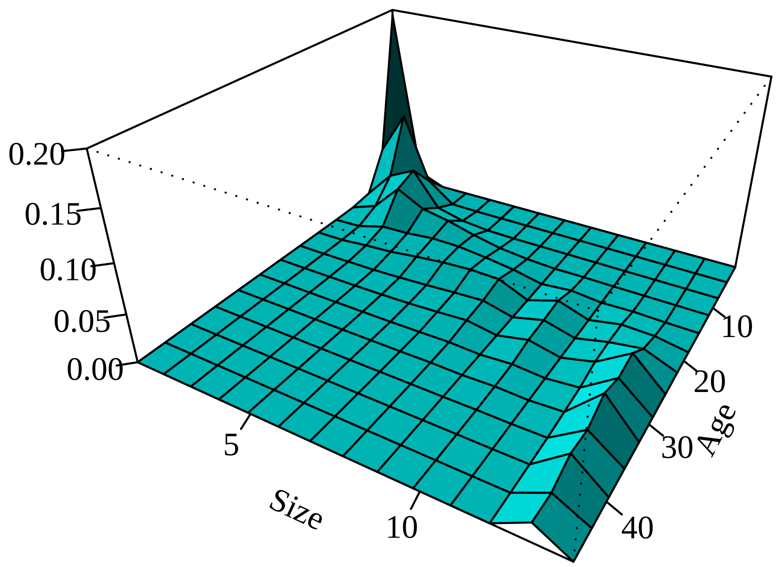

Figure 10: Age-Size density (limited enforcement)

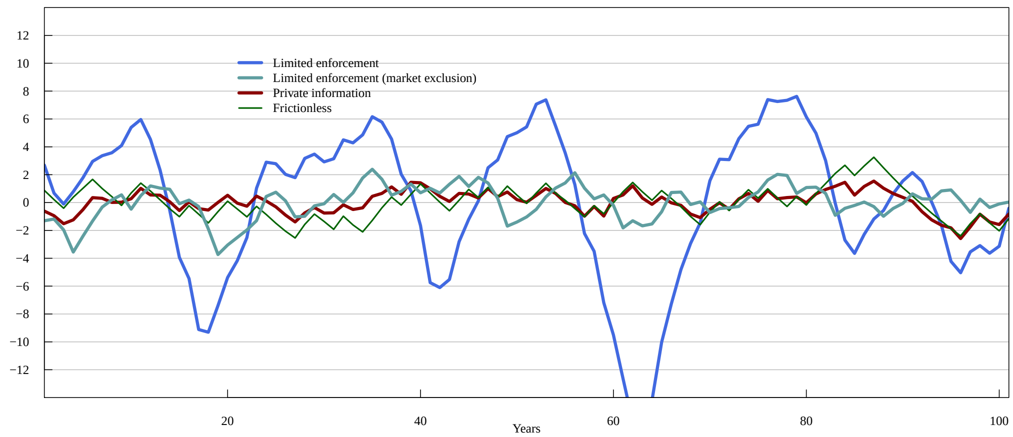

Figure 11: log-aggregate output (mean removed)

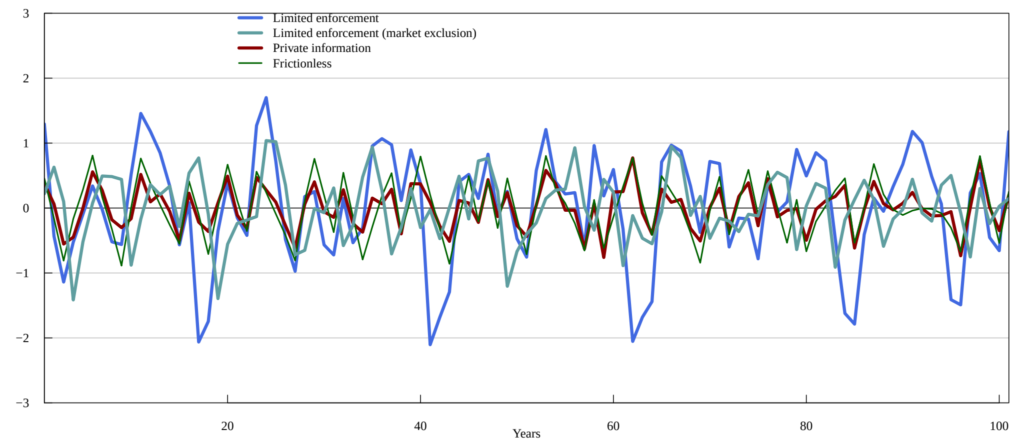

Figure 12: log-aggregate output (HP Filter with $\lambda = 6.5$, cycles)

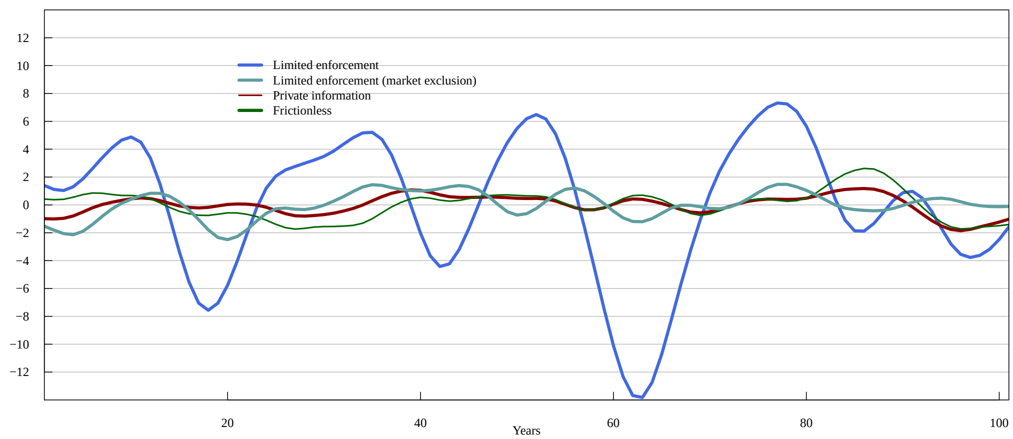

Figure 13: log-aggregate output (HP Filter with $\lambda = 6.5$, trends)