

Welfare Improvement from Restricting the Liquidity of Nominal Bonds*

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Abstract

In this paper I examine whether a society can improve welfare by imposing a legal restriction to forbid the use of nominal bonds as a means of payments for goods. To do so, I integrate a microfounded model of money with the framework of limited participation. While the asset market is Walrasian, the goods market is decentralized and the legal restriction is imposed only in a fraction of the trades. I show that the legal restriction can improve the society's welfare. In contrast to the literature, this essential role of the legal restriction persists even in the steady state and it does not rely on households' ability to trade unmatured bonds for money after observing the taste (or endowment) shocks.

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1. Introduction

The co-existence of money and nominal bonds is a classical issue in monetary economics (e.g. Hicks 1939). When nominal bonds are default-free, they have all the essential features that money has. However, nominal bonds do not act as a medium of exchange to the same extent as money does and that they are discounted. A typical resolution of this anomaly is to assume that there are legal restrictions that reduce the liquidity of bonds by limiting the use of bonds as a medium of exchange (e.g., Wallace, 1983). Traditional models of money have employed this assumption implicitly, in the form of cash in advance or money in the utility function. However, the legal restrictions in those models are not “essential” to the society; rather, they reduce agents’ utility. An essential role of the legal restrictions has been elusive in the literature.

There are two reasons why it is important to find an essential role of the legal restrictions. First, without such a role, it is difficult to justify why the legal restrictions should be imposed or to explain why the return dominance of money by bonds has survived so long. Second, there is a large literature that uses traditional models to analyze monetary policy such as open market operations (e.g., Lucas, 1990). Because the legal restrictions implicit in those models reduce welfare, the effects of policy are suboptimal outcomes. Eliminating the legal restrictions could generate the efficient allocation in those models, but then open market operations would not have any effect on the real activity. If we can justify the legal restrictions on the efficiency ground, then we can ensure that the real effects of open market operations are optimal responses of the economy to monetary policy.

In this paper, I construct a model to show that the legal restrictions can improve welfare in the steady state. The model is a hybrid of the deterministic version of Lucas’s (1990) model of limited participation and the search model of money in Shi (1997). The bonds market is centralized (Walrasian), but the goods market is decentralized with random matching. The goods market induces a demand for a medium of exchange. A legal restriction forbids the use of bonds as a means of payments in the trades of one group of goods labelled “red” goods. Such a trade is called a *restricted trade*, in which the buyer can use only money to buy goods. In the trades of another group of goods labelled “green” goods, the buyer can use both money and bonds to buy goods. Such a trade is called an *unrestricted trade*. Whether the goods are red or green is determined by a shock that is realized after agents are matched. For the same level of consumption, the relative utility of consuming red goods to consuming green goods is θ .

When θ is either small or large, the legal restriction does not affect the real allocation. When θ has intermediate values, the legal restriction reduces the quantity of (red) goods traded in restricted matches and increases the quantity of (green) goods traded in unrestricted matches. If θ is less than 1, these changes in the quantities shift consumption from the goods of low marginal utility to the goods of high marginal utility, and hence reduce the gap between the marginal utilities of consuming the two groups of goods. As a result, the legal restriction improves expected utility in this case. The legal restriction also increases the nominal interest rate.

Kocherlakota (2001) seems the first to show that a legal restriction that prevents bonds from being used as a means of payments for goods can improve welfare. Although there are some similarities between my paper and his, there are also substantial differences. First, the results differ. In Kocherlakota's model, the welfare-improving role of the legal restriction lasts for only one period, because the households differ in tastes in only one period. Making the difference in tastes permanent does not make the essential role of the legal restriction persist; to the contrary, it will eliminate the essential role. Introducing idiosyncratic shocks to tastes every period may enable the essential role to persist, but it will also make the model intractable by introducing non-degenerate distributions of asset holdings. My model provides a tractable framework in which the legal restriction improves welfare even in the steady state.

Second, the mechanisms differ in the two models. In Kocherlakota's model, the legal restriction improves welfare by smoothing the marginal utility of consumption *between* households who have received different taste shocks. For the essential role to occur, it is critical that the taste shocks are realized before the households go to the goods market, so that the households with high tastes are able to trade bonds for money with the households with low tastes. In contrast, the legal restriction improves welfare in my model by smoothing the marginal utility of goods obtained in different matches for the same household. This essential role of illiquid bonds remains the same regardless of whether households can trade assets before going to the goods market. In fact, I will assume that the taste shocks are realized after agents have gone to the market, so that all households go to the goods market with the same portfolio of assets.¹

The first paper that examines the competition between money and nominal bonds in a search

¹There are two recent papers that generate an essential role for illiquid bonds, Boel and Camera (2005) and Sun (2005). In Boel and Camera, the households differ in the rate of time preference as well as the marginal utility of consumption. In Sun, the tastes of the households alternate between odd and even periods. As Kocherlakota's model, these two models build the essential role of the legal restrictions on the assumption that the households can trade bonds and money after the taste shocks are realized but before going to the goods market.

model of money is Aiyagari et al. (1996). They assume that money and bonds are indivisible and that agents cannot always redeem matured bonds even when they want to. These assumptions restrict the ability of bonds to compete against money and make the results difficult to interpret. In a precursor to this paper (Shi, 2005), I eliminate these assumptions and show that even an arbitrarily small legal restriction can prevent matured bonds from circulating as a medium of exchange. However, the legal restriction does not improve welfare there.

2. A Search Economy with the Legal Restriction

2.1. Households, Matches and Assets

Consider a discrete-time economy with many types of households. The number of households in each type is large and normalized to one. All households have the same discount factor $\beta \in (0, 1)$. The households of each type are specialized in producing a good which they do not consume and which they exchange for a consumption good in the market. Goods are perishable between periods. Each good has two colors, “green” and “red”, which are indexed by $i \in \{G, R\}$. The utility of consuming a non-consumption good is zero. The utility of consuming a consumption good of color i is $\theta^i u(c^i)$, where $\theta^G = 1$ and $\theta^R = \theta (> 0)$. Assume that $u' > 0$, $u'' < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$.² The cost (disutility) of production is $\psi(\cdot)$, which has the following properties: $\psi(0) = 0$, $\psi'(0) = 0$, $\psi'(q) > 0$ and $\psi''(q) \leq 0$ for all $q > 0$.

There are two assets in the economy, fiat money and nominal bonds issued by the government. These objects can be stored without cost. Both are intrinsically worthless; i.e., they do not yield direct utility or facilitate production. Bonds are default-free and so risks are not the reason for bonds to be discounted in this model. As described later, the separation between the bonds market and the goods market makes it impossible for households to take newly issued bonds to the goods market in the same period. To allow bonds to perform the role of a medium of exchange before the maturity, the length of the maturity must be two periods or longer. The simplest bonds that can perform this role are two-period, pure discount bonds. These bonds will be the object analyzed in this paper. The bonds one period after the issuing date are called *unmatured bonds*. At the maturity, each bond can be redeemed for one unit of money.

Also, I assume that the government does not redeem bonds that have passed the maturity. This assumption is innocuous in the described economy because it is optimal for a household to

²All the analytical results hold for a more general specification $u(c^i, \theta^i)$, where the derivative of u with respect to c has the additional feature that it increases in θ .

redeem all matured bonds immediately at maturity rather than keeping them for the future as a medium of exchange (see Shi, 2005).

The government sells bonds in a centralized competitive market. As in Lucas (1990), households cannot bring goods into the asset market, and so the asset market involves only the exchange between different assets. Let zM be the amount of new bonds sold in the bonds market, where $z \in (0, \infty)$ is a constant and M is the aggregate stock of money. Denote the market price of newly issued (two-period) bonds as S . The two-period (net) nominal interest rate is $r = 1/S - 1$. Also, households can bring unmatured bonds into the asset market to exchange for money or newly issued bonds. Let S^u be the nominal price of unmatured bonds in the bonds market. Then, $(1/S^u - 1)$ is the one-period interest rate implied by bonds of the old vintage.

The exchange in the goods market is decentralized and described by random bilateral matches. There is no chance of a double coincidence of wants in a meeting to support barter or public record-keeping of transactions to support credit trades. As a result, every trade entails a medium of exchange, which can be money or unmatured bonds or both. Call an agent in the goods market a buyer if he holds money or bonds, and a seller if he holds no asset. A seller produces and sells goods, and a buyer purchases consumption goods. Let σ be the (fixed) fraction of sellers and $(1 - \sigma)$ the fraction of buyers in the market. Of interest are the meetings of a single coincidence of wants, i.e., meetings in which one and only one agent can produce the partner's consumption goods. Call such a meeting a *trade match*. A buyer encounters a trade match in a period with probability $\alpha\sigma$ and a seller with probability $\alpha(1 - \sigma)$, where $\alpha > 0$ is a constant.

Random matching can generate non-degenerate distributions of money holdings and consumption. To maintain tractability, I assume that each household consists of a large number of members (normalized to one) who share consumption each period and regard the household's utility function as the common objective. This assumption makes the distribution of money holdings across households degenerate so that I can select an arbitrary household as the representative household.³ In each household, there are a measure σ of sellers and $(1 - \sigma)$ of buyers. A buyer brings a combination of money and unmatured bonds into a trade.

There are two possible types of trade matches, red and green. With probability p all members of the household (buyers and sellers) will be in red matches and with probability $1 - p$ they will be all in green matches. As described earlier, the goods in the two types of matches yield different

³The assumption of large households is a modelling device extended from Lucas (1990), which is meant to capture an individual agent's allocation of time over different activities. See Faig (2004) for an alternative interpretation.

levels of marginal utility. The matching shocks are independent across households and over time. Although a household experiences both red and green trades over time, it experiences only one of the two types in any given period. This modelling method simplifies the analysis by reducing the number of different types of matches that a household will face in a period.⁴

A legal restriction forbids the use of bonds as a means of payments for goods. For the legal restriction to have a real effect, it cannot be enforced on all trades – A universal restriction would amount to rescaling the stock of assets used in the goods market and hence would not affect real activities. Thus, I assume that the legal restriction is enforced only in the trades of red goods. A trade of red goods is then called a restricted trade and a trade of green goods is called an unrestricted trade.

Moreover, I assume that the matching shocks are realized after the households have already chosen the portfolio and gone to the markets (see the later description of the timing). In particular, the household’s decisions in the asset market cannot depend on the matching shocks in the current period. This is a deliberate assumption for two reasons. First, it simplifies the analysis by eliminating the possibility of non-degenerate distributions of asset holdings across households. Second, the assumption ensures that the welfare-improving role of the legal restriction will remain even if the trading of unmatured bonds for money is shut down.

2.2. Choices and the Timing of Events

To describe the timing of events, pick an arbitrary period t , suppress the time index t , and shorten the subscript $t \pm j$ as $\pm j$. Also, pick an arbitrary household as the representative household. Lower-case letters denote the decisions of this household and capital-case letters other households’ decisions or aggregate variables. I depict the timing of events in a period in Figure 1.

At the beginning of the period the household redeems bonds that were issued two periods ago and receives lump-sum monetary transfers, T . After these events, the household’s holdings of money are denoted m and of unmatured bonds b . Monetary transfers keep money holdings per household growing at a constant (gross) rate γ .

The household divides the assets into two parts. A fraction a of money and a fraction l of unmatured bonds are allocated to the goods market, where l indicates “liquid” bonds, while the remaining assets are allocated to the bonds market. The household divides the assets for

⁴This modeling method is not critical for the analytical results. See section 5 for the alternative modelling where each household experiences both types of trades in a period.

the goods market evenly among the buyers. Each buyer carries $am/(1 - \sigma)$ units of money and $lb/(1 - \sigma)$ units of unmatured bonds into the goods market.

At the time of choosing the portfolio divisions (a, l) , the household also chooses the quantities of trades in the two markets. In the goods market, I assume that the buyers make take-it-or-leave-it offers. Such an offer consists of the amount of goods to be purchased, q^i , and the amount of assets to be spent, x^i , conditional on the type of trade $i \in \{G, R\}$. The quantities of trade in the bonds market are the amount of new bonds to be bought, d , and the amount of unmatured bonds with which the household will exit the bonds market, b^u . These quantities in the bonds market cannot depend on the matching shock in the goods market because the household cannot communicate between the two markets.

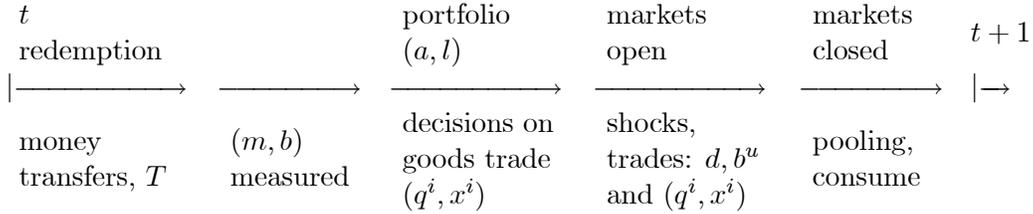


Figure 1 Timing of events in a period

Next, the two markets open simultaneously and separately. In particular, the matching shocks in the goods market are realized and an amount zM of new bonds are sold in the bonds market, where $z > 0$. The members trade according to the quantities chosen by the household. After the trade, the household pools the receipts from the trades and allocates consumption evenly among the members. After consumption, time proceeds to the next period.

As in Lucas (1990), the temporary separation between the two markets implies a discount on new bonds because by bringing money to buy the new bonds, the household has to forego the opportunity of using the money to buy goods. In contrast to Lucas's model, the goods market is decentralized here, rather than centralized. Also, Lucas assumes that money is the only medium of exchange. In contrast, households in the current model can use unmatured bonds, as well as money, to buy goods in some trades.

2.3. Quantities of Trade in the Goods Market

Normalize all nominal variables by the aggregate stock of money holdings per household. Let m be the household's holdings of money and b the holdings of unmatured bonds at the beginning of a period after the household has redeemed matured bonds and received monetary transfers. Let

$v(m, b) : R_+ \times R_+ \rightarrow R$ be the household's value function. Suppose that the household receives the matching shock $i \in \{G, R\}$ in the current period. Let m_{+1}^i and b_{+1} be the household's holdings of money and unmatured bonds next period. (The amount of unmatured bonds next period does not depend on i because it is determined by the trading decisions in the bonds market which are made before the matching shock is realized.) Let ω^{ji} be the shadow value of next period's asset j ($= m, b$), discounted to the current period by β and the money growth rate γ . That is,

$$\omega^{mi} = \frac{\beta}{\gamma} v_1(m_{+1}^i, b_{+1}), \quad \omega^{bi} = \frac{\beta}{\gamma} v_2(m_{+1}^i, b_{+1}), \quad i \in \{G, R\}, \quad (2.1)$$

where the subscripts of v indicate partial derivatives. Other households' values of the two assets are denoted similarly with the capital-case Ω .

In each trade of type i , the buyer makes a take-it-or-leave-it offer, (q^i, x^i) . The offer must satisfy two types of constraints. One is the *asset constraint*, i.e., that the amount of assets offered cannot exceed the amount that the buyer can use. Because of the legal restriction, this constraint is different for a restricted trade and an unrestricted trade. I write the constraint for the two types of trades, respectively, as follows:

$$x^R \leq \frac{am}{1 - \sigma}, \quad (2.2)$$

$$x^G \leq \frac{am + lb}{1 - \sigma}. \quad (2.3)$$

In an unrestricted trade, it is unnecessary to specify how the assets offered by the buyer are divided between money and unmatured bonds. The two assets have the same continuation value: Upon exiting from the trade, the only thing the household can do with the assets is to bring them to the next period, at which time the bonds will mature and can be redeemed for money at par.⁵ More precisely, after the trade is completed, the two assets have the same marginal value ω^{mi} to the buyer and Ω^{mi} to the seller, conditional on the current matching shock i .

Another constraint on the offer is that the offer must give the seller a non-negative surplus in order to induce the seller to participate in the trade. The seller's surplus is equal to the value of the assets received in the trade, $\Omega^m x^i$, minus the cost of production, $\psi(q^i)$. Because it is optimal for the buyer to squeeze the seller's surplus to zero, I can write the constraint as:

$$x^i = \psi(q^i) / \Omega^{mi}, \quad i = G, R. \quad (2.4)$$

⁵For the same reason, a trade in the goods market between a money holder and a bond holder is inconsequential, and so it is omitted here.

Note that the seller's valuation of the asset is indexed by the same index i as the realization of the buyer's matching shock. This is because all members of the seller's household are assumed to experience the same type of trades in a period. If one particular seller is in a type i trade, then all other trades that his household experiences in the period are also of type i .

2.4. A Household's Decision Problem

The household's choices in each period are the portfolio divisions, (a, l) , the quantities of trade, (q^i, x^i) , the amount of new bonds to purchase, d , the amount of unmatured bonds to hold exiting the bonds market, b^u , consumption, c^i , and future asset holdings, (m_{+1}^i, b_{+1}) . Taking other households' choices and aggregate variables as given, the choices solve the following problem:

$$(PH) \quad v(m, b) = \max \left[pW^R + (1-p)W^G \right] \quad (2.5)$$

where

$$\begin{aligned} W^i &= \theta^i u(c^i) - \alpha\sigma(1-\sigma)\psi(Q^i) + \beta v(m_{+1}^i, b_{+1}) \\ c^i &= \alpha\sigma(1-\sigma)q^i, \quad i \in \{R, G\} \end{aligned}$$

and the constraints are as follows:

- (i) the constraints in the goods market, (2.2), (2.3) and (2.4);
- (ii) the constraints in the bonds market: $b^u \geq 0$ and

$$Sd \leq (1-a)m + S^u [(1-l)b - b^u]; \quad (2.6)$$

- (iii) the laws of motion of asset holdings:

$$b_{+1} = \frac{d}{\gamma}, \quad (2.7)$$

$$m_{+1}^i = \frac{1}{\gamma} \left\{ m - Sd + S^u [(1-l)b - b^u] + \alpha\sigma(1-\sigma) (X^i - x^i) + (lb + b^u) \right\} + T_{+1}; \quad (2.8)$$

- (iv) other constraints: $0 \leq a \leq 1$ and $0 \leq l \leq 1$.

Consumption is equal to the amount of goods obtained by the buyers in the period, where the total number of such trades is $\alpha\sigma(1-\sigma)$. The disutility of production is computed similarly, with Q replacing q . The constraints in (i) and (iv) are self-explanatory.

The constraint $b^u \geq 0$ in (ii) requires that the household should not sell more unmatured bonds than the amount it brought into the bonds market.⁶ The constraint (2.6) states that

⁶This constraint arises because the government does not buy back unmatured bonds. Individual households can issue private bonds. However, because all households are symmetric and because the shocks are iid over time, such private bonds do not affect the equilibrium.

money spent on newly issued bonds comes from money that the household brings into the bonds market plus the receipt from selling unmatured bonds.

The law of motion of unmatured bonds, (2.7), is straightforward – newly issued bonds in this period become unmatured bonds next period. The factor $1/\gamma$ appears on the right-hand sides of (2.7) because nominal variables in each period are normalized by the money stock per household in that period. To explain the law of motion of money, (2.8), recall that the household's money holdings are measured at the time immediately after receiving monetary transfers and redeeming matured bonds (see Figure 1). These holdings can change between two adjacent periods as a result of the following transactions: purchasing newly issued bonds, selling unmatured bonds in the bonds market, selling and buying goods, redeeming matured bonds and receiving monetary transfers next period, T_{+1} . The terms following m on the right-hand side of (2.8) list the net changes in money holdings from these five types of transactions.

Remark 1. *In a symmetric equilibrium, all households make the same choices. In particular, $x^i = X^i$ for $i = R, G$. Then, (2.8) shows $m_{+1}^R = m_{+1}^G$. That is, the amount of money holdings at the end of a period is independent of the matching shocks that the household receives. Because the holdings of unmatured bonds at the end of a period is also independent of the matching shocks, all households hold the same portfolio of assets at the end of a period. Thus, the use of a representative household can be maintained over time and I can suppress the superscripts (R, G) on $(m, \omega^m, \omega^b, \Omega^m, \Omega^b)$.*

2.5. Definition of a Symmetric Equilibrium and Restrictions

A symmetric equilibrium consists of a sequence of a representative household's choices, $(a, l, q, x, d, b^u, c, m_{+1}, b_{+1})$, the value function v , the implied shadow values of assets (ω^m, ω^b) , and other households' choices (capital-case variables) such that the following requirements are met. (i) Optimality: given other households' choices, the household's choices solve (PH) with given initial holdings (m_0, b_0) and the value function satisfies (2.5); (ii) symmetry: the choices (and shadow prices) are the same across households; (iii) clearing of the market of newly issued bonds: $d = z$, with $0 < z < \infty$; (iv) clearing of unmatured bonds in the bonds market: $b^u = (1 - L)B$; (v) positive and finite values of assets: $0 < \omega_{-1}^m m < \infty$ and $0 < \omega_{-1}^b b < \infty$ if $m, b > 0$; (vi) stationarity: all real variables and the values $(\omega_{-1}^m m, \omega_{-1}^b b)$ are constant.

Note that symmetry implies $m = M = 1$ and that the requirement (iii) requires the choice of

d to be interior. Stationarity implies $\omega_{-1}^m = \omega^m$ and $\omega_{-1}^b = \omega^b$.

Money is said to generate *liquidity services* in the goods market if either the asset constraint (2.2) or (2.3) binds. In contrast, unmatured bonds yield liquidity services only if (2.3) binds. Unmatured bonds are said to be *perfect substitutes* for money if they have the same value as money, i.e., if $\omega^b = \omega^m$. As I will show later, if unmatured bonds are perfect substitutes for money, then they must generate liquidity services. But the reverse is not necessarily true.

I have restricted the amount of newly issued bonds to be a constant fraction of the money stock. The total value of each asset is restricted to be positive and finite, in order to examine the coexistence of money and bonds.⁷ Furthermore, I restrict attention to the equilibria in which money serves as a medium of exchange. This restriction imposes two requirements. First, the money growth rate must satisfy $\gamma > \beta$. If $\gamma = \beta$, then money would not generate liquidity services; if $\gamma < \beta$, then a monetary equilibrium would not exist. Second, $a > 0$; otherwise money would not be used in the goods market. Note that $a < 1$ from the market clearing condition for newly issued bonds. Hence, $0 < a < 1$.

3. The Stationary Equilibrium

3.1. Optimal Choices

To characterize the equilibrium, let me first analyze the representative household's optimal decisions. Let ρ be the Lagrangian multiplier of the constraint in the bonds market, (2.6). Let λ^R be the shadow price of the asset constraint on a trade of red goods, (2.2). To simplify the formulas, multiply λ^G by the expected number of such trades, $\alpha\sigma(1-\sigma)p$, before incorporating the constraint into the maximization problem. Similarly, let λ^G be the shadow price of (2.3) and multiply it by $\alpha\sigma(1-\sigma)(1-p)$. The household's optimal decisions are characterized by the following conditions.

(i) For q^i :

$$\theta^i u'(c^i) = (\omega^m + \lambda^i) \frac{\psi'(q^i)}{\Omega^m}, \quad i = R, G. \quad (3.1)$$

(ii) For (a, d) :

$$\alpha\sigma \left[p\lambda^R + (1-p)\lambda^G \right] = \rho, \quad (3.2)$$

$$\omega^b = (\omega^m + \rho)S. \quad (3.3)$$

⁷The value of each asset must be bounded in order to ensure that the household's optimal decisions are indeed characterized by the first-order conditions.

(iii) For (l, b^u) :

$$\omega^m + \alpha\sigma(1-p)\lambda^G = (\omega^m + \rho)S^u \quad \text{if } l \in (0, 1), \quad (3.4)$$

$$\omega^m = (\omega^m + \rho)S^u \quad \text{if } b^u > 0. \quad (3.5)$$

In each of these conditions, the choice variable attains the lowest value in the specified domain if the condition is replaced by “<”, and the highest value if “>”.

(iv) For (m, b) (envelope conditions):

$$\frac{\gamma}{\beta}\omega_{-1}^m = \omega^m + \rho, \quad (3.6)$$

$$\frac{\gamma}{\beta}\omega_{-1}^b = (1-l)(\omega^m + \rho)S^u + l[\omega^m + \alpha\sigma(1-p)\lambda^G]. \quad (3.7)$$

The condition (3.1) requires that a buyer’s net gain from asking for an additional amount of goods be zero. By getting an additional unit of good in a type i trade, the household’s utility increases by $\theta^i u'(c^i)$. The cost is to pay the additional amount $\psi'(q^i)/\Omega^m$ of assets in order to induce the seller to trade (see (2.4)). By giving one additional unit of asset, the buyer foregoes the discounted future value of the asset, ω^m , and causes the asset constraint in the trade to be more binding. Thus, $(\omega^m + \lambda^i)$ is the shadow cost of each additional unit of asset to the buyer’s household and the right-hand side of (3.1) is the cost of getting an additional unit of good.

To interpret the conditions in (ii), recall that optimal choices of a and d are both interior. For the optimal allocation of money to be interior, (3.2) requires that money should generate the same amount of liquidity services in the two markets by relaxing the asset constraints. For the optimal amount of the purchase of the new bonds to be interior, (3.3) requires that the expected future value of these bonds be equal to the cost of money that is used to acquire them, including the shadow cost of the money constraint in the bonds market (ρ).

The choices l and b^u are not necessarily interior. For the allocation of unmatured bonds, (3.4) compares the shadow values of unmatured bonds in the two markets. The shadow value of an unmatured bond in the goods market is $[\omega^m + \alpha\sigma(1-p)\lambda^G]$, because the bond may generate liquidity services in unrestricted trades and can be carried over to the next period. The shadow value of an unmatured bond in the bonds market is $(\omega^m + \rho)S^u$, because the bond can be sold for S^u units of money and each unit of money has a shadow value $(\omega^m + \rho)$ in the bonds market. If the optimal choice of l is interior, then these two shadow values must be equal to each other. For the re-balancing of unmatured bonds, (3.5) compares the shadow value of keeping such bonds for

redemption next period and the shadow value of selling it now for money in the bonds market. The household chooses to carry a positive amount of unmatured bonds out of the bonds market only if these two shadow values are equal to each other.

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the future value of the asset and the expected liquidity services generated by the asset in the current markets. Take money for example. The current value of money is given by the left-hand side of (3.6), where ω_{-1}^m is multiplied by γ/β because ω_{-1}^m is defined as the value of money discounted to one period earlier. The right-hand side of (3.6) consists of the (discounted) future value of money, ω^m , and weighted sum of liquidity services generated by money in the two markets. The weights are a for liquidity services in the goods market and $(1 - a)$ for liquidity services in the bonds market. By (3.2), this weighted sum of liquidity services is equal to ρ .

3.2. Existence of the Equilibrium

The equilibrium must have $\rho > 0$; otherwise, money would not generate liquidity services and a stationary equilibrium would exist only if $\gamma = \beta$. By (3.2), money must generate liquidity services in at least one of the two types of trades in the goods market. That is, the asset constraint binds either in restricted trades ($\lambda^R > 0$) or in unrestricted trades ($\lambda^G > 0$) or in both. To describe these cases specifically, define a constant $\mu > 0$ and a function $f(\cdot)$ as follows:

$$\mu \equiv \frac{1}{\alpha\sigma} \left(\frac{\gamma}{\beta} - 1 \right), \quad (3.8)$$

$$\frac{u'(\alpha\sigma(1 - \sigma)f(k))}{\psi'(f(k))} = k, \text{ for } k > 0. \quad (3.9)$$

Note that f is well defined for all $k \in (0, \infty)$ and that it is a decreasing function. Also, $\mu > 0$ because $\gamma > \beta$. The three cases of the equilibrium are as follows.

Case PS (Perfect Substitutability): $\lambda^R = 0$ and $\lambda^G > 0$. The features are:

- (i) Unmatured bonds are perfect substitutes for money: $\omega^b = \omega^m$;
- (ii) A household takes all unmatured bonds to the goods market: $l = 1$;
- (iii) The price of newly issued two-period bonds is $S = \beta/\gamma$;
- (iv) The price of unmatured bonds is indeterminate: $\beta/\gamma \leq S^u \leq 1$;
- (v) The quantities of goods traded in matches are $q^G = q_1^G$ and $q^R = q_1^R$, where

$$q_1^G \equiv f \left(1 + \frac{\mu}{1 - p} \right), \quad q_1^R \equiv f \left(\frac{1}{\theta} \right); \quad (3.10)$$

(vi) The allocation of money is $a = 1 - z\beta/\gamma$.

Case IS (Imperfect Substitutability): $\lambda^R > 0$ and $\lambda^G > 0$. The features are:

- (i) Unmatured bonds are imperfect substitutes for money: $\omega^b < \omega^m$;
- (ii) A household takes all unmaturred bonds to the goods market: $l = 1$;
- (iii) The price of newly issued bonds is unique and lies in $((\beta/\gamma)^2, \beta/\gamma)$;
- (iv) The price of unmaturred bonds is indeterminate;
- (v) The quantities of goods traded in matches are $q^G = q_2^G(a)$ and $q^R = q_2^R(a)$, where

$$q_2^G(a) \equiv f\left(1 + \frac{D(a)}{1-p}\right), \quad q_2^R(a) \equiv f\left(\frac{1}{\theta}\left(1 + \frac{\mu - D(a)}{p}\right)\right), \quad (3.11)$$

$$D(a) = \frac{1}{\alpha\sigma} \left[\frac{1-a}{z} \left(\frac{\gamma}{\beta}\right)^2 - 1 \right]; \quad (3.12)$$

(vi) The allocation of money lies in $(1 - z\beta/\gamma, 1 - z(\beta/\gamma)^2)$ and is uniquely given by the solution to the following equation:

$$\frac{\psi(q_2^G(a))}{\psi(q_2^R(a))} - \left(1 + \frac{z}{\gamma a}\right) = 0. \quad (3.13)$$

Case BS (Bad Substitutability): $\lambda^R > 0$ and $\lambda^G = 0$. The features are:

- (i) Unmatured bonds are imperfect substitutes for money: $\omega^b < \omega^m$;
- (ii) A household takes only a fraction of unmaturred bonds to the goods market: $0 \leq l < 1$;
- (iii) The price of newly issued two-period bonds is $S = (\beta/\gamma)^2$;
- (iv) The price of unmaturred bonds is unique: $S^u = \beta/\gamma$;
- (v) The quantities of goods traded in matches are $q^G = q_3^G$ and $q^R = q_3^R$, where

$$q_3^G \equiv f(1), \quad q_3^R \equiv f\left(\frac{1}{\theta}\left(1 + \frac{\mu}{p}\right)\right); \quad (3.14)$$

(vi) The allocation of money is $a = 1 - z(\beta/\gamma)^2$.

To determine when each case occurs, define the following numbers and functions:

$$Q_1 = \psi^{-1}\left(\psi(q_1^G) \left/ \left(1 + \frac{z}{\gamma - z\beta}\right)\right.\right), \quad (3.15)$$

$$\Theta_1 = \frac{\psi'(Q_1)}{u'(\alpha\sigma(1-\sigma)Q_1)}, \quad (3.16)$$

$$Q_3(l) = \psi^{-1} \left(\psi(q_3^G) / \left(1 + \frac{lz}{\gamma - z\beta^2/\gamma} \right) \right), \quad (3.17)$$

$$\Theta_3(l) = \left(1 + \frac{\mu}{p} \right) \frac{\psi'(Q_3(l))}{u'(\alpha\sigma(1-\sigma)Q_3(l))}. \quad (3.18)$$

Note that $0 < \Theta_1 < 1$. Also, $\Theta_3(l) > \Theta_1$ and $\Theta_3'(l) < 0$ for all $l \in [0, 1]$. I prove the following proposition in Appendix A:

Proposition 3.1. *Assume that $0 < z < \gamma/\beta$. An equilibrium exists and is characterized by one of the three cases listed above. Case PS occurs for $\theta \leq \Theta_1$. Case BS occurs for $\theta \geq \Theta_3(1)$. Case IS occurs for $\Theta_1 < \theta < \Theta_3(1)$. The equilibrium is unique for $\theta \leq \Theta_3(1)$. When $\theta > \Theta_3(1)$, there is a continuum of equilibria (Case BS) that differ in the value of l but have the same values of (q^G, q^R, S, a) .*

When the tastes for red goods are very low, the economy is in Case PS. In this case, the legal restriction in the goods market does not bind, because the marginal utility of red goods is so low that the buyer does not spend all the money in a restricted trade. As a result, unmatured bonds are perfect substitutes for money in the goods market. A household takes all unmatured bonds to the goods market. Newly issued two-period bonds are still discounted, but the discount arises entirely from the one-period separation between the bonds market and the goods market. The discount is a compensation for the foregone liquidity services that could be generated if the amount of money is brought to the goods market instead.

When the tastes for red goods are very strong, the equilibrium is Case BS. This case is opposite to Case PS. In Case BS, the asset constraint binds in a restricted trade (for red goods) but not in an unrestricted trade (for green goods). Unmatured bonds do not yield liquidity services and so they are poor substitutes for money. A positive discount on unmatured bonds is necessary for the equilibrium, which induces a deeper discount on newly issued two-period bonds than in Case PS. Thus, the bond price is lower than in Case PS. Moreover, because unmatured bonds do not generate liquidity services, a household is indifferent at the margin about sending more unmatured bonds into either of the two markets. There are a range of values of l that are consistent with equilibrium.

Case IS is between Cases PS and BS, and it occurs when θ has intermediate values. In this case, the asset constraints bind in both types of trades and a household takes all unmatured bonds

to the goods market, but unmatured bonds are not perfect substitutes for money. Unmatured bonds are discounted, but not as deeply as in Case BS. Similarly, the discount on newly-issued two-period bonds is smaller than in Case BS but greater than in Case PS.

In Case BS, the price of unmatured bonds is $S^u = \beta/\gamma$. In contrast, S^u is indeterminate in both Cases PS and IS: Because the amount of unmatured bonds carried to the bonds market is a corner solution 0, the Walrasian price of these bonds is not unique. However, this indeterminacy has no consequence on real activities. Moreover, because unmatured bonds generate liquidity services in Cases PS and IS but newly issued bonds do not, the price of unmatured bonds exceeds the price of newly issued one-period bonds, the latter of which is β/γ .

Because of the above difference in the price of unmatured bonds in the three cases, the term structure of interest rates also differs in these cases. In Case BS, the yield curve is flat, since the price of two-period bonds is equal to the square of the price of one-period bonds. In Cases PS and IS, the yield curve is negatively sloped. The slope is steeper in Case PS than in Case IS, because unmatured bonds yield higher liquidity services in Case PS.

4. Welfare-Improving Role of the Legal Restriction

In this section, I show that the legal restriction can improve welfare. Welfare is measured as the following steady state utility per period:

$$(1 - \beta)v = (1 - p) \left[u(\alpha\sigma(1 - \sigma)q^G) - \alpha\sigma(1 - \sigma)\psi(q^G) \right] + p \left[\theta u(\alpha\sigma(1 - \sigma)q^R) - \alpha\sigma(1 - \sigma)\psi(q^R) \right].$$

To begin, note that unmatured bonds are perfect substitutes for money in an economy without the legal restriction. In such an economy, the price of newly issued bonds is $S = \beta/\gamma$ and $a = 1 - z\beta/\gamma$. Then, the real allocation without the legal restriction is equivalent to the one in an economy without nominal bonds (up to rescaling the money stock), i.e., the economy with $z = 0$. Taking the limit $z \rightarrow 0$ in Cases PS, IS and BS, I obtain the real allocation without the legal restriction as follows:

Case A: $\theta \leq \left(1 + \frac{\mu}{1-p}\right)^{-1}$. In this case $q^G = q_1^G$ and $q^R = q_1^R$.

Case B: $\left(1 + \frac{\mu}{1-p}\right)^{-1} < \theta < 1 + \frac{\mu}{p}$. In this case, $q^G = q^R = f\left(\frac{1+\mu}{1-p+p\theta}\right)$.

Case C: $\theta \geq 1 + \frac{\mu}{p}$. In this case, $q^G = q_3^G$ and $q^R = q_3^R$.

Note that the real allocation is the same in Case A as in Case PS, and the same in Case C as in Case BS. Thus, the legal restriction does not affect the real allocation when the tastes for the two types of goods are far from being symmetric. However, the allocation in Case IS is different from that in Case B. So, the restriction affects the allocation when θ has intermediate values.

It is meaningful to use the parameter z to measure the extent of the legal restriction, because z is the amount of unmatured bonds that are prevented by the restriction from acting as a medium of exchange.⁸ I will examine small legal restrictions first and then large legal restrictions. The proofs of the propositions in this section appear in Appendix B.

4.1. Small Legal Restrictions

A slightly positive z represents a small legal restriction. The legal restriction changes both the region of existence of Case IS and the quantities of goods traded in matches in that case. On the region of existence, a small legal restriction reduces both the lower bound, Θ_1 , and the upper bound, $\Theta_3(1)$, of the region in which Case IS occurs. The effects on the quantities of goods traded and on welfare are summarized as follows:

Proposition 4.1. *In comparison with an economy without the legal restriction, a small legal restriction reduces a , increases the quantity of (green) goods traded in an unrestricted match, and reduces the quantity of (red) goods traded in a restricted match. The legal restriction improves welfare if and only if $\left(1 + \frac{\mu}{1-p}\right)^{-1} < \theta < 1$.*

The legal restriction can improve welfare when red goods generate a lower marginal utility than green goods and when this asymmetry in the tastes is not very large. The legal restriction has this essential role because it shifts the purchasing power of the assets from restricted trades to unrestricted trades. If the (green) goods in unrestricted trades yield a higher marginal utility than the (red) goods in restricted goods, then this shift reduces the gap in the marginal utility of consumption of the two types of goods, and hence increases expected utility.

To see how the legal restriction shifts the purchasing power between the two types of goods, let me examine how the restriction affects the amount of assets allocated to the goods market. This amount is $am + lb$, which is equal to $a + z/\gamma$ in Case IS of the equilibrium. An increase in z

⁸In contrast, the parameter p is not a suitable one with which one conducts comparative statics on the legal restriction. A change in p changes not only the coverage of the legal restriction, but also preferences. Even without the legal restriction, a change in p affects the quantities of goods traded in matches.

has two opposite effects on this amount. One effect is to reduce a , i.e., to shift money from the goods market to the bonds market. The other effect is to increase b . It can be shown that when z is near 0, the effect through b dominates and so the total amount of assets in the goods market increases with z . In an unrestricted trade, this larger amount of assets allows a buyer to buy a larger quantity of goods. The effect is opposite in a restricted trade. There, because a buyer can only use money to buy goods and because the legal restriction reduces the amount of money in the goods market, the buyer in a restricted trade can only afford a smaller quantity of goods. Thus, there is a shift of the purchasing power from restricted trades to unrestricted trades.

Note that prices do adjust to the increased amount of assets in the goods market. Express prices of goods in terms of utility, i.e., by multiplying prices by the marginal value of money ω^m . Then, the price of the (green) goods in an unrestricted trade increases to respond to the increased amount of assets in the trade, and the price of the (red) goods in restricted trades falls to respond to the reduced amount of money in the trade. However, these responses of prices do not fully offset the shift of the purchasing power between the two types of matches.

Also note that the welfare-improving role of the legal restriction necessarily comes with a higher interest rate. In the case where the legal restriction can improve welfare (i.e., Case IS), the bond price is between $(\beta/\gamma)^2$ and β/γ . In the absence of the legal restriction, the bond price is β/γ . Thus, the legal restriction reduces the bond price and hence increases the interest rate.

The welfare-improving mechanism has some interesting differences from that in Kocherlakota (2003). In Kocherlakota, the legal restriction has a different form – it applies to all trades rather than a fraction of the trades. (If such a universal legal restriction were imposed in the current model, it would not affect the real allocation.) To generate an essential role for the restriction, Kocherlakota assumes that households receive the taste shocks first and trade assets before going to the goods market. As a result, different households with different tastes choose different portfolios of assets before they go to purchase goods. Such trading in the asset market prior to the goods market is critical there for the legal restriction to reduce (or smooth) the gap between different households' marginal utilities of consumption. In contrast, the households in my model receive matching shocks after they have already chosen the portfolio of assets, and so all households hold the same portfolio entering the markets. In addition, all trades in the goods market occur between households who receive the same matching shock. A household smoothens the marginal utility of consumption not by trading with other households that have different taste

shocks, but by smoothing the marginal utility between the two types of matches.

To see the above difference between the two models in another way, suppose that the households are not allowed to trade unmatured bonds. In the current model, because net trading of unmatured bonds in the bonds market is zero anyway, shutting down such trading does not affect the allocation or the welfare-improving role of the legal restriction. In contrast, shutting down the trading between unmatured bonds and money in Kocherlakota's model eliminates the welfare-improving role of the legal restriction.⁹

Above all, the most important difference between the two models is that the welfare-improving role of the legal restriction lasts for only one period in Kocherlakota's model, but the role sustains in the steady state in the current model.

4.2. Large Legal Restrictions

I now allow z to be significantly different from 0. The following proposition extends the essential role of the legal restriction from small values of z to large values:

Proposition 4.2. *For any given $0 < z < \gamma/\beta$, there exists $\theta_A > \Theta_1$ such that the legal restriction improves welfare for $\theta \in (\Theta_1, \theta_A)$.*

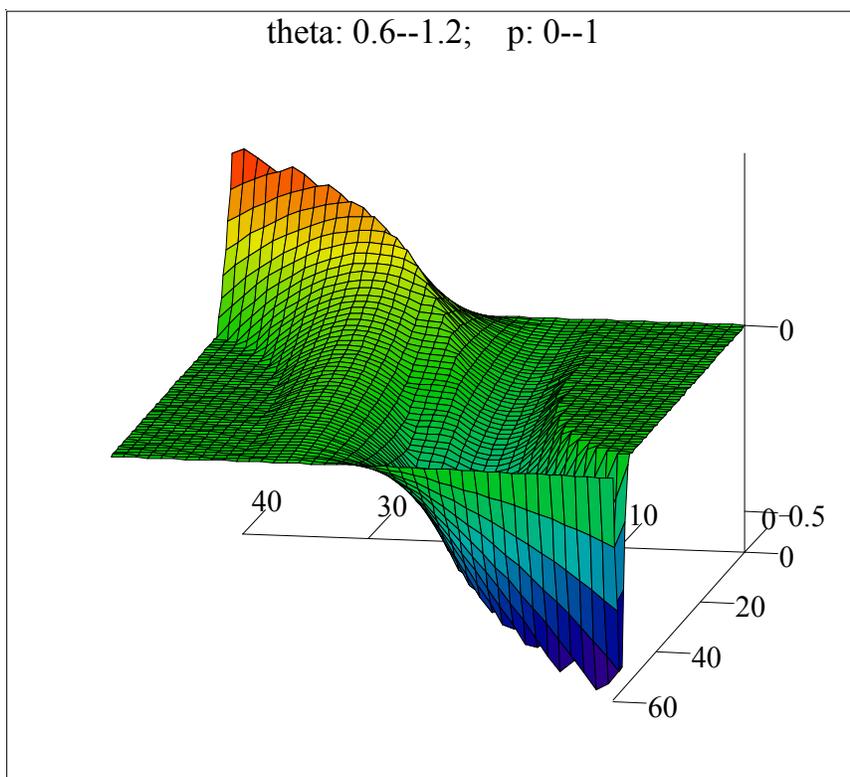
This proposition does not indicate how wide the region is in which the restriction improves welfare or how this essential role of the restriction depends on the parameter p . To illustrate these aspects of the equilibrium, consider the following example:

Example 4.3. *Let $u(c) = \ln(c)$ and $\psi(q) = q$. Choose $\beta = 0.995$, $\gamma = 1.005$, $\alpha = 0.5$, and $\sigma = 0.5$. Let $z = 0.2$. The length of a period is chosen to be one and a half month. The values of β and γ are chosen to match the real interest rate and the inflation rate in a period. The value of z reflects a significantly large legal restriction.*

The parameters θ and p are not given particular values. Instead, I will let θ vary between 0.6 and 1.2, and p between 0 and 1. The solutions to the variables will be expressed as functions of these two parameters. I measure the welfare cost of the legal restriction in the standard way as

⁹This difference between the two mechanisms may be important for the following reason. If trading in the asset market prior to the goods market is the way to achieve the smoothing of marginal utility of consumption, then there may be other ways that can achieve better allocations than the legal restriction. Examples include discount windows operated by the government and lending and borrowing between households.

the percentage of consumption that a household is willing to give up in order to eliminate the restriction. Denote this cost as $\Delta c(\theta, p)$.



ΔC

$$\Delta C(\theta, p): \theta = 0.6 + 0.015i, \quad p = 0.017j;$$

The axes on the plane: $i = 0 - 40, j = 0 - 60$.

Figure 2a Welfare cost (% of consumption) of the legal restriction

Figure 2a depicts the welfare cost of the legal restriction as a function of θ and p . In the two flat sections in the diagram, the legal restriction does not affect the real allocation. The flat section with low values of (θ, p) corresponds to Case PS and the flat section with high values of (θ, p) corresponds to Case BS. The “hill” in the diagram is the region of (θ, p) in which the legal restriction reduces welfare. The “valley” is the region with negative welfare costs, in which the legal restriction improves welfare. Similar to the case of small legal restrictions, the essential role of large legal restrictions occurs when the relative tastes for the (red) goods in restricted trades have intermediate values. As before (but not shown here), the legal restriction increases the quantity of (green) goods traded in an unrestricted match and reduces the quantity of (red) goods traded in a restricted match. Thus, the restriction improves welfare by reducing the gap

between the marginal utility of consumption of the two types of goods.

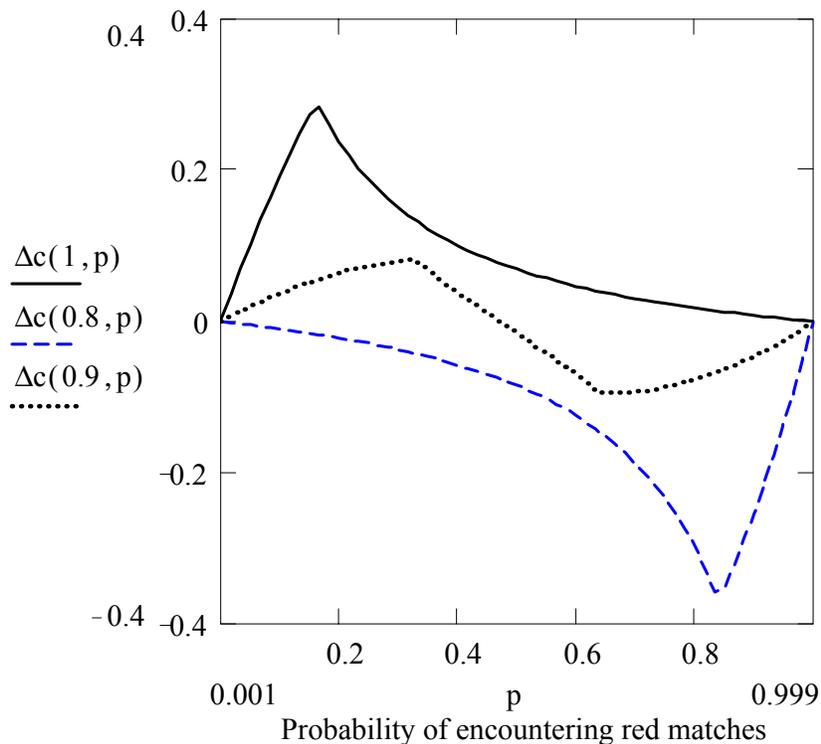


Figure 2b Welfare costs (% of consumption) with three values of θ

To see the welfare cost of the legal restriction in a different way, I depict three sections of Figure 2a in Figure 2b. These sections correspond to $\theta = 1, 0.8,$ and 0.9 respectively. There are a few noteworthy aspects of Figure 2b. First, for all three values, the welfare cost is zero at the two ends, $p = 0$ and $p = 1$. When $p = 0$, the legal restriction is inactive. When $p = 1$, the legal restriction is imposed on all trades, which changes only the price level without any real consequence. Second, when $\theta = 1$, the legal restriction always reduces welfare, provided $p \neq 0$ or 1 . The explanation is that, when $\theta = 1$, the economy without the restriction has already equalized the marginal utility of consumption of the two types of goods. Since the restriction shifts consumption from the restricted goods to the unrestricted goods, it widens the gap between the marginal utility of consumption of the two goods, and hence reduces welfare. Third, when $\theta = 0.8$, the legal restriction always improves welfare. This is because the restriction shifts consumption from the goods which the household values less to the goods which the household values more. Finally, when $\theta = 0.9$, the restriction reduces welfare when p is low and improves welfare when p is high.

In the above example, the size of the welfare gain or cost of the legal restriction is small, around 0.5% of consumption. However, the gain can increase substantially with z . For example, when $z = 1$, the gain from the legal restriction at $\theta = 0.2$ and $p = 0.98$ is 4% of consumption.

5. Discussions

5.1. Robustness of the Results

The welfare-improving role that the legal restriction has in this model relies on two features of the model. First, trading in the goods market is decentralized and the legal restriction is imposed in only a fraction of the trades. Second, at the time of choosing the amount of assets to be brought into the goods market, an agent does not know which type of trades he will be involved. Given these features, the essential role of the legal restriction can survive several modifications of the model. I have already discussed one modification which shuts down the trading between unmaturred bonds and money in the asset market – This modification leaves the results intact.

Another modification is to change the way in which a household experiences the matching shocks. In previous sections, I have assumed that a household encounters either the trades of red goods or the trades of green goods but not both in a period. An alternative specification of the matching shocks is as follows: After a match is formed, a shock is realized to determine whether the seller in the match can produce red goods or green goods, where the probability with which the seller can produce red goods is p . As before, the shocks are independent across the sellers and over time. With this specification, a household experiences both types of trades in a period. Among the household's buyers who trade, a fraction p of them purchase red goods and the other fraction $(1 - p)$ purchase green goods.

If each subgroup of the buyers who experience the same shock share consumption among themselves but not with others, then expected utility of consumption in a period is:

$$p\theta u(\alpha\sigma(1 - \sigma)q^R) + (1 - p)u(\alpha\sigma(1 - \sigma)q^G).$$

The equilibrium in this economy is identical to the one analyzed in previous sections, and so the essential role of the legal restriction remains. If, instead, all consumption goods are shared among all members in the household, as in previous sections, then consumption of red goods per member is $c^R = p\alpha\sigma(1 - \sigma)q^R$ and consumption of green goods per member is $c^G = (1 - p)\alpha\sigma(1 - \sigma)q^G$. Expected utility of consumption in the household in a period is $\theta u(c^R) + u(c^G)$. Wherever u'

appears in section 3, change its argument to the new expressions for c^i . Then, the characterization of the equilibrium there is still valid. Propositions 4.1 and 4.2 can be modified to show that the legal restriction continues to improve welfare for some values of (θ, p) .¹⁰

5.2. Restrictions on Lump-Sum Taxes

The welfare-improving role of the legal restriction exists only when $\mu > 0$, i.e., when $\gamma > \beta$. It vanishes in the limit $\gamma \downarrow \beta$, because Case IS no longer exists in this limit. The limit $\gamma \downarrow \beta$ corresponds to the so-called Friedman rule, because the nominal interest rate on a newly-issued one-period bond is $(\gamma/\beta - 1)$. The legal restriction does not improve welfare in this limit because the asset constraints in the goods market do not bind, in which case whether bonds can be used to purchase goods or not is irrelevant for the quantities of goods traded in matches.

Although a stationary equilibrium requires $\gamma > \beta$, the limit case $\gamma \downarrow \beta$ still deserves attention. The issue at stake is whether the legal restriction can improve welfare when the money growth rate is set optimally. In the economy without the legal restriction, it is easy to see that expected utility (or welfare) decreases in μ and hence in γ . Thus, welfare is maximized at $\gamma = \beta$. In the economy with the legal restriction, expected utility decreases in γ in Cases PS and BS, but changes ambiguously with γ in Case IS. However, when γ is sufficiently close to β , expected utility decreases in γ . Thus, it is possible that welfare in such an economy is also maximized at $\gamma = \beta$. If this is the case, then setting money growth at the optimal rate in the two economies would eliminate the welfare-improving role of the legal restriction.

One way to exclude the Friedman rule is to restrict the government's ability to collect lump-sum taxes, as in Kocherlakota (2003). Such a restriction is reasonable in the described economy because agents are anonymous and trades are decentralized in the goods market. The law of motion of money holdings implies that the amount of lump-sum transfers satisfies: $\gamma^2 T = \gamma(\gamma - 1) - z(1 - \gamma S)$. Because the bond price lies in the interval $[(\beta/\gamma)^2, \beta/\gamma]$ in Case IS, a necessary condition for $T \geq 0$ is $\gamma(\gamma - 1) - z(1 - \beta) \geq 0$. This condition can be written as $\gamma \geq \gamma_0$, where $\gamma_0 > 1$.

However, the above restriction on lump-sum taxes is much stronger than it is needed for preserving the essential role of the legal restriction. As Proposition 4.2 states, there exists an

¹⁰After I wrote the first version of the current paper in 2002, I became aware of a paper by Rocheteau (2002), who uses a search model to examine the legal restrictions in the goods market. His model is different from mine. Also, his result on the welfare-improving role of the legal restriction is largely numerical.

interval of θ in which the legal restriction improves welfare, provided that lump-sum taxes do not induce $\gamma = \beta$ or they are costly to collect.

6. Conclusion

In this paper I examine whether a society can improve welfare by imposing a legal restriction to forbid the use of nominal bonds as a means of payments. To do so, I integrate a microfounded model of money with the framework of limited participation. While the asset market is Walrasian, the goods market is decentralized and the legal restriction is imposed only in a fraction of the trades. I show that the legal restriction can improve welfare of the society. In contrast to some previous results (see the introduction), this essential role of the legal restriction persists even in the steady state and it does not rely on households' ability to trade unmatured bonds for money after observing the taste (or endowment) shocks. This robust role of the legal restriction can be construed as a justification for why bonds should be made less liquid than money.

The current model can be useful for analyzing monetary policy. In particular, the framework of limited participation has been popular for analyzing the effects of open market operations (see Lucas, 1990). The current model provides a microfoundation of the role that the legal restrictions play in the framework, and hence of the real effects of open market operations there. Moreover, the integrated model uncovers a new mechanism that propagates monetary policy. That is, open market operations can affect future activities by changing the amount of unmatured bonds that will be used as payments for goods in a fraction of the trades in the future. To explore this new mechanism fully, I need to make the model stochastic to capture the so-called liquidity effect of monetary shocks. This task is left for a sequel.¹¹

¹¹Williamson (2005) constructs a different model of limited participation to prolong the real effects of monetary injection. However, he does not examine the essentiality of illiquid bonds.

A. Proof of Proposition 3.1

Consider Case PS first. To show $l = 1$ in this case, suppose $l < 1$ to the contrary. The market clearing condition for unmatured bonds implies $b^u > 0$. Because $\lambda^G > 0$ in this case, then (3.4) would imply $S^u > \omega^m/(\omega^m + \rho)$ which would contradict (3.5). Thus, $l = 1$. To show $\omega^b = \omega^m$, note that $\rho = \alpha\sigma(1-p)\lambda^G$ in the current case (see (3.2)). Because $l = 1$, then (3.6) and (3.7) yield $\omega^b = \omega^m = (\omega^m + \rho)\beta/\gamma$, where I have used the stationarity of the equilibrium. Substituting this result into (3.3), I get $S = \beta/\gamma$. Because the asset market constraint, (2.6), implies $a = 1 - zS$, then $a = 1 - z\beta/\gamma$. Under the assumption $0 < z < \gamma/\beta$, a indeed lies in the interior of $(0, 1)$. The feature $l = 1$ implies $S^u \leq 1$ from (3.4) and the feature $b^u = 0$ implies $S^u \geq \beta/\gamma$. To solve for the quantities of goods traded in matches, solve λ^i from (3.1), substitute the result into (3.2) to obtain ρ , and then substitute ρ into (3.6). This procedure yields $q^G = q_1^G$ and the feature $\lambda^R = 0$ yields $q^R = q_1^R$, where q_1^G and q_1^R are defined in (3.10).

Now I find the restriction on θ that indeed delivers $\lambda^G > 0$ and $\lambda^R = 0$. Because $\lambda^G > 0$ and $\lambda^R = 0$, the two asset constraints (2.2) and (2.3) induce the relationship: $1 + \frac{lb}{am} \leq \psi(q^G)/\psi(q^R)$. Substituting $l = 1$, $m = 1$, $b = z/\gamma$ and $a = 1 - z\beta/\gamma$, this condition becomes $q_1^R \leq Q_1$, where Q_1 is defined by (3.15). Substituting q_1^R , this condition is equivalent to $\theta \leq \Theta_1$, where Θ_1 is defined in (3.16).

Next, consider Case BS. Again, use the market clearing condition for unmatured bonds, $b^u = (1 - l)b$. If $l < 1$, then $b^u > 0$, and so (3.5) implies $(\omega^m + \rho)S^u = \omega^m$. If $l = 1$, then (3.4) again implies $(\omega^m + \rho)S^u = \omega^m$. Substituting this result and $\lambda^G = 0$ into (3.7) and using the stationary requirement $\omega_{-1}^b = \omega^b$, I get $\omega^b = \omega^m\beta/\gamma$. Clearly, $\omega^b < \omega^m$. Substituting ω^b and (3.6) and using stationarity, I can derive the bond price from (3.3) as $S = (\beta/\gamma)^2$. The constraint in the asset market then implies $a = 1 - z(\beta/\gamma)^2 \in (0, 1)$. The quantities of goods traded in matches can be solved by following the same procedure as the above one for Case PS. This procedure now yields $q^R = q_3^R$ and $q^G = q_3^G$, which are defined in (3.14).

To find where Case BS exists, divide the two assets constraints in the goods market to obtain $1 + \frac{lb}{am} \geq \psi(q^G)/\psi(q^R)$, where the inequality comes from $\lambda^R > 0$ and $\lambda^G = 0$. Substituting $m = 1$, $b = z/\gamma$ and $a = 1 - z(\beta/\gamma)^2$, the condition becomes $q_3^R \geq Q_3(l)$, where $Q_3(l)$ was defined in (3.17). Substituting q_3^R , this condition is equivalent to $\theta \geq \Theta_3(l)$, where $\Theta_3(l)$ is defined in (3.18).

Turn to Case IS. As in Case PS, $\lambda^G > 0$ implies $l = 1$. Because $\lambda^G > 0$ and $\lambda^R > 0$, then

(3.6) and (3.7) yield $\omega^b < \omega^m$. Also, (3.6) implies that $\omega^m + \rho = \omega^m \gamma / \beta$, where I have used the stationary requirement $\omega_{-1}^m = \omega^m$. Then, I can rewrite (3.3) as $S = (\beta/\gamma)(\omega^b/\omega^m)$. Because $\omega^b < \omega^m$, then $S < \beta/\gamma$. Substituting ω^b from (3.7) generates $S = (\beta/\gamma)^2 [1 + \alpha\sigma(1-p)\lambda^G/\omega^m]$. The feature $\lambda^G > 0$ implies $S > (\beta/\gamma)^2$. Because $a = 1 - zS$, then $1 - z\beta/\gamma < a < 1 - z(\beta/\gamma)^2$. Substituting the formula of S into the formula of a yields $\lambda^G/\omega^m = D(a)/(1-p)$, where $D(a)$ is defined by (3.12). Solving λ^i/ω^m from (3.1) and substituting λ^G yields $q^G = q_2^G(a)$. Substituting (λ^G, λ^R) into (3.2) to obtain ρ and then substituting into (3.6), I obtain $q^R = q_2^R(a)$.

Because the two asset constraints in the goods market hold with equality in this case, dividing the two constraints and using the fact $l = 1$ yields (3.13), which determines a . Rewrite the requirement $\lambda^i > 0$ as $u'(c^i) > \psi'(q^i)$. Then, $\lambda^G > 0$ and $\lambda^R > 0$ if and only if $0 < D(a) < \mu$. Equivalently, these requirements ask a to lie in the interval $[1 - z\beta/\gamma, 1 - z(\beta/\gamma)^2]$.

Finally, I find the region of θ in which the solution for a lies in the above interval and show that such a solution is unique. Note that $q_2^G(a)$ is an increasing function and $q_2^R(a)$ a decreasing function. Denote the left-hand side of (3.13) temporarily as $LHS(a)$. Then, $LHS'(a) > 0$. So, if there is a solution to (3.13), then the solution is unique. For a solution to exist and to lie in the interval specified above, the necessary and sufficient conditions are that $LHS(1 - z\beta/\gamma) < 0$ and $LHS(1 - z(\beta/\gamma)^2) > 0$. When $a = 1 - z(\beta/\gamma)^2$, I have $q_2^G(a) = q_3^G$ and $q_2^R(a) = q_3^R$. Then, $LHS(1 - z(\beta/\gamma)^2) > 0$ iff $q_3^R < Q_3(1)$. This condition is equivalent to $\theta < \Theta_3(1)$, where $\Theta_3(1)$ can be found by setting $l = 1$ in (3.18). Similarly, when $a = 1 - z\beta/\gamma$, I have $q_2^G(a) = q_1^G$, $q_2^R(a) = q_1^R$. So, $LHS(1 - z\beta/\gamma) < 0$ iff $\theta > \Theta_1$, where Θ_1 is defined by (3.16). Therefore, Case IS exists iff $\Theta_1 < \theta < \Theta_3(1)$. **QED**

B. Proofs for Section 4

To prove Proposition 4.1, differentiate (3.11) with respect to z . I get:

$$\frac{dq_2^G}{dz} = \frac{f'}{1-p} \left(\frac{dD}{dz} \right), \quad \frac{dq_2^R}{dz} = -\frac{f'}{\theta p} \left(\frac{dD}{dz} \right).$$

Here and in other parts of the proof of Proposition 4.1, the argument of f and f' is $(1 + \mu)/(1 - p + p\theta)$. I will verify later that $z \frac{dD}{dz} \rightarrow 0$ when $z \rightarrow 0$. Using this result, differentiating (3.12) with respect to z and evaluating the result at $z = 0$, I get:

$$\frac{da}{dz} = -(\alpha\sigma D + 1) \left(\frac{\beta}{\gamma} \right)^2 < 0.$$

Also, when $z \rightarrow 0$, $D \rightarrow (1-p)[p(1-\theta) + \mu]/[1-p + \theta p]$. Differentiating (3.13) and evaluating the result at $z = 0$ yields:

$$\frac{dD}{dz} = \frac{\psi(f)}{\gamma\psi'(f)f'} \left(\frac{\theta p(1-p)}{\theta p + 1 - p} \right) < 0.$$

Indeed, $z \frac{dD}{dz} \rightarrow 0$ when $z \rightarrow 0$. Substituting dD/dz into the expressions for dq_2^G and dq_2^R yields:

$$\frac{dq_2^G}{dz} = \frac{\psi(f)}{\gamma\psi'(f)} \left(\frac{\theta p}{1-p + \theta p} \right) > 0,$$

$$\frac{dq_2^R}{dz} = -\frac{\psi(f)}{\gamma\psi'(f)} \left(\frac{1-p}{1-p + \theta p} \right) < 0.$$

It is then easy to verify that steady state utility in Case IS increases in z iff $\theta < 1$. Because Case IS requires $\left(1 + \frac{\mu}{1-p}\right)^{-1} < \theta < 1 + \frac{\mu}{p}$ when $z \rightarrow 0$, then the legal restriction improves welfare iff $\left(1 + \frac{\mu}{1-p}\right)^{-1} < \theta < 1$. This completes the proof of Proposition 4.1.

To prove Proposition 4.2, note that $\Theta_1 < \theta_1$ for any $z > 0$. Consider values of θ in the interval $(\Theta_1, \Theta_1 + \varepsilon)$, where $0 < \varepsilon < \theta_1 - \Theta_1$ is small. In this interval, the equilibrium is Case IS in the economy with the legal restriction. Substituting q_2^G and q_2^R from (3.11), I can write the steady state utility in this economy as a function of D , say, $v(D)$. The equilibrium in the economy without the legal restriction is Case A, where the quantities of goods traded in the two types of matches are q_1^G and q_1^R , respectively. Let v_0 denote the steady state utility in this economy, which does not depend on D . When $\varepsilon \downarrow 0$, I get $D \uparrow \mu$, $q_2^G \downarrow q_1^G$, $q_2^R \uparrow q_1^R$, and $v(D) \rightarrow v_0$. If $v'(D) < 0$, then $v(D) > v_0$ for sufficiently small $\varepsilon > 0$. In this case, there exists $\theta_A > \Theta_1$ such that $v > v_0$ for $\theta \in (\Theta_1, \theta_A)$, as stated in the proposition.

To show $v'(D) < 0$ for small $\varepsilon > 0$, differentiate $v(D)$ and use the definition of f in (3.9). I get:

$$\begin{aligned} \frac{(1-\beta)v'(D)}{\alpha\sigma(1-\sigma)} = & \frac{D}{1-p}\psi'(q_2^G)f' \left(1 + \frac{D}{1-p}\right) \\ & - \frac{\mu-D}{\theta p}\psi'(q_2^R)f' \left(\frac{1}{\theta}\left(1 + \frac{\mu-D}{p}\right)\right). \end{aligned}$$

Because $D \approx \mu$ when ε is sufficiently small, the second term on the right-hand side is close to zero. Then $v'(D) < 0$ follows from the fact that $f' < 0$. **QED**

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