The War of Information†

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Abstract

We analyze a model of political campaigns to examine how parties' access to funding affects outcomes and voter welfare. Two parties with opposing interests provide costly information to a voter who must choose between two policies. The flow of information is modeled as a continuous-time process that stops when neither party incurs the cost of information provision. The parties' actions are strategic substitutes, i.e., a lower cost of party 1 leads to more information provision by party 1 and less information provision by party 2. For the voter, the parties' actions are complements and the voter is best served when the parties' costs are similar. We also examine the case where a party has private information about the merits of the policy. We show that this leads to a *signaling barrier*: a belief-threshold beyond which information unfavorable to the informed party is ignored by voters.

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1. Introduction

A new policy is proposed by a political party, for example, a new plan to organize health care. Interest groups in favor of and opposed to the plan provide information to convince voters of their respective positions. This process continues until polling data suggest that voters decisively favor or oppose the new policy. At that point, Congress enacts the policy favored by voters. Recent health care debates in the US, or the debate on social security during the Bush administration are prominent examples of this pattern.

A key question is how asymmetric access to funds affects the outcome of such campaigns. For example, proponents of health care reform often cite the superior funding of their opponents as the main reason health care reform failed during the Clinton administration. Hence, the question is to what degree superior funding can determine the outcome of political campaigns, and whether asymmetric access to funds can be detrimental to voter welfare. To address this and related questions, we formulate a model of competitive advocacy that captures salient features of political campaigns.

For the most part, we assume that parties cannot distort information and focus on the trade-off between the cost of information provision and the probability of convincing the median voter. In actual campaigns, parties may present distorted and biased information. Voters, in turn, anticipate distortions and discount the information generated during the campaign accordingly. If voters are not systematically fooled by distortions then their effect is a coarsening of the information available to voters. A variant of our main model analyzes the consequence of such a coarsening.

The underlying uncertainty is about the preferences of the median voter. There are two states, one in which the (median) voter prefers party 1's policy and one in which the voter prefers party 2's policy. We begin with the case of symmetric information. The two parties and the voter are uncertain about the state and learn as information about the policies is revealed. The assumption of public learning reflects the fact that parties take frequent opinion polls to track their standing during a campaign.

We model the flow of information as a continuous-time process. As long as one of the parties provides information, all players observe an informative signal that takes the form of a Brownian motion with a state-dependent drift. Parties must bear the cost of information provision and the game stops when no party is willing to incur that cost. At that point, the median voter picks his preferred policy based on his posterior beliefs. We refer to this game as the "war of information."

The war of information has a unique subgame perfect equilibrium. In that equilibrium each party chooses a *belief threshold* and stops providing information if the posterior belief (of the median voter) is less favorable (from the perspective of the party) than the belief threshold.

The lower a party's cost, the more aggressive is its equilibrium belief threshold and the higher is its probability of winning. When viewed as a game between the two parties, the war of information is a game of strategic substitutes: in response to a more aggressive threshold by the opponent the party chooses a less aggressive threshold. Hence, one party's easy access to resources will have a deterrent effect on its opponent. When parties are patient, the effect of resources on the outcome is limited: even if one of the parties has unlimited access to resources it cannot guarantee a win. When parties discount future payoffs, then a party with unlimited access to resources is guaranteed to win against an opponent with a positive cost.

For voters the parties' thresholds are complements. A more aggressive threshold of one party raises the marginal benefit from an increased threshold by the other party. This complementarity implies that the voters' payoff is highest when the campaigns are "balanced," that is, when they feature two parties with similar costs of providing information. When the parties have sufficiently asymmetric access to resources, voters benefit from regulation that raises the cost of the low-cost party. Such regulation reduces the amount of information provided by the low cost party but increases the amount of information provided by the high cost party. We demonstrate that the latter effect dominates (from the perspective of the voter) if costs are sufficiently asymmetric. We also show that if the regulation benefits voters it must also increase the overall resources spent on the campaign. Hence, regulation that benefits voters reduces the combined payoff of parties.

In US political campaigns substantial effort is devoted to fundraising. The cost parameter in our model should be interpreted as measuring the efficiency of a candidate's (or an advocacy group's) fundraising effort. By placing limits on the amount of money an individual donor can give to a political campaign, US election laws limit the efficiency of fundraising. Moreover, this regulation can be expected to disproportionately affect the low cost party. Consider the case of two parties with an equally large group of supporters. If the supporters of party 1 are wealthier than the supporters of party 2 then limitations on the maximum donation will disproportionally affect the low cost party. Our results show that the median voter may benefit from this type of regulation.

To this point, we have assumed that parties are symmetrically informed. In section 5, we consider a setting where one party is informed of the true state of the world. This describes a situation where the party advocating a policy knows its merit. To provide information to voters, the party must utilize a noisy signal as in the case of symmetric information. One interpretation is that policy makers cannot communicate directly with voters but must rely on noisy intermediaries to transmit information. Alternatively, it may be that voters can understand the details of the proposed policy only over time. In either case, parties cannot simply "disclose" their information. Instead, they must convey information through a costly and noisy process of campaigning.[1]

Specifically, there are two types of parties. A type 1 party advocates the ban of an unsafe technology in accordance with the voters preference while a type 0 party advocates the ban of a safe technology contrary to the voters preference. The voter's prior considers the technology safe enough and therefore the party must provide information to convince the voter to adopt the ban. As before, the party provides hard information in the form of a Brownian motion with a type-dependent drift. However, voters must now take into account that the party has private information and, in particular, must take into account information contained in the party's decision to quit or continue providing information. The natural direction of inference in this game is to interpret a party's decision to continue as evidence in favor of an unsafe technology. We assume that voters beliefs must be monotone in this sense on and off the equilibrium path. We show there is a unique equilibrium satisfying this restriction and refer to it as the *monotone equilibrium.*[2]

[1] The literature on strategic transmission of verifiable information (Milgrom and Roberts (1996), Austen-Smith (1992)) has focused on the incentive to disclose a known signal. This literature assumes that disclosure is costless.

[2] There are also non-monotone equilibria. We discuss non-monotone equilibria at the end of Section 7.

In a monotone equilibrium, type 1 never quits irrespective of the voter's beliefs. Type 0 provides information as long as the voter's belief that the technology is unsafe remains are above a threshold p. When the belief reaches the threshold, type 0 randomizes between quitting and not quitting. The randomization is calibrated so that further evidence of a safe technology is exactly compensated and the voters belief never drops below p. Asymmetric information therefore leads to a *signaling barrier*, i.e., a lower bound on beliefs that cannot be crossed as long as the party provides information. Once the party drops out, the party's type is revealed and voters know the technology is safe.

As long as the party does not quit, the voter remains unconvinced of the technology's safety – even after an overwhelming amount of direct information suggesting that the technology is safe. Moreover, the signaling barrier implies an asymmetric response of voters to information. Information unfavorable to the informed party is discounted while favorable information is not. A relatively small amount of recent information in favor of a ban may offset a large amount of information against a ban. Therefore, the voter may choose in favor of the informed party even if in a non-strategic situation he would not. That is, even if the total public information suggests that the technology is safe, the fact that the informed party continues to provide information renders it unsafe and the voter chooses to ban it.

The probability of an incorrect choice (the ban of a safe technology) depends on the voter's initial prior but is independent of the party's cost. A change in the cost affects the location of signaling barrier and the expected duration of the game but does not affect the probability of an incorrect choice. An increase in the cost has two offsetting effects. On the one hand, it makes it more costly to continue providing information (hence lowering the incentive to provide information) while on the other hand it makes the signal of continuing more informative.

1.1 Related Literature

The war of information resembles a war of attrition. However, there are two key differences. First, in a war of information, players can temporarily quit providing information (when they are ahead) and resume at a later date. In a war of attrition both players must bear a cost for the duration of the game. Second, the resources spent during a war of information generate a payoff relevant signal. If the signal were uninformative and both players incurred the cost for the duration of the game then the war of information reduces to a war of attrition with a public randomization device. The war of information is similar in structure to models of contests (Rosenthal and Rubinstein (1984), Dixit (1987), and rent seeking games (Tullock (1980)). The key difference is that in a war of information the resources are spent to generate decision-relevant information for the voter.

Austen-Smith and Wright (1992) examine strategic information transmission between two competing lobbies and a legislator. They consider a static setup where lobbies may provide single binary signal. Their focus is on whether and when lobbies provide useful information for the legislator. A key incentive problem in Austen-Smith and Wright (1992) and in Austen-Smith (1994) is whether the informed party discloses the information. In our model, this incentive problem is absent. In our symmetric information model all players are uncertain about the median voters preferences and learn as the campaign unfolds. In our asymmetric information model, the party knows which policy the median voter prefers. However, it can only convey this information using a noisy signal and the party does not know in advance how the voters will interpret the signal. Our model fits situations where the informed party cannot simply disclose information but must convey information through a costly and noisy process of campaigning. The Austen-Smith and Wright setting is appropriate when an informed lobby interacts with a sophisticated policy maker to whom information can be conveyed at no cost and without noise.

The literature on strategic experimentation (Harris and Bolton (1999, 2000), Cripps, Keller and Rady (2005)) analyzes situations where agents must incur costs to learn the true state but can also learn from the behavior of others. This leads to a free-riding problem that is the focus of this literature. The information structure in our paper is similar to Harris and Bolton (1999); the signal takes the form of a Brownian motion with unknown drift.[3] However, the incentive problem analyzed in the war of information differs. Parties in the war of information would like to deter opponents from providing information and therefore benefit from a low cost beyond the direct cost saving. In a model of strategic

[3] See also Moscarini and Smith (2001) for an analysis of the optimal level of experimentation in a decision problem.

experimentation, agents have an incentive to free-ride on other players and therefore would like to encourage opponents to provide information.

Our model is related to work in campaign advertising, most notably, Prat (2002).[4] Prat considers a setting where campaigns expenditures are not inherently informative but may signal private information about the candidate's ability. Our model of asymmetric information can be seen as a hybrid between a model of informative advertising and Prat's model of campaign spending. Prat provides an alternative argument for restrictions on political advertising. In Prat's model, a party must choose its policy to cater to the needs of a privately informed campaign donor and the resulting policy bias may lead to lower welfare than in a setting where political advertising is prohibited. By contrast, in our model campaign spending may bias the information provided in equilibrium to the detriment of voters. Clearly, both effects play a role in public policy debates about campaign finance regulation.

Yilankaya (2002) provides a model of evidence production and an analysis of the optimal burden of proof. The model assumes an informed defendant, an uninformed prosecutor and an uninformed judge. This corresponds to our setting with asymmetric information. Yilankaya's model is static; that is, parties commit to a fixed expenditure at the beginning of the game. Yilankaya explores the trade-off between an increased burden of proof and increased penalties for convicted defendants. He shows that an increased penalty may lead to larger errors, i.e., a larger probability of convicting innocent defendants or acquitting guilty defendants. In our model, an increased penalty for a convicted defendant is equivalent to a lower cost of information provision for the defendant. If the defendant is informed then in our case this has no effect on the probability of convicting an innocent defendant or acquitting a guilty defendant.

[4] See also Potters, Soof and Van Winden (1997).

2. The War of Information

The *War of Information* is a three-person, continuous-time game. We refer to players 1 and 2 as *parties* and player 3 as the *voter*. Nature endows one of the two parties with the correct (voter-preferred) position. Then, the parties decide whether or not to provide information about their positions. Once the flow of information stops, the voter makes a decision in favor of one of the two parties. The voter's payoff is 1 if he chooses the party with the correct position and 0 otherwise. A party receives a payoff of 1 if it (or its policy) is chosen by the voter and 0 otherwise. Furthermore, party i incurs a flow cost $k_i/2$ while providing information.

Players are symmetrically informed.[5] Let p_t denote the probability that the voter (and parties) assigns at time t to party i having the correct position and let T denote the time at which the flow of information stops. It is optimal for the voter to decide in player 1's favor if and only if $p_T \geq 1/2$. We say that player 1 (2) is *trailing* at time t if $p_t < 1/2$ ($p_t \geq 1/2$).

We assume that only the trailing player may provide information at time t. Hence, the game stops whenever the trailing player quits. The equilibria analyzed below remain equilibria when this assumption is relaxed and players are allowed to provide information when they are ahead. We discuss the more general case in detail at the end of this section.

We say that the game is running at time t if, at no $\tau \leq t$, a trailing player has quit. As long as the game is running, all three players observe the process X where

$$X_t = \mu t + Z_t \tag{2}$$

and Z is a Wiener process. Hence, X is a Brownian motion with drift μ and variance 1. Players are symmetrically informed and uncertain about the drift μ. The common prior assigns probability 1/2 to the values $\mu = 1/2$ and $\mu = -1/2$. The realization $\mu = 1/2$ implies that party 1 holds the correct position, while $\mu = -1/2$ implies that party 2 holds the correct position. Let

$$p(x) = \frac{1}{1 + e^{-x}} \tag{3}$$

[5] See Section 7 for the case of asymmetric information.

for all $x \in I\!R$; for $x = -\infty$, we set $p(x) = 0$ and for $x = \infty$, we set $p(x) = 1$. A straightforward application of Bayes' Law yields

$$p_t := \text{ Prob}\{\mu = 1/2 \,|\, X_t\} = p(X_t)$$

and therefore, i is trailing if and only if

$$(-1)^{i-1} X_t < 0 \tag{4}$$

In this section, we restrict both parties to stationary, pure strategies. We call the resulting game the *war of information.* In Appendix D we show that this restriction is without loss of generality. Specifically, we show that players choose stationary strategies in any subgame perfect equilibrium of a dynamic game that allows players to revise their strategies at discrete points in time.

A stationary pure strategy for player 1 is a real number $y_1 \leq 0$ ($y_1 = -\infty$ is allowed) such that player 1 quits providing information as soon as X reaches y_1. That is, player 1 provides information when $y_1 < X_t < 0$ and quits as soon as $X_t = y_1$.[6] Similarly, a stationary pure strategy for player 2 is an extended real number $y_2 \geq 0$ such that player 2 provides information when $0 \leq X_t < y_2$ and quits as soon as $X_t = y_2$. Let

$$T = \inf\{t > 0 \,|\, X_t - y_i = 0 \text{ for some } i = 1, 2\} \tag{5}$$

if $\{t \,|\, X_t = y_i \text{ for some } i = 1, 2\} \neq \emptyset$ and $T = \infty$ otherwise. Observe that the game runs until time T. At time $T < \infty$, player 3 rules in favor of player i if and only if $X_T = y_j$ for $j \neq i$. If $T = \infty$, we let $p_T = 1/2$ and assume that both players win.[7] Let $y = (y_1, y_2)$ and let $v_1(y)$ denote the probability that player 1 wins given the strategy profile y; that is, $v_1(y) = \text{Prob}\{p_T > 1/2\}$. The probability that 2 wins is $v_2(y) = 1 - v_1(y)$.

To compute the parties' cost associated with the strategy profile y, define $C : [0, 1] \to \{0, 1\}$ such that

$$C(s) = \begin{cases} 1 & \text{if } s < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

[6] If $y_1 = 0$ then player 1 never provides information.

[7] This specification of payoffs for $T = \infty$ has no effect on the equilibrium outcome since staying in the game forever is not a best responses to any opponent strategy for any probability of winning at $T = \infty$. We chose this particular specification to simplify the notation and exposition.

The expected cost of party 1 given the strategy profile y is

$$c_1(y) = \frac{k_1}{2}\, E \int_0^T C(p_t)dt \tag{6}$$

and the corresponding cost for party 2 is

$$c_2(y) = \frac{k_2}{2}\, E \int_0^T (1 - C(p_t))dt \tag{7}$$

The parties' expected utilities are

$$U_i(y) = v_i(y) - c_i(y) \tag{8}$$

while the voter's expected utility, is

$$U_3(y) = E[\max\{p_T, 1 - p_T\}] \tag{9}$$

Note that $1/2 - p(y_1)$ is the range of beliefs over which party 1 provides information while $p(y_2) - 1/2$ is the corresponding range for party 2. It is convenient to describe the parties' behavior in terms of these ranges. Let

$$\alpha_i := (-1)^{i-1}\big(1 - 2p(y_i)\big)$$

Hence, $\alpha_1 = 1-2p(y_1) \in [0,1]$ and $\alpha_2 = 2p(y_2)-1 \in [0,1]$. For both players, higher values of α_i indicate a greater willingness to bear the cost of information provision. If α_i is close to 0, then party i is not willing to provide much information; i quits at y_i close to zero. Conversely, if $\alpha_i = 1$, then party i continues to provide information no matter how far behind i is (i.e., $y_1 = -\infty$ or $y_2 = \infty$). Without risk of confusion, we write $U_i(\alpha)$, where $\alpha = (\alpha_1, \alpha_2) \in [0,1]^2$ in place of $U_i(y)$.

Lemma 1 below derives a simple expression for the players' payoffs.

Lemma 1: *Player $i = 1, 2$ wins with probability $\alpha_i/(\alpha_1 + \alpha_2)$ and*

$$U_i(\alpha) = \frac{\alpha_i}{\alpha_1 + \alpha_2}\Big(1 - k_i \alpha_j \ln \frac{1+\alpha_i}{1-\alpha_i}\Big) \text{ for } i, j \in \{1,2\} j \neq i$$
$$U_3(\alpha) = \frac{1}{2} + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$$

If $\alpha_i = 1$, *then* $U_i(\alpha) = -\infty$.

The expression for the win-probabilities in Lemma 1 follows from the fact that $p(X_t)$ is a martingale and therefore

$$\Pr(1 \text{ wins })p(y_2) + \Pr(2 \text{ wins })p(y_1) = {}^1\!/\!_2 \tag{10}$$

where the right hand of the above equation is the prior. Substituting $(1-\alpha_1)/2$ for $p(y_1)$ and $(1+\alpha_2)/2$ for $p(y_2)$ yields the expression for the win-probabilities in Lemma 2.

Lemma 2 below utilizes Lemma 1 to establish that the best response of player i to a_j is well-defined, single valued, and differentiable. Furthermore, the war of information is dominance solvable. In Appendix D, we use this last fact to show that the war of information has a unique subgame perfect Nash equilibrium even if nonstationary strategies are permitted. Specifically, we prove uniqueness for a discrete-time approximation of the war of information where parties can revise their strategies at discrete points in time.

The function $B_i : (0,1] \to (0,1]$ is party 1's best response function if

$$U_1(B_1(\alpha_2), \alpha_2) > U_1(\alpha_1, \alpha_2)$$

for all $\alpha_2 \in (0,1]$ and $\alpha_1 \neq B_1(\alpha_2)$. Party 2's best response function is defined in an analogous manner. Then, α_1 is a Nash equilibrium strategy for party 1 if and only if it is a fixed-point of the mapping ϕ defined by $\phi(\alpha_1) = B_1(B_2(\alpha_1))$. Lemma 2 below ensures that ϕ has a unique fixed-point.

Lemma 2: *There exist differentiable, strictly decreasing best response functions* B_i : $(0,1] \to (0,1]$ *for both parties. Furthermore, if* $\alpha^1 \in (0,1)$ *is a fixed-point of* ϕ, *then* $0 < \phi'(\alpha^1) < 1$.

Using Lemma 2, Proposition 1(i) below establishes that the war of information has a unique equilibrium. Proposition 1(ii) shows that as the cost of player i decreases, i becomes more aggressive while his opponent becomes less aggressive. The equilibrium strategy of player i converges to 0 as the cost of that player goes to infinity and converges to 1 as the cost goes to zero. It follows that any strategy profile $\alpha \in (0,1)^2$ can be attained for appropriate costs (k_1, k_2).

Proposition 1: *(i) The war of information has a unique Nash equilibrium. (ii) The equilibrium strategy α_i is strictly decreasing in k_i and strictly increasing in k_j. (iii) For every $\alpha \in (0,1)^2$ there exist (k_1, k_2) such that α is the equilibrium of the war of information with cost (k_1, k_2).*

Proof: Appendix A.

We have assumed that the states have equal prior probability. To capture a situation with an arbitrary prior π, we can choose the initial state $X_0 = x$ so that $p(x) = \pi$. The equilibrium strategies are unaffected by the choice of the initial state and hence if (α_1, α_2) is the equilibrium for $X_0 = 0$ then (α_1, α_2) is also an equilibrium for $X_0 = x$.

The prior affects equilibrium payoffs and win probabilities. For example, if the initial prior is not equal to 1/2 then one of the parties may quit at time 0. If (α_1, α_2) are the equilibrium strategies then for

$$\pi \leq \frac{1-\alpha_1}{2}$$

party 1 quits at time 0 whereas for

$$\pi \geq \frac{1+\alpha_2}{2}$$

party 2 quits at time 0. In those cases, the initial prior is so lopsided that the trailing party does not find it worthwhile to conduct the campaign. The game ends in period 0 and the voter chooses the policy favored by the prior.

When $\pi \in \left[\frac{1-\alpha_1}{2}, \frac{1+\alpha_2}{2}\right]$, the win probabilities must satisfy the following modified version of Equation (10) above:

$$\Pr(1 \text{ wins })p(y_2) + \Pr(2 \text{ wins })p(y_1) = \pi \tag{11}$$

Recall that $p(y_1) = (1-\alpha_1)/2$ and $p(y_2) = (1+\alpha_2)/2$ and therefore

$$\Pr(1 \text{ wins }) = \frac{2\pi - 1 + \alpha_1}{\alpha_1 + \alpha_2}$$

when the prior is π.

The war of information assumes that the drift of X_t is $\mu \in \{-1/2, 1/2\}$ and the variance of X_t is 1. All these assumptions are normalizations that are without loss of generality. Let $\mu_1 > \mu_2$ be the drift parameters and let σ^2 be the variance. Define

$$\delta = \frac{\sigma^2}{(\mu_1 - \mu_2)^2}$$

As we show in Appendix B, the parties' payoffs in the game with arbitrary σ^2, μ_1, μ_2 are

$$U_i(\alpha) = \frac{\alpha_i}{\alpha_1 + \alpha_2}\left(1 - \delta k_i \alpha_j \ln \frac{1+\alpha_i}{1-\alpha_i}\right) \tag{11}$$

while the voter's payoff is unchanged. The difference between (11) and the payoffs described in Lemma 1 is that the cost k_i is multiplied by the parameter δ. Hence, the analysis above extends immediately to the case of general μ_1, μ_2, and σ^2 provided we multiply the cost k_i by δ. The parameter δ measures of the informativeness of the underlying signal. When δ is close to zero then the signal is very informative and when $\delta \to \infty$ the signal becomes uninformative.

2.1 Both Parties Provide Information

Throughout, we have assumed that only the trailing party provides information. We made this assumption to simplify the analysis, in particular, to avoid having to specify the process of information when both advocates provide information.

A general model would allow the leading party to provide information as well. The simplest extension is a model where no additional information is generated when both parties provide information. In that case, it is a dominant strategy for the leading advocate not to provide information if players cannot observe *who* provides information. It can be shown that even if the identity of the information provider is observable, the equilibrium of the simple war of information remains the unique subgame perfect equilibrium.

Perhaps a more natural extension to the case where multiple parties may provide information is to adopt the formulation of Moscarini and Smith (2001) and assume the following signal process:

$$dX_t = \mu + \frac{\sigma}{\sqrt{n_t}} dZ_t$$

where n_t is the number of parties that provide information at time t.[8] In this case, we interpret dX_t as the sample mean of n independent signal draws. If the sample increases from 1 to 2 then the signal variance is reduced by a factor of 2. Hence, multiple parties providing information results in a reduction of the signal variance.

The equilibrium of the war of information characterized above remains an equilibrium in this case. Given a stationary strategy by the opponent, parties have a strict incentive *not* to provide information when they are ahead. To see this, note that p_t is a martingale and therefore a player cannot increase the probability of winning (even if it may change the *speed* of learning) by providing information when he is ahead. Since information provision is costly, it follows that such a deviation would lower the party's payoff and, therefore, the equilibrium strategies of the simple war of information remain equilibrium strategies in this setting. However, we do not have a result that shows uniqueness of equilibrium for this case.

3. Resources, Outcomes and Welfare

The parameters k_1 and k_2 measure the parties' respective costs of campaigning. We interpret these parameters as measuring the effort a party or a candidate must exert to raise funds during a campaign. If k_i is small then the party has easy access to funds while a high k_i indicates that the party finds it difficult to raise money. Proposition 1 implies that party 1's chance of winning is decreasing in k_1 and increasing in k_2. Hence, the party with easier access to resources is more likely to win.

However, the effect of superior access to resources is limited. Consider the case where the prior is 1/2 and payoffs and win probabilities are as described in Lemma 1. Suppose that party 1 has unlimited access to resources and hence k_1 is close to zero. For any fixed k_2 the probability that party 2 wins remains bounded away from zero. To see this, note $\alpha_2 \geq B_2(1)$ and therefore party 2's win-probability $\alpha_2/(\alpha_1 + \alpha_2)$ satisfies

$$\alpha_2/(\alpha_1 + \alpha_2) \geq B_2(1)/(1 + B_2(1))$$

[8] Moscarini and Smith (2001) use this model to analyze the optimal level of experimentation in a decision problem with unknown drift. In their case, n_t represents the number of signals the agent acquires.

Since $B_2(1)$ is strictly positive for every k_2, party 2 wins with a probability that is bounded away from zero.

Next, we examine how the parties' win-probabilities and payoffs are affected by the informativeness of the signal, as measured by $\delta = \sigma^2/(\mu_1 - \mu_2)$. The following proposition shows that if the signal is very informative ($\delta \to 0$) then both parties' payoffs converge to 1/2. In that case, all information is revealed and both parties win with equal probability. If the signal is very uninformative ($\delta \to \infty$) then the parties' payoffs depend on the ratio of costs $r = k_2/k_1$. Let $h : I\!R_+ \to [0, 1]$ be defined as

$$h(r) = \frac{1}{3r}\left(r + 2\sqrt{1 - r + r^2} - 2\right)$$

and note that h is increasing with $h(0) = 0, h(1) = 1/3$ and $h \to 1$ as $r \to \infty$. The following proposition shows that if $k_2/k_1 = r$ then the payoff of party 1 in an uninformative war of information is $h(r)$ and the payoff of party 2 is $h(1/r)$.

Proposition 2: *Let $\alpha = (\alpha_1, \alpha_2)$ be the unique equilibrium of the war of information with the cost structure (k_1, k_2) and precision $1/\delta$. Then,*

$$\text{(i) } \lim_{\delta \to 0} U_j(\alpha) = 1/2;$$
$$\text{(ii) } \lim_{\delta \to \infty} U_i(\alpha) = h(k_j/k_i);$$

for $i, j = 1, 2, j \neq i$. The utility of the voter is decreasing in δ with $\lim_{\delta \to 0} U_3(\alpha) = 1$ and $\lim_{\delta \to \infty} U_3(\alpha) = 1/2$.

Proof: Appendix A.

Since $h(0) = 0$, Proposition 2 (ii) reveals that in the case of an uninformative signal a party's win probability converges to one as its cost goes to zero (and the opponent's cost stays fixed). If the two parties are evenly matched, then both prefer a very informative to a very uninformative signal. To see this, note that for $k_1 = k_2$ the payoff is 1/3 if the signal is uninformative and 1/2 if the signal is very informative. An informative signal leads to a quick resolution of the war of information and the cost of providing information is a vanishing fraction of the overall surplus. By contrast, 1/3 of the overall surplus is spent providing information if the signal is uninformative.

To determine the value of the campaign for the voter, note that without the campaign the voter's payoff is 1/2. Hence the *value of the campaign* W is

$$\begin{aligned} W &= U_3 - 1/2 \\ &= \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \end{aligned}$$

The above expression illustrates the complementary value of the parties' actions for voters. If one party does not provide information ($\alpha_i = 0$) then the campaign has no value for the voter. This complementarity suggests that the voter are best served by "balanced" campaigns, that is, campaigns where the costs of candidates are not too dissimilar. Our next results confirm this intuition.

Let $U_3^*(k_1, k_2)$ denote the voter's equilibrium payoff and let $c^*(k_1, k_2)$ denote the parties' expected equilibrium campaign expenditure as a function of the costs (k_1, k_2). We say that f is a threshold function if it satisfies the following properties:

(i) $f(k) < k$ and there is $z < \infty$ such that $f(k) = 0$ if and only if $k < z$.

(ii) f is strictly increasing for $k \geq z$ and unbounded;

Proposition 3: *There is a threshold function f such that*

(i) $U^*(k_1, k_2)$ is increasing in k_1 if $k_1 < f(k_2)$ and decreasing in k_1 if $k_1 > f(k_2)$

(ii) $c^*(k_1, k_2)$ is decreasing in k_1 if $k_1 < f(k_2)$.

Proof: Appendix A.

Proposition 3 (i) shows that when parties' costs are sufficiently asymmetric, regulation that raises the low-cost party's cost increases voter welfare. Since $f(k) < k$ only the low-cost party can be below the threshold and hence raising the cost of the high-cost party never benefits the voter. Moreover, if the high-cost party has low costs (costs below z) then the threshold is zero. In that case, the voter is always harmed by regulation that raises the cost of campaigns. Figure 1 below illustrates the relation between costs and voter utility.

Proposition 3(ii) shows that if the regulation benefits the voter then it must also raise the resources spent on the campaign.[9] As a result of the regulation, the low-cost

[9] In this interpretation, we assume that the cost is affected through regulation. If the cost were raised through a tax then we must subtract the tax revenue from the cost of campaigning. In that case, we cannot sign the effect of the policy on the (net) campaign expenditure.

party chooses a smaller threshold while the high cost party chooses a larger threshold (by Proposition 1). This implies that the high-cost party's payoff increases while the low cost party's payoff decreases. Proposition 3 (ii) implies that the sum of parties' payoffs decreases as a consequence of any regulation that benefits voters.

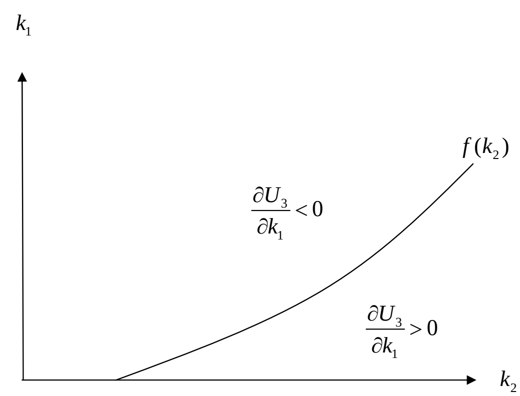

Figure 1

In some situations, regulation cannot target the low-cost party but affects both parties. Our next result shows that a tax on both parties will benefit the voter if the high-cost party's cost is sufficiently large.

Proposition 4: *For every k_1 there is $\bar{k}_2$ such that for $k_2 > \bar{k}_2$*

$$\frac{dU_3^*(k_1 + t, k_2 + t)}{dt}\Big|_{t=0} > 0$$

Proof: Appendix A.

Propositions 3(i) and 4 consider a single voter who is indifferent between the two parties when the states are equally likely. We can interpret this voters' utility as the median utility of a population of voters. Each voter has a threshold γ_i such that for $p = \gamma_i$ voter i is indifferent. If party 1 is the low-cost party, voters with threshold above 1/2 prefer the low-cost party's policy at the median's threshold and voters with threshold below 1/2 prefer the high-cost party at the median's threshold. In that case, any regulation that increases the utility of the median voter also increases the utility of all voters with threshold smaller than 1/2. Thus, a majority of voters benefit from the regulation. However, voters who have a sufficiently strong preference for the low-cost party's policy at the median threshold will be harmed. Thus, in a setting with a population of voters with diverse preferences, Propositions 3(i) and 4 imply only that a majority of voters benefits from the regulation under the stated conditions.

Together, Propositions 3 and 4 provide a rationale for increasing the cost of information provision by parties even when parties provide undistorted information to voters. The key insight is that the war of information is a game of strategic substitutes between parties. Raising the cost of the low-cost party will raise the equilibrium action of the high-cost party. For the median voter the actions of parties are complements and, as a result, the median voter prefers balanced campaigns. However, as we show in Proposition 3 (ii), a regulation that raises the median voter's utility also raises the resources spent during the campaign.

4. Distorted Information

We have assumed that the information generated by the two parties is communicated to the voter without distortion. In this section, we consider a simple variation of the model when parties need not truthfully communicate information.

If information is *verifiable* then by a standard unravelling argument (Milgrom and Roberts (1986)) the parties will truthfully reveal all information to voters. Hence, for a potential distortion to have an effect, it is necessary to assume that some aspects of the parties' information are not verifiable. We will assume that the voter can verify bounds for the realization of the signal process X but not its precise value. There is $\Delta > 0$ such that the voter can verify three ranges for the process X_t: (i) $X_t \leq -\Delta$, (ii) $X_t \in (-\Delta, \Delta)$ and (iii) $X_t \geq \Delta$. In case (i) the signal is unambiguously favorable for party 2; in case (ii) the signal is ambiguous and may favor either party; in case (iii) the signal is unambiguously favorable for party 1. Let $Z_t \in \{-1, 0, 1\}$ be the signal defined as

$$Z_t = \begin{cases} -1 & \text{if } X_t \leq -\Delta \\ 0 & \text{if } X_t \in (-\Delta, \Delta) \\ 1 & \text{if } X_t \geq \Delta \end{cases}$$

Parties observe the signal X_t but can credibly reveal only the coarser signal Z_t to voters. Notice that party 1 always has the incentive to report a larger X_t while party 2 always has the incentive to report a smaller X_t. Thus, no information beyond Z_t can be credibly communicated to voters. On the other hand, by the standard unravelling argument, parties cannot withhold the signal Z_t from voters. Therefore, the communication between parties and voters will result in voters observing the signal Z_t and parties observing the signal X_t. Next, we analyze the war of information under this modified information structure.

Notice that the resulting game is one of asymmetric information between the parties and the voter. The voter must form beliefs about the value of X_t when the game ends. If $Z_t \neq 0$ then the voter prefers one or the other policy for all values X_t. Hence, the details of the voter's beliefs are decision-irrelevant in those cases. By contrast, if the game ends when $Z_t = 0$ then the voter's beliefs matter for the optimal decision. To simplify the exposition, we postulate a decision rule for the voter and thereby turn the game into a symmetric information game between the two parties. We demonstrate below that the postulated behavior for the voter is indeed an equilibrium behavior in an appropriately specified dynamic game. Moreover, we show in appendix C that if we perturb the game to allow a small probability of termination then the postulated behavior is the unique equilibrium behavior for the voter.

The voter's decision rule is as follows: Let τ be the last time prior to period t that $X_\tau \in \{-\Delta, \Delta\}$. Note that τ is well defined since $X_0 = -\Delta$. Let $Y_t = X_\tau$. The voter chooses candidate 1 if $Y_t > 0$ and candidate 2 if $Y_t < 0$. Hence, the voter chooses the candidate favored by the most recent non-zero realization of the signal Z_t.

Given this decision rule for the voter, we define the simple war of information with distortion information. For convenience, we assume that the initial state is $x_0 = -\Delta$. As in section 2, a stationary strategy for party 1 is a real number $y_1 \leq 0$ and a stationary strategy for party 2 is a real number $y_2 \geq 0$. Let T be the random time as defined in (5). Let

$$\xi(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the cost of player 1 is

$$c_1(y) = \frac{k_1}{2} \int_0^T \xi(Y_t)dt \tag{5d}$$

and the cost of player 2 is

$$c_2(y) = \frac{k_2}{2} \int_0^T (1 - \xi(Y_t))dt \tag{6d}$$

The parties' expected utilities are

$$U_i(y) = v_i(y) - c_i(y)$$

where $v_i(y)$ is the probability that party i wins, as defined in section 2.

Let α_i be as defined in section 2 and let

$$\beta := \frac{1 - e^{-\Delta}}{1 + e^{-\Delta}}$$

Notice that β measures the size of the ambiguous interval. If $\beta = 0$ then $\Delta = 0$ and if $\beta = 1$ then $\Delta = \infty$.

Lemma 1B provides a closed form expression for the player's payoffs. For $\beta = 0$, Lemma 1B reduces to Lemma 1 in section 2. In that case, the coarse information has no effect on the voter's behavior. After all, the voter can identify the correct decision for every realization of the signal. If $\Delta > 0$ then the voter's coarse information affects the parties' payoff.

Lemma 1B: *(i) If $\alpha_i \geq \beta$ then party $i \in \{1, 2\}$ wins with probability $\alpha_i/(\alpha_1 + \alpha_2)$ and*

$$U_i(\alpha_1, \alpha_2) = \frac{\alpha_i - \beta}{\alpha_1 + \alpha_2} - k_i \frac{\alpha_j + \beta}{\alpha_1 + \alpha_2} \left(\alpha_i \ln \frac{1 + \alpha_i}{1 - \alpha_i} - \beta \ln \frac{1 + \beta}{1 - \beta} \right)$$

(ii) If $\alpha_1 < \beta$ then party 2 wins with probability 1 and $U_1 = 0, U_2 = 1$.

(iii) If $\alpha_1 \geq \beta$ and $\alpha_2 < \beta$ then party 1 wins with probability $(\alpha_1 - \beta)/(\alpha_1 + \beta)$, the payoff of 1 is $U_1(\alpha_1, \beta)$ and the payoff of player 2 is $2\beta/(\alpha_1 + \beta)$

Proof: See Appendix C.

In the following, we assume that both parties have costs below a threshold $\bar{k}$. This ensures that parties' best responses are greater than β and therefore guarantees an interior equilibrium. Specifically, we assume that

$$k_i < \frac{(1-\beta)}{8\beta} \tag{$*$}$$

As in the case without distortion, the game has a unique equilibrium. The equilibrium is similar to the equilibrium of the simple war of information. Both parties choose a stationary cutoff rules $y_1 < -\Delta, \Delta < y_2$. The difference is that the interval of signal realizations for which parties provide information overlap: Starting from an initial condition $x \leq -\Delta$ party 1 provides information as long as $X_t \in [y_1, \Delta]$. Conversely, starting from an initial condition $x \geq \Delta$ party 2 provides information as long as $X_t \in [-\Delta, y_2]$. Hence, the trailing party provides information until either y_i is reached and the party gives up or the voter has verifiable information that the party no longer trails.

Proposition 5: *Assume costs satisfy* $(*)$*. Then, (i) the war of information with distorted information has a unique Nash equilibrium. (ii) Voter utility is decreasing in* Δ*.*

Proof: Appendix C.

The parties' ability to distort information together with the voter's inability to verify the signal process leads to a form of inertia in the voter's behavior. The trailing party must provide unambiguous evidence that its policy is the better choice before the voter is willing to switch action. Effectively, this raises the cost of the trailing party and, for a given opponent strategy, reduces its willingness to provide information. There is also an equilibrium effect. Parties' strategies are strategic substitutes and, therefore, the lower threshold of the opponent may offset the effect of an increase in Δ. We can show that, in equilibrium, the high cost party always reduces the threshold in response to an increase in Δ. The low cost party may increase its threshold if the asymmetry in costs is sufficiently large. As Proposition 5 (ii) shows, the combined effect of an increase in Δ on the voter's utility is always negative. Even though the low cost party may choose a higher equilibrium strategy, the benefit of this increase is more than outweighed by the reduction in the equilibrium strategy of the high cost party.

Next, we argue that the postulated voter behavior is the equilibrium behavior of an appropriately defined dynamic game. When the costs satisfy (*) then, along the equilibrium path, the game never terminates in the interval $(-\Delta, \Delta)$. Therefore, standard equilibrium concepts cannot tie down the voter's beliefs if the game terminates due to a deviation by one of the parties. This implies that the specified strategy can be rationalized as equilibrium behavior: when the game unexpectedly terminates in the interval $(-\Delta, \Delta)$ the voter has beliefs $p_t < 1/2$ if $Y_t < 0$ and $p_t > 1/2$ if $Y_t > 0$. However, it also suggests that there are other equilibria supported by different voter decision rules. For example, consider the following voter decision rule: if the process terminates with $Z_t = 0$ then the voter chooses party 1 if the time spent in the interval $(-\Delta, \Delta)$ is greater than 1 and party 2 if the time spent in the interval $(-\Delta, \Delta)$ is less than 1. As long as equilibrium strategies are such that the process is not expected to terminate in the interval $(-\Delta, \Delta)$, the voter can assign beliefs that rationalize this behavior after an unexpected termination of the game.

To deal with this multiplicity, we consider the following perturbation of the game. With the constant hazard rate $\lambda > 0$ the game terminates and the voter must choose one of the alternatives. In Appendix C we consider the limit of a sequence of appropriately defined dynamic games as λ converges to zero. We show that, under this perturbation, the voter decision rule described above is the unique equilibrium strategy for the voter.

To see why exogenous termination selects the proposed equilibrium, fix the parties' strategies $y_1 < -\Delta, \Delta < y_2$. Since neither party quits in the interval $[-\Delta, \Delta]$ any termination of the game must be for exogenous reasons. We can calculate the voter's beliefs given $Z_t = 0$ and $Y_t \in \{-\Delta, \Delta\}$. If $Y_t < 0$ the voter must form beliefs p_t conditional on the information $X_\tau = -\Delta$ and $X_{t'} \in [-\Delta, \Delta)$ for $t' \in [\tau, t]$. We claim that $p_t < 1/2$. To see this, we can partition the possible paths $\{X_{t'} | t' \in [\tau, t]\}$ into two sets. Let I_- be all the paths such that $X_{t'} < 0$ for all $t' \in [\tau, t]$ and let I_0 be all the paths such that $X_{t'} = 0$ for some $t' \in [\tau, t]$. Conditional on I_- the probability of state 1 is strictly less than 1/2 and conditional on I_0 the probability of state 1 is equal to 1/2. The latter assertion follows from the symmetry of the interval $(-\Delta, \Delta)$ and the fact that both states are equally likely when $X_{t'} = 0$. Since I_- must have strictly positive probability, it follows that $p_t < 1/2$.[10]

5. Extensions

Next, we consider two extensions of the simple war of information. In the model analyzed above, parties have a constant cost of information provision. If we interpret the cost as measuring a party's ability to raise funds then it seems plausible that the cost depends on a party's standing in the polls. In our model, this can be captured by allowing the cost k_i to be a function of the voter's belief p_t. Section 5.1 below considers a log-linear specification for this cost function. We show how the party's payoff functions must be modified and demonstrate that our analysis is robust to this modification.

We have assumed that parties are patient and there is no cost of delay. Section 5.2 considers the war of information with discounting. The discounted case differs qualitatively from the undiscounted case when one of the parties has unlimited resources (near zero costs). As we show below, a party with near-zero costs captures all the surplus in the discounted case and therefore wins with near certainty. As we have shown in section 3, this is not the case in the undiscounted case.

5.1 State Dependent Cost

In this subsection, we assume that party 1's cost of providing information is decreasing in p_t while party 2's cost of providing information is increasing in p_t. To get a closed form expression, similar to the one in Lemma 1, we assume that costs are linear functions of the log-likelihood ratio $\ln \frac{p_t}{1-p_t}$. Since $X_t = \ln \frac{p_t}{1-p_t}$ this implies that costs are a linear function of the signal X_t. Hence, if party 1 provides information (and $X_t < 0$) then party 1 incurs the cost

$$k_1 \cdot (-X_t)$$

[10] The symmetry of the ambiguous interval is essential for this conclusion. If the ambiguous interval is asymmetric (for example $(-\Delta, 2\Delta)$) then the voter's strategy would be time-dependent. After X has spent a certain amount of time in the interval the voter may switch from one party to the other. This would complicate the computation of payoffs but we conjecture that the qualitative properties of the equilibrium would stay unchanged.

and if party 2 provides information (and $X_t \geq 0$) then party 2 incurs the cost

$$k_2 \cdot X_t$$

Therefore, the cost functions (equations (6) and (7)) must be modified to

$$c_1(y) = k_1\, E \int_0^T \ln \frac{1-p_t}{p_t} C(p_t) dt \tag{6b}$$

and

$$c_2(y) = k_2\, E \int_0^T \ln \frac{p_t}{1-p_t} (1 - C(p_t)) dt \tag{7b}$$

The game is unchanged in all other respects. Lemma 1C below shows how player's payoffs must be modified to account for this simple version of a belief-dependent cost.

Lemma 1C: *Player $i = 1, 2$ wins with probability $\alpha_i/(\alpha_1 + \alpha_2)$ and*

$$U_i(\alpha) = \frac{\alpha_i}{\alpha_1 + \alpha_2}\left(1 - k_i \alpha_j \left(\left(\ln \frac{1+\alpha_i}{1-\alpha_i}\right)^2 + \frac{2}{\alpha_i} \ln \frac{1+\alpha_i}{1-\alpha_i} - 4\right)\right) \text{ for } i, j \in \{1, 2\} j \neq i$$

$$U_3(\alpha) = \frac{1}{2} + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$$

If $\alpha_i = 1$, then $U_i(\alpha) = -\infty$.

The payoffs described in Lemma 1b have qualitatively similar properties as the ones described in Lemma 1. The game is dominance solvable and therefore has a unique equilibrium. The comparative statics described in Proposition 1 continue to hold.

5.2 Discounting

In this subsection, we extend the war of information to include a cost of delay. Players discount future payoffs at a common rate $r > 0$. The game is as described in section 2 with payoffs modified to incorporate discounting.

Let $y_1 \leq 0 \leq y_2$ be stationary strategies of players 1 and 2 respectively. Given y_1, y_2 the random time T is the time at which the game ends (that is, one player quits) and p_t is the probability of the high-drift state. Below, we provide the modified expressions for the party's cost with discounted payoffs. Party 1's expected discounted cost is

$$c_1(y) = \frac{k_1}{2}\, E \int_0^T e^{-rt} C(p_t) dt \tag{5c}$$

Party 2's expected discounted cost is

$$c_2(y) = \frac{k_2}{2} E \int_0^T e^{-rt}(1 - C(p_t))dt \tag{6c}$$

The overall payoff of party 1 is

$$U_1(y) = E\left[e^{-rT}(1 - C(p_T))\right] - c_1(y)$$

The overall payoff of party 2 is

$$U_2(y) = E\left[e^{-rT} C(p_T)\right] - c_2(y)$$

where $C : [0,1] \to [0,1]$ is as defined in section 2. In appendix D we provide closed form expressions for the payoff functions of players 1 and 2. The following result extends Proposition 1 to the discounted war of information.

Proposition 6: *(i) The war of information with discounting has a unique Nash equilibrium. (ii) The absolute value of the equilibrium strategy $|y_i|$ is strictly increasing in k_i and strictly decreasing in k_j for $j \neq i$.*

The next result describes the key difference between the discounted and the undiscounted case. Fix the cost of player 2 and let the cost of player 1 converge to zero. Like in the undiscounted case, the corresponding equilibrium strategy of player 1 converges to $-\infty$, i.e., player 1 never gives up. In the undiscounted case, the equilibrium strategy of player 2 stays bounded away from zero, that is, player 2 provides information even if the opponent has unlimited resources. By contrast, in the discounted case, the equilibrium strategy of player 2 converges to zero, i.e., player 2 gives up immediately. Hence, when a player (in this case player 1) has unlimited resources (but his opponent does not) then this player is almost sure to win the war of information. Moreover, no information is provided and therefore the voter is no better off than in the event where player 1's favorite policy is chosen by default.

Proposition 7: *Let $(k_1, k_2) \to (0, z)$ for some $z > 0$. Then, $y_1(k_1, k_2) \to -\infty$ and $y_2(k_1, k_2) \to 0$.*

To see the intuition for Proposition 6, note that the threshold of the low cost player (player 1) must converge to $-\infty$ as the cost of that player converges to zero. The reason is that the marginal benefit of extending the threshold remains positive for all finite values of the threshold while the cost goes to zero. Since $y_1 \to -\infty$ it follows that the random time at which player 2 can win must converge to infinity (almost surely). Since player 2 discounts future payoffs it follows that the value of winning goes to zero. However, the cost of any strictly positive threshold stays bounded away from zero and therefore the optimal strategy for player 2 is to quit immediately.

6. Asymmetric Information

In this section, we consider the case where party 1 is informed of the state (the drift parameter) while the voter remains uninformed. In this setting, the party may signal a favorable state of the world by not quitting the game despite the arrival of negative information.

To simplify the analysis, we consider the case where party 2 does not provide information. Hence, we consider a one-sided war of information between party 1 and the voter. For the remainder of this section, we refer to party 1 simply as *the party* and denote its cost with $k/2$. Extending the analysis to include an uninformed party 2 is straightforward.[11] Since the behavior of the uninformed party is analogous to the case of symmetric information we omit this extension.

The party is informed and knows the drift of the Brownian motion. Let $X^i, i = 0, 1$, be a Brownian motion with drift $\mu_i = i - 1/2$ and variance 1. Let $\{\Omega, \mathcal{F}, P^i\}$ be a probability space for X^i and let $\mathcal{F}_t^i$ denote the filtration generated by X^i. Type 0 (of party 1) knows that the signal is X^0 (with drift $\mu = -1/2$) and type 1 knows that the signal is X^1 (with drift $\mu = 1/2$). The voter is uncertain about the drift and has a prior $\pi < 1/2$ that the drift is μ_1. The initial state of the signal X_0 is such that $\pi = 1/(1 + e^{-X_0})$. Since $\pi < 1/2$ it follows that $X_0 < 0$. As before, the voter prefers the policy of the party if the probability of state 1 is at least 1/2 and prefers the alternative otherwise. The game ends if the party quits or if the voter's belief reaches the threshold 1/2.

[11] The uninformed party would have the same information as the voter.

To analyze the war of information with asymmetric information, we must introduce mixed strategies. A mixed strategy in this game is described by a stochastic process Q_t^i. Given any sample path ω the value $Q_t^i(\omega)$ specifies the probability that type i quits by time t. Hence, $Q^i(\omega)$ is a cumulative distribution function on $[0, \infty)$ that captures the random quitting time for a given sample path of the Brownian motion. The strategy Q^i is right-continuous and non-decreasing in t for every ω. Moreover, Q_t^i is measurable with respect to $\mathcal{F}_t^i$ to ensure that type i's behavior depends only on information available to the type at time t.

Given a strategy profile $Q = (Q^0, Q^1)$ we can determine the belief of the voter that the party's position is correct. Along histories where $Q_t^i(\omega) < 1$ for some $i = 0, 1$, $p_t(\omega)$ is determined by Bayes' rule:

$$p_t(\omega) = \frac{1 - Q_t^1(\omega)}{1 - Q_t^1(\omega) + (1 - Q_t^0(\omega))e^{-X_t(\omega)}}$$

We say that a belief process p is consistent with the strategy profile Q if the belief process satisfies the above equality for all (ω, t) with $Q_t^i(\omega) \neq 1$ for some $i = 0, 1$. A belief process p is *monotone* if for all $t > \tau$

$$p_t(\omega) \geq \frac{p_\tau(\omega)}{p_\tau(\omega) + (1 - p_\tau(\omega))e^{X_\tau(\omega) - X_t(\omega)}} \tag{M}$$

Hence, a monotone belief process has the property that "not quitting" can never be evidence of type 0. A *monotone equilibrium* is a perfect Bayesian Nash equilibrium consistent with monotone beliefs.

The game ends if $p_t \geq 1/2$ or if the party quits. For a strategy-belief pair (Q, p) define

$$v_t(\omega) = \begin{cases} 1 & \text{if } p_\tau(\omega) \geq 1/2 \text{ for some } \tau \leq t \\ 0 & \text{otherwise} \end{cases}$$

For a given sample path ω, if the game ends in period t, then the payoff of the party is

$$v_t(\omega) - tk/2$$

Let

$$D_t^i(\omega) = (1 - v_t(\omega))Q_t^i(\omega) + v_t(\omega)$$

Hence, $D_t^i(\omega)$ is the probability that the game ends at time t given ω and given that the party is type i. The payoff of type i is therefore given by

$$E\int_{\tau=0}^{\infty}(v_\tau(\omega)-\tau k/2)\,dD_\tau^i$$

where the expectation is taken with respect to P^i. Similarly, for any history $\{X_\tau(\omega)\}_{\tau\leq t}$ the payoff given that history is

$$E\int_{\tau=t}^{\infty}(v_\tau(\omega)-\tau k/2)\,dD_\tau^i$$

where the expectation is taken with respect to $P^i(\cdot|\{X_\tau(\omega)\}_{\tau\leq t})$.

Next, we define a class of strategies Q^z indexed by the parameter $z\in I\!R$. Let $Y_t = \inf_{\tau<t} X_\tau$ and, for any real number z, let $Y_t^z = \min\{0, Y_t - z\}$. Define Q^z to be the strategy that satisfies

$$Q_t^z(\omega) = 1 - e^{Y_t^z(\omega)}$$

Consider the strategy profile $Q = (Q^0, Q^1)$ where $Q_t^0 = Q_t^z$ for some $z<0$ and $Q_t^1 \equiv 0$. In this profile, type 1 never quits and type 0 follows the mixed strategy Q_t^z. To determine a belief process consistent with this profile, first note that $p_t = 0$ conditional on the party quitting. Along any path where the party does not quit at or before t, a straightforward application of Bayes' Law yields that

$$\begin{aligned} p_t &= \frac{1-Q_t^1}{1-Q_t^1+(1-Q_t^0)e^{-X_t}} \\ &= \frac{1}{1+e^{Y_t^z-X_t}} \end{aligned}$$

Note that

$$p_t \geq p(z) = \frac{1}{1+e^{-z}}$$

and, therefore, conditional on the party not quitting, beliefs are bounded below by $p(z)$. We refer to $p(z)$ as the *signaling barrier.* In terms of the associated beliefs, the strategy profile $Q^0 = Q^z, Q^1 = 0$ has the property that *conditional on not quitting* beliefs are bounded below by $p(z)$. This bound is achieved by type 0 quitting at $p_t = p(z)$ with a probability that offsets any negative information from the signal X_t.

Define z^* to be the solution of the following equation:

$$e^{-z^*} + z^* = \frac{k_1 + 1}{k_1} \tag{S}$$

and note that $z^* < 0$. The following proposition shows that there is a unique monotone equilibrium in the one-sided war of information with asymmetric information. In that equilibrium, type 1 never quits and type 0 follows the strategy Q^{z^*}.

Proposition 8: *The strategy profile $Q^0 = Q^{z^*}, Q^1 \equiv 0$ is the unique monotone equilibrium of the one-sided war of information with asymmetric information.*

Proof: Appendix F.

The equilibrium in the asymmetric information case is similar to that of the symmetric information case when $p_t > p(z^*)$. In that case, beliefs are affected only by the information generated by the signal X_t. When $p_t = p(z^*)$ beliefs are also affected by the quit decision of type 0. In fact, type 0 quits at a rate that exactly offsets any negative information revealed by the signal X_t. If the party has not quit and $X_t = x < z^*$ then one of the following must be true: either it is type 1 or it is type 0 but by chance the random quitting strategy had the party continue until time t. The probability of type 0 quitting by time t is $1 - e^{x-z^*}$. Hence, if x is "very negative," the party counters the public information $X_t = x$ with his private information.

An observer who ignores the signaling component might incorrectly conclude that the voter chooses the wrong position. Evidence that in a nonstrategic environment would indicate that the party holds the incorrect position (i.e., $X_T < 0$) will result in the voter adopting the party's favored position. Hence, if the signaling component is ignored, it appears that the voter's posterior is biased in favor of the party conducting the campaign.

When the posterior depends only on the direct information revealed by the signal (as in the case of symmetric information) only the value X_t matters for the posterior at time t and the posterior is independent of the path $(X_\tau)_{\tau<t}$. By contrast, the posterior is path-dependent when the signaling component is taken into account. In particular, recent (positive) public information is given greater weight than past negative information conditional on the party not having quit. To see this, note that for a given value $X_t = x$ the

belief p_t is decreasing in $Y_t := \inf_{\tau \leq t} X_t$. Thus, when the signaling component is ignored the voter appears biased because he puts too much weight on recent information.

The location of the signaling barrier depends on the cost k. Equation (S) reveals that z^* is increasing in k. Notice that the location of the signaling barrier is independent of the initial prior π. When π is below the signaling barrier, that is, $\pi < p(z^*)$, then type 0 quits with strictly positive probability at time 0 so that conditional on not quitting the voter's beliefs jump to $p(z^*)$.

Next, we calculate the win-probability of player 1 as a function of the type. It is easy to see that type 1 (who holds the correct position) wins with probability 1. This follows since type 1 never quits. To compute the win-probability of type 0 we use the martingale property of the voter's beliefs. When the game terminates, the voter's beliefs are either 1/2 (in case the party wins) or 0 (in case the party quits an is revealed to be type 0). Therefore, the following equation must hold:

$$\pi = \frac{1}{2} \cdot (\pi \Pr(\text{type 1 wins}) + (1 - \pi) \Pr(\text{type 0 wins}))$$

Since Pr(type 1 wins) = 1 we have

$$\Pr(\text{type 0 wins}) = \frac{\pi}{1 - \pi}$$

Thus, we have demonstrated the following corollary:

Corollary 1: *The probability that player* 0 *wins in the one-sided war of information is independent of k and given by $\frac{\pi}{1-\pi}$.*

The corollary implies that the amount of information revealed in equilibrium is independent of the cost k. A higher cost has two exactly offsetting effects. On the one hand, it increases the cost of information provision but on the other hand strengthens the signal of not-quitting.

6.1 Non-Monotone Equilibria

We have restricted voter beliefs to be monotone (i.e., satisfy inequality (M)). As is typical in signaling games, there are other equilibria that violate monotonicity. Let z^* be as defined above and recall that $p(z^*)$ is the signaling barrier in a monotone equilibrium. Consider any $z \in [z^*, 0]$. There is a non-monotone equilibrium of the game with the following (pure) strategies. Both types provide information when $X_t \in [z, 0]$ and quit once X_t reaches z. The voter decides against the party if the process terminates at z and in favor of the party if the process terminates at 0. If (out of equilibrium) information is provided when $X_t < z$ then the voter believes to be in state 0 (the low-drift state) with probability 1.

Note that, given these beliefs, both types have an incentive to quit at z. Conversely, since $z \geq z^*$ it follows that both types have an incentive to provide information as long as $X_t \geq z$ and, therefore, those strategies are equilibrium strategies. Notice, that in these equilibria *less* information is provided than in the corresponding game with an uninformed party. The reason is that the informed party is deterred from continuation by the out of equilibrium beliefs of voters. By contrast, in a monotone equilibrium the party only quits if it is the low type and therefore more information is provided than in the corresponding equilibrium with an uninformed party.

7. Conclusion

We analyzed political campaigns in a model where two parties provide information to convince a voter of the merit of their respective policies. A key feature of our model is that information is conveyed to voters through a continuous process. This feature adds tractability but also has substantive implications.

Consider the case where only one party provides information (as in our analysis of asymmetric information). It is optimal for the party to stop the flow of information once the voter is convinced that the proposed policy is as good as the alternative, i.e., when the voter is just indifferent. Because information arrives continuously, the party can indeed stop the flow of information at this point. As a result, the voter receives no surplus: the policy that was optimal given the prior remains optimal at the end of the campaign. Thus, competition between parties is necessary for voters to benefit from campaigns. We show that voters benefit most when parties are equally matched - providing a rationale for regulating political campaigns.

When a party knows the state of the world, the indirect inference from a party's campaign spending will interact with the direct information provided. When the strategic interaction between the party and voters is ignored, the voters' posteriors seem biased in favor of the party conducting the campaign. In particular, we show that no matter how much unfavorable direct information is revealed, voter posteriors cannot drop below a threshold we call the signaling barrier.

8. Appendix A

8.1 Proof of Lemma 1

Let $E\big[C(X_t)|\,\mu = r\big]$ be the expected cost incurred by player 1 given the strategy profile $y = (y_1, y_2)$ and $\mu = r$. (Recall that $\sigma^2 = 1$.) Hence, the expected delay cost of player 1 is:

$$E\big[C(X_t)\big] = 1/2\,E\big[C(X_t)\big|1/2\big] + 1/2\,E\big[C(X_t)\big| -1/2\big] \qquad (A1)$$

First, we will show that

$$E\big[C(X_t)|\mu\big] = \frac{1}{2\mu^2}\left(\frac{1-e^{-2\mu y_2}}{1-e^{-2\mu(y_2-y_1)}}\right)\big(1-e^{2\mu y_1}(1-2\mu y_1)\big) \qquad (A2)$$

For $z_1 \leq 0 \leq z_2$, let $P(z_1, z_2)$ be the probability that a Brownian motion X_t with drift μ and variance 1 hits z_2 before it hits z_1 given that $X_0 = 0$. Harrison (1985) (p. 43) shows that

$$P(z_1, z_2) = \frac{1-e^{2\mu z_1}}{1-e^{-2\mu(z_2-z_1)}} \qquad (A3)$$

For $z_1 \leq 0 \leq z_2$, let $T(z_1, z_2)$ be the expected time a Brownian motion with drift μ spends until it hits either z_1 or z_2 given that $X_t = 0$. Harrison (1985) shows (p. 52) that

$$T(z_1, z_2) = \frac{z_2 - z_1}{\mu}P(z_1, z_2) + \frac{z_1}{\mu}$$

To compute $E\big[C(X_t)|\mu\big]$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{-y_1, \epsilon\}$. If $X_t = \epsilon$ then player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$. If $X_{t+\tau} = 0$ then process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{-y_1, \epsilon\}$ and so on. Clearly, this yields an upper bound to $E\big[C(X_t)|\mu\big]$. Let T^ϵ denote that upper bound and note that

$$T^\epsilon = T(y_1, \epsilon) + P(y_1, \epsilon)(1 - P(-\epsilon, y_2 - \epsilon))T^\epsilon$$

Substituting for $T(y_1, \epsilon)$ and $P(y_1, \epsilon)$ we get

$$\mu T^\epsilon = \left(\frac{(\epsilon - y_1)(1-e^{2\mu y_1})}{1-e^{-2\mu(\epsilon-y_1)}} + y_1\right)\left(1 - \frac{(1-e^{2\mu y_1})(e^{-2\mu\epsilon} - e^{-2\mu y_2})}{(1-e^{-2\mu(\epsilon-y_1)})(1-e^{-2\mu y_2})}\right)$$

and therefore

$$E\big[C(X_t)|\mu\big] \leq \lim_{\epsilon \longrightarrow 0} T^{\epsilon} = \frac{1}{2\mu^2}\left(\frac{1-e^{-2\mu y_2}}{1-e^{-2\mu(y_2-y_1)}}\right)\big(1-e^{2\mu y_1}(1-2\mu y_1)\big)$$

Choosing $\epsilon < 0$ we can compute an analogous lower bound which converges to the right hand side of (A2) as $\epsilon \rightarrow 0$. This establishes (A2).

Recall that $p(y_i) = \frac{1}{1+e^{-y_i}}$ and $\alpha_1 = 1-2p(y_1), \alpha_2 = 2p(y_2)-1$. Then, $(A1), (A2)$ yield

$$E\big[C(X_t)\big] = \frac{2\alpha_1 \cdot \alpha_2}{\alpha_1+\alpha_2} \ln \frac{1+\alpha_1}{1-\alpha_1}$$

Let v be the probability that player 1 wins. Since p_T is a martingale and $T < \infty$

$$vp(y_2) + (1-v)p(y_1) = E(p_T) = 1/2$$

Hence,

$$v = \frac{\alpha_1}{\alpha_1+\alpha_2}$$

The last two display equations yield

$$U_1(\alpha) = \frac{\alpha_1}{\alpha_1+\alpha_2}\left(1 - k_1\alpha_2 \ln \frac{1+\alpha_1}{1-\alpha_1}\right) \tag{A4}$$

A symmetric argument establishes yields the desired result of U_2. □

8.2 Proof of Lemma 2

By Lemma 1, party i's utility is strictly positive if and only if

$$\alpha_i \in \left(0, \frac{e^{\frac{1}{k_i\alpha_j}}-1}{e^{\frac{1}{k_i\alpha_j}}+1}\right)$$

Furthermore, throughout this range, $U_i(\cdot, \alpha_j)$ is twice continuously differentiable and strictly concave in α_i. To verify strict concavity, note that U_i can be expressed as the product of two concave functions f, g that take values in $I\!R_+$, where one function is strictly increasing and the other strictly decreasing. Hence, $(f \cdot g)^{''} = f^{''}g + 2f^{'}g^{'} + fg^{''} < 0$.

Therefore, the first order condition characterizes the unique best response of player i to α_j. Player i's first order condition is:

$$U_i = \frac{2\alpha_i^2 k_i}{1-\alpha_i^2} \tag{A5}$$

Note that ($A5$) implicitly defines the best response functions B_i. Equation ($A5$) together with the implicit function and the envelop theorems yield

$$\frac{dB_i}{d\alpha_j} = \frac{\partial U_i}{\partial \alpha_j} \cdot \frac{(1-\alpha_i^2)^2}{4\alpha_i k_i} \tag{A7}$$

Equation ($A5$) implies

$$\frac{\partial U_i}{\partial \alpha_j} = -\frac{1}{\alpha_1+\alpha_2}\left(U_i + k_i \ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)\right) \tag{A7}$$

Note that ($A7$) implies $\frac{\partial U_i}{\partial \alpha_j} < 0$. The three equations ($A5$), ($A6$), and ($A7$) yield

$$\frac{dB_i}{d\alpha_j} = -\frac{\alpha_i(1-\alpha_i^2)}{2(\alpha_1+\alpha_2)} \cdot \left(1 + \frac{1-\alpha_i^2}{2\alpha_i}\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)\right) \tag{A8}$$

Then using the fact that $\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right) \leq \frac{2\alpha_i}{1-\alpha_i}$ yields

$$\frac{dB_i}{d\alpha_j} \geq -\frac{\alpha_i(1-\alpha_i^2)(2+\alpha_i)}{2(\alpha_1+\alpha_2)} \tag{A9}$$

Hence, since $\phi' = \frac{dB_1}{d\alpha_2}\frac{dB_2}{d\alpha_1}$ we have

$$0 < \phi'(\alpha_1) \leq \frac{\alpha_1(1-\alpha_1^2)(2+\alpha_1)\alpha_2(1-\alpha_2^2)(2+\alpha_2)}{4(\alpha_1+\alpha_2)^2} \tag{A10}$$

Note that the $\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2} \leq 1/2$ and, hence, $\phi'(\alpha_1) < 1$ if

$$(1-\alpha_i^2)(2+\alpha_i) < 2\sqrt{2}$$

It is easy to verify that the left-hand side of the equation above reaches its maximum at $\alpha_i < 1/2$. At such α_i, the left-hand side is no greater than $5/2 < 2\sqrt{2}$, proving that $0 < \phi'(\alpha_1) < 1$. □

8.3 Proof of Proposition 1

Part (i): By Lemma 2, B_i are decreasing, continuous functions. It is easy to see that $B_i(1) > 0$ and $\lim_{r\to 0} B_i(r) = \sqrt{\frac{1}{1+2k_i}}$ (Note that $U_i \to 1$ if $\alpha_j \to 0$ for $j \neq i$). Hence, we can continuously extend B_i and ϕ to the compact interval $[0,1]$, so that ϕ must have a fixed-point. Since B_i is strictly decreasing, $B_i(0) < 1$ implies that 1 is not a fixed-point. Since $B_i(1) > 0$ every fixed-point of ϕ must be in the interior of $[0,1]$. Let r be the infimum of all fixed-points of ϕ. Clearly, r itself is a fixed-point and hence $r \in (0,1)$. Since $\phi'(r) < 1$, there exists $\varepsilon > 0$ such that $\phi(s) < s$ for all $s \in (r, r+\varepsilon)$. Let $s^* = \inf\{s \in (r,1) \,|\, \phi(s) = s\}$. If the latter set in nonempty, s^* is well-defined, a fixed-point of ϕ, and not equal to r. Since $\phi(s) < s$ for all $s \in (r, s^*)$, we must have $\phi'(s^*) \geq 1$, contradicting Lemma 2. Hence, $\{s \in (r,1) \,|\, \phi(s) = s\} = \emptyset$ proving that r is the unique fixed-point of ϕ and hence the unique equilibrium of the war of information.

Part (ii): Consider party 1's best response as a function of both α_2 and k_1. The analysis in Lemma 2 ensures that $B_1 : (0,1] \times I\!R^+ \backslash \{0\} \to (0,1]$ is differentiable. Hence, the unique equilibrium of the war of information is characterized by

$$B_1(B_2(\alpha_1), k_1) = \alpha_1$$

Taking a total derivative and rearranging terms yields

$$\frac{d\alpha_1}{dk_1} = \frac{\frac{\partial B_1}{\partial k_1}}{1 - \frac{d\phi}{d\alpha_1}}$$

where $\frac{d\phi}{d\alpha_1} = \frac{\partial B_1}{\partial \alpha_2} \cdot \frac{dB_2}{d\alpha_1}$. By Lemma 1, $\phi' < 1$. Taking a total derivative of $(A6)$ (for fixed α_2) establishes that $\frac{\partial B_1}{\partial k_1} < 0$ and hence $\frac{d\alpha_1}{dk_1} < 0$ as desired. Then, note that k_1 does not appear in $(A6)$ for player 2. Hence, a change in k_1 affects α_2 only through its effect on α_1 and therefore has the same sign as

$$\frac{dB_2}{dk_1} = \frac{dB_2}{d\alpha_1} \cdot \frac{d\alpha_1}{dk_1} > 0 \qquad (A11)$$

By symmetry, we also have $\frac{d\alpha_2}{dk_2} < 0$ and $\frac{d\alpha_1}{dk_2} > 0$.

Part (iii): By (ii), as k_i goes to 0, the left-hand side of $(A6)$ is bounded away from 0. Hence, $\frac{2\alpha_i^2}{1-\alpha_i^2}$ must go to infinity and therefore α_i must go to 1. Since $U_i \leq 1$ it follows from (A6) that $k_i \to \infty$ implies $\alpha_i \to 0$. Fix (α_1, α_2) and note that $B_i(\alpha_j, \cdot)$ is a continuous function and hence by the above argument there is k_i such that $B_i(\alpha_j, k_i) = \alpha_i$. $\square$

8.4 Proof of Proposition 2

For part (i) suppose i chooses the strategy $\alpha_i = 1 - \epsilon$. Then, for δ sufficiently small we have $U_i \geq \frac{1-\epsilon}{2-\epsilon} - \epsilon$ for $i = 1, 2$. Since ϵ can be chosen arbitrarily small, it follows that $U_i \to 1/2$ as $\delta \to 0$. The first order condition $(A5)$ implies that $\alpha_i \to 1$ which in turn implies that $U_3 \to 1$.

For part (ii) note that $\alpha_i \to 0$ as $\delta \to \infty$. Let $r = k_2/k_1$ and define $a = \alpha_1/\alpha_2$ and $z = \alpha_1^2 \delta k_1$. Then, the first order condition $(A5)$ can be re-written as

$$\frac{1}{1+a}\left(1 - az\frac{\ln\left(\frac{1+\alpha_1}{1-\alpha_1}\right)}{\alpha_1}\right) = \frac{2z}{1-\alpha_1^2}$$
$$\frac{1}{1+a}\left(1 - azr\frac{\ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)}{\alpha_2}\right) = \frac{2azr}{1-\alpha_2^2}$$

These two equations imply that z, a must be bounded away from zero and infinity for large δ. Moreover as $\delta \to \infty$ it must be that $\alpha_i \to 0$ for $i = 1, 2$. Therefore,

$$\frac{\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)}{\alpha_i} \to 2$$

And therefore, the limit solution to the above equations must satisfy

$$\frac{1}{1+a}(1 - 2az) = 2z$$
$$\frac{1}{1+a}(1 - 2azr) = 2azr$$

We can solve the two equations for a, z and find U_i from the first order condition. For party 1, we get $U_1 = 2z = \frac{1}{3r}\left(r + 2\sqrt{1 - r + r^2} - 2\right)$ and for party 2 we get $U_2 = 2azr = \frac{1}{3(1/r)}\left((1/r) + 2\sqrt{1 - (1/r) + (1/r)^2} - 2\right)$.

(iii) To demonstrate that U_3 is decreasing in δ, note that an increase in δ is equivalent to an increase in the costs of both parties by a common factor δ. Therefore, it is sufficient to show that α_i is decreasing in δ if the cost of party i is δk_i. The first order conditions for parties i can be written as:

$$\frac{1}{\delta k_i} = \alpha_j \ln \frac{1+\alpha_i}{1-\alpha_i} + 2\alpha_i \frac{\alpha_1+\alpha_2}{1-\alpha_i^2}$$

From this first order condition, it is straightforward to derive an expression for $\frac{d\alpha_i}{d\delta}$ as a function of α_i, α_j. It can be shown that $\frac{d\alpha_i}{d\delta} < 0$ for all α_1, α_2. This implies part (iii). □

8.5 Proof of Propositions 3 and 4

From Lemma 1, we have:

$$\frac{dU_3}{dk_1} = \frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_2}{dk_1}$$

Since $\alpha_2 = B_2(\alpha_1)$, (A8) and (A11) imply $\frac{dU_3}{dk_1} < 0$ if and only if

$$\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{2(\alpha_1+\alpha_2)} \cdot \left[1 - \alpha_2^2 + \frac{(1-\alpha_2^2)^2}{2\alpha_2} \ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)\right] > 0 \qquad (A12)$$

Define $g : (0,1] \to (0,1]$ by $g(\alpha_1) = \alpha_2$ where

$$\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{2(\alpha_1+\alpha_2)} \cdot \left[1 - \alpha_2^2 + \frac{(1-\alpha_2^2)^2}{2\alpha_2} \ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)\right] = 0$$

First, we show that g is well-defined. For any fixed α_1 the right hand side of (A12) is negative for α_2 sufficiently close to zero and strictly positive for $\alpha_2 = \alpha_1$. Note that $\frac{\alpha_1}{2(\alpha_1+\alpha_2)}$, $1-\alpha_2^2$, and the last term inside the square bracket are all decreasing in α_2. Hence g is well defined.

Note also that the left hand side of (A12) is decreasing in α_1. Hence, g must be increasing.

Since the terms in the brackets add up to less than 1 it follows that $g(z) < z$. Setting $\alpha_1 = 1$, define $\hat{\alpha}_2$ such that the left hand side of (A12) is zero. By the monotonicity of the right hand side of (A12) in α_1 it follows that $g \leq \hat{\alpha}_2$.

Proof of Proposition 3 (i): By part (ii) of Proposition 1, the first term on the left-hand side of $(A12)$ is increasing in k_1. Similarly, $\frac{\alpha_1}{2(\alpha_1+\alpha_2)}$, $1-\alpha_2^2$, and the last term inside the square bracket are all decreasing in k_1. Furthermore, the terms inside the square bracket add up to a quantity between 0 and 1. Hence, $g(z) < z$ and $\alpha_1 \le \alpha_2$ for $k_1 \ge k_2$.

Next, note that, as k_1 goes to 0, α_1 goes to 1 (by Proposition 1(iii) above). Setting $\alpha_1 = 1$, define $\hat{\alpha}_2$ such that the left hand side of (A12) is zero. Let $\bar{r}$ be such that $B_2(1) = \hat{\alpha}_2$. By the monotonicity of the left hand side of (A12) in α_1 it follows that (A13) is greater than zero for all $\alpha_2 > \hat{\alpha}_2$. Conversely, for $\alpha_2 < \hat{\alpha}_2$ there is a unique $\alpha_1 \in (0,1)$ such that the left hand side of (A12) is zero. It follows that there is $f(k_2) > 0$ such that at $k_1 = f(k_2)$, $\frac{dU_3}{dk_1} = 0$. Clearly, there can be at most one such $f(k_2)$. The monotonicity of g and the monotonicity of α_i in k_i imply that $\frac{dU_3}{dk_1} < 0$ for $k_1 > f(k_2)$, $\frac{dU_3}{dk_1} > 0$ for $k_1 < f(k_2)$, and $\frac{dU_3}{dk_1} = 0$ at $f(k_2)$.

The function f is continuous, since $f(k_2)$ is well-defined, (α_1, α_2) are continuous functions of (k_1, k_2), and g is continuous. That f is strictly increasing follows from the strict monotonicity of g.

That $f \to \infty$ as $r \to \infty$ follows from the fact that for every $\alpha_1 > 0$ the left hand side of (A12) is strictly negative for α_2 sufficiently small. □

Proof of Proposition 3(ii): Let party 1 be the low-cost party and let party 2 be the high-cost party. First, we show that the cost of the weak party is increasing in k_2 under the conditions stated in Proposition 3(ii). By Proposition 3(i), we know that $\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)}$ is increasing in k_1. Moreover, by Proposition 1, it follows that α_2 is increasing in k_1. Therefore,

$$c_2(\alpha_1, \alpha_2) = k_2 \frac{\alpha_1\alpha_2}{(\alpha_1 + \alpha_2)} \ln \frac{1+\alpha_2}{1-\alpha_2}$$

must be increasing in k_1.

Next, we show that $c_1(\alpha_1, \alpha_2)$ is increasing in k_1. First, note that inequality (A12) holds if $\alpha_1 \le \frac{3}{2}\alpha_2$ and therefore $\alpha_1 > \frac{3}{2}\alpha_2$ under the hypothesis of Proposition 3(ii). Using the first order condition (A6) to solve for k_1 we find that, in equilibrium,

$$c_1(\alpha_1, \alpha_2) = \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2 \alpha_2 \ln \frac{1+\alpha_1}{1-\alpha_1} \left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} \ln \frac{1+\alpha_1}{1-\alpha_1} + 2\frac{\alpha_1^2}{1-\alpha_1^2}\right)^{-1}$$

We must show that c_1 is increasing if k_1 increases. Note that an increase in k_1 implies a decrease in α_1 and an increase in α_2 by Proposition 1. Hence, it is sufficient to show that the above expression is increasing in α_2 and decreasing in α_1 for $\alpha_1 > 3/2\alpha_2$.

The derivative of the above expression with respect to α_1 is negative if

$$\frac{4\alpha_1^2(\alpha_1+\alpha_2)}{1+\alpha_1\alpha_2} + \left(\ln\frac{1+\alpha_1}{1-\alpha_1}\right)^2 \frac{\alpha_2^2(1-\alpha_1)^2(1+\alpha_1)^2}{(1+\alpha_1\alpha_2)(\alpha_1+\alpha_2)} < 4\alpha_1^2 \ln\frac{1+\alpha_1}{1-\alpha_1}$$

Since the left hand side is increasing in α_2 it suffices to show this inequality for $\alpha_1 = \frac{3}{2}\alpha_2$ and all $\alpha_1 \in (0,1]$. A straightforward calculation reveals this to be the case. A similar calculation shows that the derivative with respect to α_2 is positive for $\alpha_1 \geq \frac{3}{2}\alpha_2$. □

Proof of Proposition 4: Note that since α_1 is increasing in k_2 it follows that

$$\left.\frac{dU_3^*}{dt}\right|_{t=0} \geq \frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2}\left(\frac{d\alpha_2}{dk_1} + \frac{d\alpha_2}{dk_2}\right)$$

From Proposition 4 we know that

$$\left.\frac{dU_3^*}{dt}\right|_{t=0} \geq \frac{\alpha_2^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1+\alpha_2)^2}\frac{d\alpha_2}{dk_1} > 0$$

for k_2 sufficiently large. Since $\frac{d\alpha_1}{dk_1}$ is bounded away from zero for all k_2 it suffices to show that

$$\left(\frac{d\alpha_2}{dk_2}\right) / \left(\frac{d\alpha_2}{dk_1}\right) \to 0$$

as $k_2 \to \infty$ and since $\frac{d\alpha_2}{dk_1} = \frac{d\alpha_2}{d\alpha_1}\frac{d\alpha_1}{dk_1}$ with $\frac{d\alpha_1}{dk_1}$ bounded away from zero for all k_2 it suffices to show that

$$\left(\frac{d\alpha_2}{dk_2}\right) / \left(\frac{d\alpha_2}{d\alpha_1}\right) \to 0$$

Recall that the first order condition is

$$\frac{\alpha_2}{\alpha_1+\alpha_2}(1 - k_2\alpha_1 \ln\frac{1+\alpha_2}{1-\alpha_2}) = \frac{2\alpha_2^2 k_2}{1-\alpha_2^2}$$

and therefore:

$$\left|\left(\frac{d\alpha_2}{dk_2}\right) / \left(\frac{d\alpha_2}{d\alpha_1}\right)\right| = \left|(\alpha_2+\alpha_1)\frac{-2\alpha_2(\alpha_1+\alpha_2) - \alpha_1 \ln\frac{1+\alpha_2}{1-\alpha_2} + \alpha_2^2\alpha_1 \ln\frac{1+\alpha_2}{1-\alpha_2}}{(1-\alpha_2^2)(k_2\alpha_2 \ln(\frac{1+\alpha_2}{1-\alpha_2}+1)}\right|$$

Note that $\alpha_2 \to 0$ as $k_2 \to \infty$ and hence the right hand side of the above expression goes to zero as $k_2 \to \infty$ as desired. □

9. Appendix B

Let $\hat{X}_t$ be a signal with and arbitrary value for the variance $\hat{\sigma}^2$ and arbitrary state-dependent drift $\hat{\mu}_1 > \hat{\mu}_2$. We can rescale time so that each new unit corresponds to $1/\delta = \frac{(\hat{\mu}_1 - \hat{\mu}_2)^2}{\hat{\sigma}^2}$ old units. The cost structure with the new time units is $k_i = \delta \hat{k}_i$, where $\hat{k}_i$ is the cost of party i in the old time units. Let X_i be the signal process in the new time unit and note that the state-dependent drift is $\mu_i = \delta \hat{\mu}_i$ and the variance is $\sigma^2 = \delta \hat{\sigma}^2$. Observe that $(\mu_1 - \mu_2)/\sigma = 1$.

Let

$$Z_i = \frac{\hat{\mu}_1 - \hat{\mu}_2}{\hat{\sigma}^2} \left(X_i - \frac{\mu_1 + \mu_2}{2} t \right)$$

A simple calculation shows that Z_i has the state dependent drift $\mu_1 = 1/2$ and $\mu_2 = -1/2$ and variance 1. Note that Z_i is a deterministic function of X_i. Therefore, the equilibrium with signal X_i must be the same as the equilibrium with signal Z_i. Hence, the game with time renormalized corresponds to the simple war of information analyzed in section 2.

10. Appendix C: Distorted Information

Proof of Lemma 1B: Let $E\big[C(X_T)\big]$ be the expected time player 1 incurs the cost given the strategy profile $y = (y_1, y_2)$ and let $E\big[C(X_T)|\mu\big]$ be the expected time player 1 incurs the cost given μ. Note that the initial state is $-\Delta$ and hence $p(X_0) = p(-\Delta) = 1/(1+e^{\Delta})$. Then:

$$E\big[C(X_T)\big] = p(X_0)\, E\big[C(X_T)\big|1/2\big] + (1 - p(X_0))\, E\big[C(X_T)\big| - 1/2\big] \qquad (E1)$$

For $z_1 \leq 0 \leq z_2$, let $P(z_1, z_2)$ be the probability that a Brownian motion X_t with drift μ and variance 1 hits z_2 before it hits z_1 given that $X_0 = 0$. Harrison (1985) (p. 43) shows that

$$P(z_1, z_2) = \frac{1 - e^{2\mu z_1}}{1 - e^{-2\mu(z_2 - z_1)}} \qquad (A3)$$

For $z_1 \leq 0 \leq z_2$, let $T(z_1, z_2)$ be the expected time a Brownian motion with drift μ spends until it hits either z_1 or z_2 given that $X_t = 0$ (where $z_1 \leq 0 \leq z_2$). Harrison (1985) shows (p. 52) that

$$T(z_1, z_2) = \frac{z_2 - z_1}{\mu} P(z_1, z_2) + \frac{z_1}{\mu}$$

To compute $E\big[C(X_T)|\mu\big]$, let $\Delta \in (0, y_2]$ and note that player 1 bears the cost until $X_t \in \{y_1, \Delta\}$. If $X_t = \Delta$ then player 2 bears the cost until $X_{t+\tau} \in \{-\Delta, y_2\}$. If $X_{t+\tau} = -\Delta$ then the process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{y_1, \Delta\}$ and so on. Thus,

$$E\big[C(X_T)|\mu\big] = T(y_1 + \Delta, 2\Delta) + P(y_1 + \Delta, 2\Delta)(1 - P(-2\Delta, y_2 - \Delta))E\big[C(X_T)|\mu\big]$$

Substituting for $T(y_1 + \Delta, 2\Delta)$, $P(y_1 + \Delta, 2\Delta), P(-2Delta, y_2 - \Delta)$ and then recalling $\alpha_1 = 1 - 2p(y_1), \alpha_2 = 2p(y_2) - 1, \beta = 2p(\Delta) - 1$ with $p(z) = \frac{1}{1+e^{-z}}$ we obtain

$$E\big[C(X_t)\big] = \frac{2(\beta + \alpha_2)}{\alpha_1 + \alpha_2}\left(\alpha_1 \ln \frac{1+\alpha_1}{1-\alpha_1} - \beta \ln \frac{1+\beta}{1-\beta}\right)$$

Let v be the probability that player 1 wins. Since p_t is a martingale and $T < \infty$

$$vp(y_2) + (1 - v)p(y_1) = E[p_T] = p(-\Delta)$$

Hence,

$$v = \frac{\alpha_1 - \beta}{\alpha_1 + \alpha_2}$$

The last two displayed equations yield part (i) of the Lemma. Parts (ii) and (iii) are obvious. □

10.1 Proof of Proposition 5:

We begin by deriving the best response of player 1. A direct calculation reveals that the utility function of player i is concave in α_i. For $\alpha_i > \beta$ for $i = 1, 2$ a the first order condition for an optimal choice is:

$$\frac{2\alpha_1(\alpha_1 + \alpha_2)}{(1 - \alpha_1^2)} + \alpha_2 \ln \frac{1+\alpha_1}{1-\alpha_1} = 1/k_i - \beta \ln \frac{1+\beta}{1-\beta}$$

Our bound on k_i (inequality (*) in section 4) implies that any solution to the above equation satisfies $\alpha_i > \beta$. We conclude that the best response of player i is a continuous function that maps $[\beta, 1]$ to $[\beta, 1]$. A standard fixed point argument (as in the proof of Proposition 1 above) then yields existence of an equilibrium.

To prove uniqueness, note that from the first order condition it follows that $\frac{d\alpha_i}{d\alpha_j}$ is independent of β. Therefore, the argument given in the proof of Proposition 1 applies to this case as well.

To prove part (ii) we will show that (1) $\alpha_1(\Delta) + \alpha_2(\Delta)$ is decreasing in Δ and (2) $\alpha_i(\Delta)$ is decreasing in Δ if $\alpha_i \leq \alpha_j$. It is straightforward to show that (1) and (2) imply that the utility of the voter is decreasing in Δ. Rewrite the first order condition as

$$\frac{2\alpha_1(\alpha_1 + \alpha_2)}{(1 - \alpha_1^2)} + \alpha_2 \ln \frac{1 + \alpha_1}{1 - \alpha_1} = 1/k_i - b$$

where $b = \beta \ln \frac{1+\beta}{1-\beta}$. Since b is a strictly increasing function of Δ it suffices to show that $d\alpha_1/db + d\alpha_2/db < 0$ and $d\alpha_i/db < 0$ for $\alpha_i \leq \alpha_j$. Let $g(x, y) = \frac{2x(x+y)}{(1-y^2)} + y \ln \frac{1+x}{1-x}$ and denote with g_1, g_2 the partial derivatives of g. Note that

$$g_1(x, y) = 4(x + y)/(1 - x^2)^2$$

and

$$g_2(x, w) = 2x/(1 - x)^2 + \ln \frac{1 + x}{1 - x}$$

It is straightforward to show that

$$g_1(x, y) - g_2(z, w) > 0 \tag{$**$}$$

for all $x, y, z, w \in (0, 1)$ with $z \leq x$. An application of the implicit function theorem yields

$$\frac{d\alpha_i}{db} = \frac{g_2(\alpha_i, \alpha_j) - g_1(\alpha_2, \alpha_1)}{g_1(\alpha_1, \alpha_2)g_1(\alpha_2, \alpha_1) - g_2(\alpha_1, \alpha_2)g_2(\alpha_2, \alpha_1)} \tag{$***$}$$

(1) and (2) now follow from $(**)$ and $(***)$. □

A strategy of player i is a random stopping time that describes the player's decision to quit the game. We denote with τ^i the strategy of player i. For any sample path ω, $\tau^i(\omega)$ is the time at which player i quits. The stochastic process $W_t \in \{0, 1\}$ describes the decision rule of the voter. If $W_t(\omega) = 1$ then given the sample path ω if the process stops at time t the voter chooses party 1 whereas if $W_t(\omega) = 0$ then the voter chooses party 2. We require that $W_t(\omega) = 1$ if $X_t(\omega) \geq \Delta$ and $W_t(\omega) = 0$ if $X_t(\omega) \leq -\Delta$. The reason for this restriction is that in these cases the sign of the Brownian motion is observable to voters.

Player i's quit decision leads to a termination of the game if and only if player i is trailing. Define the random stopping times T^1, T^2 as follows. If $W_{\tau^1(\omega)}(\omega) = 0$ (that is, player 1 quits when trailing) then $T^1(\omega) = \tau^1(\omega)$. If $W_{\tau^1(\omega)}(\omega) = 1$ (that is, player 1 quits when ahead) then $T^1(\omega) = \infty$. Hence, if player 1 quits when ahead then the quit decision is ignored.[12] Similarly, if $W_{\tau^2(\omega)}(\omega) = 1$ (that is, player 2 quits when trailing) then $T^2(\omega) = \tau^2(\omega)$. If $W_{\tau^2(\omega)}(\omega) = 0$ (that is, player 2 quits when ahead) then $T^2(\omega) = \infty$. Let $T = \min\{T^1, T^2\}$.

The game may also terminate for exogenous reasons. There is $\lambda > 0$ that represents the rate at which the game terminates. Whenever the game terminates the winner is player 1 if $W_t(\omega) = 1$ and player 2 if $W_t(\omega) = 0$.

The payoff of player 1 is

$$E\int_0^T \lambda e^{-\lambda t}\left(W_t - \frac{k_1}{2}\int_0^t (1-W_\tau)d\tau\right)dt + Ee^{-\lambda T}\left(W_T - \frac{k_1}{2}\int_0^T (1-W_\tau)d\tau\right)$$

and the payoff player 2 is

$$E\int_0^T \lambda e^{-\lambda t}\left(1 - W_t - \frac{k_2}{2}\int_0^t W_\tau d\tau\right)dt + Ee^{-\lambda T}\left(1 - W_T - \frac{k_2}{2}\int_0^T W_\tau d\tau\right)$$

Lemma C1: *Assume (*) holds. Let* $X_t(\omega) \in [-\Delta, \Delta]$ *and* $W_t(\omega) = 0$. *Then, there is* $\bar{\lambda} > 0$ *such that, for* $\lambda < \bar{\lambda}$ *and any optimal strategy* τ^1 *for player* 1, $\tau^1(\omega) \neq t$.

Proof: Assume that (i) player 2 never quits ($\tau^2 = \infty$) and (ii) $W_t(\omega) = 0$ for all ω with $X_t(\omega) \in [-\Delta, \Delta]$. Clearly, this yields a lower bound to the payoff of player 1. Suppose that the initial state is $x_0 = \Delta$. Let y_1 be the value of X_t when player 1 chooses to quit. We must show that $y_1 \leq -\Delta$ in any optimal plan. Since the optimal cutoff is independent of the initial condition, this proves the Lemma.

[12] We assume that if a player tries to quit when ahead then this player's quit decision is ignored for the remainder of the game. This is done to simplify the notation and has no substantive role in the analysis.

Let $\gamma = 1/(1+e^{-\Delta})$. Following the lines of the proof of Lemma 1 above and taking a limit as $\lambda \to 0$ it is straightforward to show that the cost of player 1 converges to (for $y_1 < 0$)

$$k_1\gamma(1 + e^{y_1-\Delta}(y_1 - \Delta - 1) + (1-\gamma)(\Delta - 1 - y_1 + e^{y_1-\Delta})$$

as $\lambda \to 0$. The probability that player 1 wins converges to

$$\gamma(1 - e^{y_1-\Delta})$$

as $\lambda \to 0$. Hence, the payoff of player 1 converges to

$$\gamma(1 - e^{y_1-\Delta}) - k_1\gamma(1 + e^{y_1-\Delta}(y_1 + \Delta - 1)) + (1-\gamma)(\Delta - 1 - y_1 + e^{y_1-\Delta})$$

We must show that this expression is strictly decreasing in y_1 for all $y_1 \geq -\Delta$. Since the expression is concave in y_1 it suffices to show that the derivative at $y_1 = -\Delta$ is strictly negative. A straightforward calculation reveals this to be the case if

$$k_1 < \frac{1-\gamma}{2\gamma - 1 + 2\Delta(1-\gamma)}$$

Since $\gamma = \frac{1+\beta}{2}$ and $\ln\frac{1+\beta}{1-\beta} < \frac{2\beta}{1-\beta}$ for $\beta \in (0,1)$ the right hand side of the last displayed equation is less than $\frac{1-\beta}{8\beta}$ and therefore

$$k_1 < \frac{1-\beta}{8\beta} \qquad (*)$$

is sufficient for the conclusion of the Lemma. □

Lemma C2: *$W_t(\omega) = 1$ if $Y_t(\omega) > 0$ and $W_t(\omega) = 0$ if $Y_t < 0$.*

Proof: Note that since players do not quit it follows that any termination must be due to an exogenous termination. By the argument provided in the text, this implies that $p_t - 1/2$ has the same sign as Y_t. It follows that the voter decision rule is as described in the Lemma. □

11. Appendix D: Nonstationary Strategies and Subgame Perfection

In this section, we relax the restriction to stationary strategies. Our objective is to show that the unique equilibrium of the war of information is also the unique subgame perfect equilibrium of the dynamic game.

With nonstationary strategies, it is possible to have Nash equilibria that fail subgame perfection. To see this, let $\hat{\alpha}_2 = B_2(1)$ and $\hat{\alpha}_1 = B_1(\hat{\alpha}_2)$, where B_i are the stationary best response functions analyzed in section 2. Hence, $\hat{\alpha}_2$ is party 2's best response to an opponent who never quits and $\hat{\alpha}_1$ is party 1's best response to an opponent who quits at $\hat{\alpha}_2$.

Define the function $a_i : I\!R \to [0, 1]$ as

$$a_i(x) = (-1)^{i-1}(1 - 2p(x))$$

where p is as defined in (3). Consider the following strategy profile: $\alpha_2 = \hat{\alpha}_2$ and $\alpha_1 = \hat{\alpha}_1$ if $a_2(X_\tau) < \hat{\alpha}_2$ for all $\tau < t$ and $\alpha_1 = 1$ otherwise. Hence, party 2 plays the stationary strategy $\hat{\alpha}_2$ while party 1 plays the strategy $\hat{\alpha}_1$ along any history that does not require party 2 to quit. But, if 2 deviates and does not quit when he is supposed to, then party 1 switches to the strategy of never quitting.

To see why this is a Nash equilibrium, note that 1's strategy is optimal by construction. For player 2, clearly, quitting before α reaches $\hat{\alpha}_2$ is suboptimal. Not quitting at $\hat{\alpha}_2$ is also suboptimal since such a deviation triggers $\alpha_1 = 1$. However, the strategy profile is not subgame perfect because the prescribed behavior for 1 after a deviation by 2 is suboptimal: at any X_t such that $a_1(X_t) < \hat{\alpha}_1$, party 1 would be better off quitting.

To simplify the analysis, we will utilize a discrete version of the war of information: parties may revise their action α_i only at times $t \in \{0, \Delta, 2\Delta, \ldots\}$. The initial state is x_0, i.e., $X_0 = x_0$. We refer to $t = n\Delta$ as period n. Each period n, player i chooses $\alpha_i \in [0, 1]$. The game ends at $t \in [(n-1)\Delta, n\Delta]$ if

$$t = \inf\{\tau \in [(n-1)\Delta, n\Delta] \,|\, a_i(X_\tau) \geq \alpha_i \text{ for some } i = 1, 2\}$$

If

$$\{\tau \in [(n-1)\Delta, n\Delta] \,|\, a_i(X_\tau) \geq \alpha_i \text{ for some } i = 1, 2\} = \emptyset$$

the game continues and the players choose new α_i's in every period. Note that $\alpha_i \leq a_i(x_0)$ means that player i quits immediately.

A pure strategy for player i in period n associates with every history $(X_0, \ldots, X_{n\Delta})$ an action:

Definition: *A pure strategy for player i is a sequence $f^i = (f_0^i, f_1^i, \ldots)$ such that $f_n^i : [0,1]^n \to [0,1]$ is a measurable function for all n.*

Let n^* be the smallest integer n such that for some $t \in [(n-1)\Delta, n\Delta]$ and some $i = 1, 2$

$$a_i(X_t) \geq f_n^i(X_0, \ldots, X_{(n-1)\Delta})$$

If $n^* = \infty$, set $T = \infty$. If $n^* < \infty$ let

$$T = \inf\{t \in [n^*\Delta, (n^*+1)\Delta] \,|\, a_i(X_t) \geq f_{n^*}^i(X_0, \ldots, X_{(n^*-1)\Delta}) \text{ for some } i = 1, 2\}$$

The game ends at time T. Given the definition of T, the payoffs of the game are defined as in section 2 section (expressions (6)-(9)).

The parties' payoffs following a history $\zeta = (x_0, x_1, \ldots, x_{k-1})$ are defined as follows: Let $\hat{f} = (\hat{f}^1, \hat{f}^2)$ where $\hat{f}_n^i(\hat{x}_0, \ldots, \hat{x}_n) = f_{n+k}^i(\zeta, \hat{x}_0, \ldots \hat{x}_n)$ for all $n \geq 0$. Hence, we refer to $\zeta \in [0,1]^n$ as a subgame and let $U_{(\zeta, \hat{x}_0)}^i(f) = U_{\hat{x}_0}^i(\hat{f})$.

Definition: *The strategy profile f is a subgame perfect Nash equilibrium if and only if*

$$U_{(\zeta,\hat{x}_0)}^1(f) \geq U_{(\zeta,\hat{x}_0)}^1(\tilde{f}^1, f^2)$$
$$U_{(\zeta,\hat{x}_0)}^2(f) \geq U_{(\zeta,\hat{x}_0)}^2(f^1, \tilde{f}^2)$$

for all $\tilde{f}^1, \tilde{f}^2, \zeta$.

Let E be the set of all subgame perfect Nash equilibria and let E_i be the set of al subgame perfect Nash equilibrium strategies of player i; that is,

$$E^i = \{f^i \,|\, (f^i, f^j) \in E \text{ for some } f^j, j \neq i\}$$

Let $\alpha = (\alpha_1, \alpha_2)$ be the unique equilibrium of the war of information studied in section 2. Without risk of confusion, we identify $\alpha' \in [0,1]$ with the constant function $f_n^i = \alpha'$

and the stationary strategy $f^i = (\alpha', \alpha', \ldots)$. The proposition below establishes that the stationary strategy profile α is the only subgame perfect Nash equilibrium of the game.

Proposition D1: *The strategy profile α is the unique subgame perfect Nash equilibrium of the discrete war of information.*

Next, we provide a proof of Proposition D. We begin by proving two Lemmas.

Lemma D1: *Let $f^i = (f_0^i, \ldots)$, $f^j = (f_0^j, \ldots)$, and $\tilde{f}^j = (\tilde{f}_0^j, \ldots)$ for $i = 1, 2$ and $j \neq i$ and let $\tilde{f}_n^j(\zeta) \geq f_n^j(\zeta)$ for every n and $\zeta \in [0,1]^n$. Then, $U_{x_0}^i(f^i, f^j) \geq U_{x_0}^i(f^i, \tilde{f}^j)$.*

Proof: Consider any sample path $X(\omega)$. Let $T(\omega), \tilde{T}(\omega)$ denote the termination dates corresponding to the strategy profile (f^i, f^j) and $(f^i, \tilde{f}^j)$ respectively. Note that $T(\omega) \leq \tilde{T}(\omega)$ and therefore the cost of player i is larger if the opponent chooses $\tilde{f}^j$. Furthermore, if $T(\omega) < \tilde{T}(\omega)$ then player i wins along the sample path $X(\omega)$ when the strategy profile is (f^i, f^j). Therefore, the probability of winning is higher for player i under (f^i, f^j) than under $(f^i, \tilde{f}^j)$. $\square$

Lemma D2: *Let $f^i = (a_i, a_i, \ldots)$ be a stationary strategy and for $j \neq i$, let $f^j = (f_0^j, B_j(a_i), B_j(a_i), \ldots)$. (i) If $B_j(a_i) < \alpha_j(x_0) < f_0^j(x_0)$, then $U_{x_0}^j(f^1, f^2) < 0$ and (ii) if $B_j(a_i) \geq f_0^j(x_0) > \alpha_j(x_0)$, then $U_{x_0}^j(f^1, f^2) > 0$.*

Proof: Let $V(a, b)$ denote the payoff of player j if $f_0^j = b$ and $f_n^j, n \geq 1$ are chosen optimally, $f^i = (a_i, \ldots)$ and the initial state is $\alpha^{-1}(a)$. Let $b^* = \arg\max_{b \in [0,1]} V(1/2, b)$. It is easy to see that V is continuous and hence b^* is well defined. Next, we show that $V(a, b^*) \geq V(a, b)$ for all $a \in [0,1]$ and for all $b \in [b^*, 1]$. To prove this, assume that $b > b^*$ and note that $V(a, b) - V(a, b^*) = c(a)V(b^*, b)$ where $c(a)$ is the probability that $X(t)$ reaches the state $y = \alpha_j^{-1}(b^*)$ for some $t \in [0, \Delta]$ if the initial state is $\alpha_j^{-1}(a)$. Since a is arbitrary it follows from the optimality of b^* that $V(b^*, b) \leq 0$. Since the decision problem is stationary, it follows that $f_n^j = b^*$ is a best response to $f^i = (a_i, \ldots)$. This in turn implies that $b^* = B_j(a_i)$ and $U_{x_0}^j(f^1, f^2) \leq 0$ if $B_j(a_i) < \alpha_j(x_0) < f_0^j(x_0)$. Let $b = f_0^j(x_0)$. If $U_{x_0}^j(f^1, f^2) = 0$ then by the argument above $f^j = (b, b, \cdots)$ is also a best response to $(a_i, a_i, \ldots)$. But this contradicts the fact that $B_j(a_i)$ is unique. Hence, a strict inequality must hold and part (i) of the Lemma follows. Part (ii) follows from a symmetric argument. □

Proof of Proposition C: Let

$$\bar{a}_i = \sup\{\alpha_i(x) \,|\, f^i = (f^i_0, \ldots) \in E^i, f^i(x) > \alpha_i(x) \text{ for some } x\}$$
$$\underline{a}_i = \inf\{\alpha_i(x) \,|\, f^i = (f^i_0, \ldots) \in E^i, f^i(x) \le \alpha_i(x) \text{ for some } x\}$$

Hence, $\underline{a}_i$ and $\bar{a}_i$ are, respectively, the least and most patient actions for i observed in any subgame perfect Nash equilibrium. Clearly, $\underline{a}_i \le \alpha_i \le \bar{a}_i$.

First, we show that (i) $B_2(\underline{a}_1) \ge \bar{a}_2$. To see this note that if the assertion is false, then there exists x_0,$(f^1, f^2) \in E$ such that $f^2_0(x_0) > \alpha_2(x_0) > B_2(\underline{a}_1)$. By Lemma D1 and part (i) of Lemma D2, $U^2_{x_0}(f^1, f^2) \le U_{x_0}(\underline{a}_1, f^2) < 0$, contradicting the fact that $(f^1, f^2) \in E$.

Next, we prove that (ii) $B_1(\bar{a}_2) \le \underline{a}_1$. If the assertion is false, then there exists x_0,$(f^1, f^2) \in E$ such that $f^1_0(x_0) \le \alpha_1(x_0) < B_1(\bar{a}_2)$. Then, $0 = U^1_{x_0}(f^1, f^2)$ and by Lemma D1 $U^1_{x_0}(f^1, f^2) \ge U^1_{x_0}(\tilde{f}^1, f^2) \ge U^1_{x_0}(\tilde{f}^1, \bar{a}_2)$ for all $\tilde{f}^1$. By Lemma D2 part (ii), there exists $\tilde{f}^1$ such that $U^1_{x_0}(\tilde{f}^1, \bar{a}_2) > 0$ and hence $U^1_{x_0}(f^1, f^2) > 0$, a contradiction.

The two assertions (i) and (ii) above together with the fact that B_i is nonincreasing yield $\phi(\underline{a}_1) = B_1(B_2(\underline{a}_1)) \le B_1(\bar{a}_2) \le \underline{a}_1$. Since the slope of ϕ is always less than 1 (Lemma 2), we conclude that $\alpha_1 \le \underline{a}_1$ and therefore $\alpha_1 = \underline{a}_1$. Symmetric arguments to the ones used to establish (i) and (ii) above yield $B_2(\bar{a}_1) \le \underline{a}_2$ and $B_1(\underline{a}_2) \ge \bar{a}_1$. Hence, $\phi(\bar{a}_1) = B_1(B_2(\bar{a}_1)) \ge B_1(\underline{a}_2) \ge \bar{a}_1$ and so $\alpha_1 \ge \bar{a}_1$ and therefore $\alpha_1 = \underline{a}_1 = \bar{a}_1$ proving that α_1 is the only action that 1 uses in a subgame perfect Nash equilibrium. Hence, $E_1 = \{\alpha_1\}$ and therefore $E = \{(\alpha_1, \alpha_2)\}$ as desired. □

12. Appendix E: Extensions

12.1 Proof of Lemma 1C

The proof follows the lines of the proof of Lemma 1 above. Let $c_1(y|\mu)$ be the expected cost incurred by player 1 given the strategy profile $y = (y_1, y_2)$ and the drift μ. Hence, the expected cost of player 1 is:

$$c_1(y) = 1/2c_1(y|1/2) + 1/2c_1(y|-1/2) \qquad (D1)$$

First, we will show that

$$\begin{aligned} c_1(y|1/2) =& 2k_1 \frac{1-e^{-y_2}}{1-e^{y_1-y_2}}(e^{y_1}(2-y_1^2) - 2(y_1+1)) \\ c_1(y|-1/2) =& 2k_1 \frac{e^{y_2}-1}{e^{y_2}-e^{y_1}}(2e^{y_1}(1-y_1) + y_1^2 - 2) \end{aligned} \qquad (D2)$$

For $z_1 \leq 0 \leq z_2$, let $P(z_1, z_2)$ be the probability that a Brownian motion X_t with drift μ and variance 1 hits z_2 before it hits z_1 given that $X_0 = 0$. Harrison (1985) (p. 43) shows that

$$P(z_1, z_2) = \frac{1-e^{2\mu z_1}}{1-e^{-2\mu(z_2-z_1)}} \qquad (A3)$$

For $z_1 \leq 0 \leq z_2$, let

$$C(z_1, z_2|\mu) = E\int_0^T X_t dt$$

where X_t is a Brownian motion with drift μ and T is the random time at which $X_t = z_1$ or $X_t = z_2$ (where $z_1 \leq 0 \leq z_2$). Harrison (1985) Proposition (3) on page 49 provides an expression for

$$E\int_0^T e^{-\lambda t} X_t dt$$

Taking the limit as $\lambda \to 0$ yields

$$\begin{aligned} C(z_1, z_2|1/2) =& \frac{z_2(z_2-2-2z_1) + e^{z_1}(z_2-z_1)(z_1-z_2+2) + e^{z_1-z_2}z_1(z_1+2)}{1-e^{z_1-z_2}} \\ C(z_1, z_2|-1/2) =& \frac{z_1(z_1-2) + e^{z_1-z_2}z_2(-2z_1+z_2+2) + e^{-z_2}(-z_1+z_2+2)(z_1-z_2)}{1-e^{z_1-z_2}} \end{aligned}$$

To compute $c_1(y|\mu)$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{y_1, \epsilon\}$. If $X_t = \epsilon$ then player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$. If $X_{t+\tau} = 0$

then the process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{-y_1, \epsilon\}$ and so on. Clearly, this yields an upper bound to $c_1(y|\mu)$. Let $T^\epsilon(\mu)$ denote that upper bound and note that

$$T^\epsilon(\mu) = k_1 C(y_1, \epsilon|\mu) + P(y_1, \epsilon)(1 - P(-\epsilon, y_2 - \epsilon))T^\epsilon(\mu)$$

Substituting for $C(y_1, \epsilon|\mu)$ and taking the limit as $\epsilon \to 0$ establishes the claim above. Choosing $\epsilon < 0$ we can compute an analogous lower bound which converges to the right hand side of (D2) as $\epsilon \to 0$. This establishes (D2).

Recall that $p(y_i) = \frac{1}{1+e^{-y_i}}$ and $\alpha_1 = 1 - 2p(y_1), \alpha_2 = 2p(y_2) - 1$. Substituting these expressions into $(D1), (D2)$ yields

$$c_1(y) = k_1 \frac{\alpha_1 \alpha_1}{\alpha_1 + \alpha_2} \left(\left(\ln \frac{1+\alpha_1}{1-\alpha_1} \right)^2 + \frac{2}{\alpha_1} \ln \frac{1+\alpha_1}{1-\alpha_1} - 4 \right)$$

The win probability is identical to Lemma 1. □

12.2 Discounting

We define

$$\alpha = (\mu^2 + 2r)^{1/2} - \mu = ((1/2)^2 + 2r)^{1/2} - (1/2)$$

$$\beta = (\mu^2 + 2r)^{1/2} - \mu = ((1/2)^2 + 2r)^{1/2} - (1/2)$$

and

$$x_1 = e^{y_1}, x_2 = e^{-y_2}$$

Let $y_1 < 0 < y_2$ and therefore $x_i \in [0, 1]$ with a lower x_i indicating a larger (in absolute value) threshold.

Lemma E1: *The utility of player i is*

$$\frac{1 - x_i^{\alpha+\beta}}{1 - (x_i x_j)^{\alpha+\beta}} \frac{x_j^\alpha + x_j^\beta}{2} - \frac{k_i}{4r} \frac{(1 - x_i^\alpha)(1 - x_i^\beta)(1 - x_j^{\alpha+\beta})}{1 - (x_i x_j)^{\alpha+\beta}}$$

for $i = 1, 2$, $j \neq i, j = 1, 2$.

Proof: We begin by computing the expected cost. We follow the same approach as in the proof of Lemma 1. Fix μ and let $E[C(y)|\mu]$ be the expected cost of player 1 given μ. To compute $E[C(y|\mu)]$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{-y_1, \epsilon\}$. If $X_t = \epsilon$ then player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$. If $X_{t+\tau} = 0$ then the process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{-y_1, \epsilon\}$ and so on. Clearly, this yields an upper bound to $E[C(y)|\mu]$. Let C^ϵ denote that upper bound. Let τ_1 be such that $X_{\tau_1} \in \{y_1, \epsilon\}$ given the initial state 0. Let τ_2 be the random time when $X_t \in \{0, y_2\}$ given the initial state ϵ. Then, by the strong Markov property of Brownian motion we have

$$C^\epsilon = \frac{k_1}{2} \int_0^{\tau_1} e^{-rt} dt + E[e^{-r\tau_1}; X_{tau_1} = \epsilon] E[e^{-r\tau_2}; X_{\tau_2} = 0] C^\epsilon \qquad (D1)$$

By Proposition 3-2-18 in Harrison (1985) we have

$$E[e^{-r\tau_1}] = \frac{e^{-\alpha\epsilon} - e^{\beta y_1} e^{\alpha(y_1-\epsilon)}}{1 - e^{\beta(y_1-\epsilon)} e^{\alpha(y_1-\epsilon)}}$$

and

$$E[e^{-r\tau_2}] = \frac{e^{-\beta\epsilon} - e^{-\alpha(y_2-e)} e^{-\beta y_2}}{1 - e^{-\beta y_2} e^{-\alpha y_2}}$$

and by Proposition 3-5-3 in Harrison (1985) we have

$$E[\int_0^{\tau_1} e^{-rt} dt] = \frac{1}{r} \left(1 - \frac{e^{-\alpha\epsilon} - e^{\beta y_1} e^{\alpha(y_1-\epsilon)}}{1 - e^{\beta(y_1-\epsilon)} e^{\alpha(y_1-\epsilon)}} - \frac{e^{\beta y_1} - e^{-\alpha\epsilon} e^{\beta(y_1-\epsilon)}}{1 - e^{\beta(y_1-\epsilon)} e^{\alpha(y_1-\epsilon}}\right)$$

Let

$$\bar{C}^\epsilon = 1/2 C^\epsilon(y|1/2] + 1/2 C^\epsilon(y| - 1/2]$$

Then, substituting the above expressions we get

$$\begin{aligned}
\lim_{\epsilon \to 0} \bar{C}^\epsilon =& \frac{k_i(1 - e^{-(\alpha+\beta)y_2})}{4r} \frac{1 - e^{\alpha y_2} - e^{\beta y_1} + e^{(\alpha+\beta)y_1}}{1 - (e)^{(\alpha+\beta)(y_1-y_2)}} \\
=& \frac{k_1}{4r} \frac{(1 - x_1^\alpha)(1 - x_1^\beta)(1 - x_2^{\alpha+\beta})}{1 - (x_i x_j)^{\alpha+\beta}}
\end{aligned}$$

Choosing $\epsilon < 0$ we can compute an analogous lower bound that converges to the same limit as $\epsilon \to 0$. Hence, the expression above captures the expected cost of player 1.

Next, we compute the expected utility of winning. Let T be the time when $X_T \in \{y_1, y_2\}$ (the game ends). Then,

$$
\begin{aligned}
2E[e^{-rT}; X_T = y_2] =& E[e^{-rT}; X_T = y_2|1/2] + E[e^{-rT}; X_T = y_2| - 1/2] \\
=& \frac{e^{-\alpha y_2} - e^{\beta y_1} e^{\alpha(y_1 - y_2)}}{1 - e^{\beta(y_1 - y_2)} e^{\alpha(y_1 - y_2)}} + \frac{e^{-\beta y_2} - e^{\alpha y_1} e^{\beta(y_1 - y_2)}}{1 - e^{\alpha(y_1 - y_2} e^{\beta(y_1 - y_2)}} \\
=& \frac{1 - x_1^{\alpha+\beta}}{1 - (x_1 x_2)^{\alpha+\beta}} \left(x_2^{\alpha} + x_2^{\beta} \right)
\end{aligned}
$$

This completes the proof of Lemma D1. □

Note that the coefficients α, β are functions of r. When $r \to 0$ we obtain the payoffs calculated in Lemma 1.

$$
\begin{aligned}
\lim_{r \to 0} & \left(\frac{1 - x_i^{\alpha+\beta}}{1 - (x_i x_j)^{\alpha+\beta}} \frac{x_j^{\alpha} + x_j^{\beta}}{2} - \frac{k_i}{4r} \frac{(1 - x_i^{\alpha})(1 - x_i^{\beta})(1 - x_j^{\alpha+\beta})}{1 - (x_i x_j)^{\alpha+\beta}} \right) \\
&= \frac{1 - x_i}{2(1 - x_i x_j)} \left(1 + x_j - k(1 - x_j) \ln x_i\right) \\
&= \frac{\alpha_1}{\alpha_1 + \alpha_2} \left(1 - k_i \alpha_j \ln \left(\frac{1 + \alpha_i}{1 - \alpha_i} \right) \right)
\end{aligned}
$$

where $\alpha_i = \frac{1 - x_i}{1 + x_i}$.

Proof of Proposition 6 To simplify the algebra, we rescale time and renormalize the signal so that

$$\alpha + \beta = 1$$

(The costs and the discount factor must be adjusted to account for the renormalization.)

Hence, the payoff of player i is

$$U_i(x_i, x_j) = \frac{1 - x_i}{1 - (x_i x_j)} \frac{x_j^{\alpha} + x_j^{1-\alpha}}{2} - \frac{\hat{k}_i}{4\hat{r}} \frac{1 - x_i^{\alpha} - x_i^{1-\alpha} + x_i}{1 - (x_i x_j)}$$

Let $K = \frac{\hat{k}_i}{4\hat{r}}$. Then, the first order condition can be written as:

$$x_j^{\alpha} + x_j^{1-\alpha} = K h(x_i, x_j)$$

where

$$h(x_i, x_j) = \frac{1}{x_i^{\alpha}} \left(1 - \alpha + \alpha x_i x_j + \alpha x_i^{2\alpha-1} - (1 + x_j) x_i^{\alpha} + (1 - \alpha) x_j x_i^{2\alpha} \right)$$

Let h_1 denote the partial derivative of h with respect to the first argument and let h_2 denote the partial derivative of h with respect to the second argument. We have

$$\begin{aligned} h_1 =& -\frac{1}{x_1^{\alpha+2}}\alpha(1-\alpha)(x_1+x_1^{2\alpha})(1-x_1x_2) \\ h_2 =& \frac{1}{x_1^{\alpha}}\left(\alpha x_1 - x_1^{\alpha} + (1-\alpha)x_1^{2\alpha}\right) \end{aligned}$$

Note that $h_1 < 0$ which implies that the second order condition is satisfied and that $dx_i/dK > 0$ at any solution to the first order condition. We conclude that the first order condition has a unique solution. Moreover, it is straightforward to show that $x_i > 0$ for all $x_j \in [0,1]$ and $K > 0$ and that $x_i < 1$ for all $x_j > 0$.

Next, we show that $dx_i/dx_j > 0$ and find a convenient bound for $|dx_i/dx_j|$.

$$\frac{dx_i}{dx_j} = -\frac{Kh_2 - \alpha x_j^{\alpha-1} - (1-\alpha)x_j^{-\alpha}}{Kh_1} < 0$$

since $h_2 < 0$ (which in turn follows from the fact that $0 < \alpha < 1$) and $h_1 < 0$. Note that $\alpha x_j^{\alpha-1} + (1-\alpha)x_j^{-\alpha} \leq \frac{x_j^{\alpha}+x_j^{1-\alpha}}{2x_j}$. Therefore, using the first order condition it follows that

$$\left|\frac{dx_i}{dx_j}\right| \leq \frac{h_2 - h/(2x_j)}{h_1}$$

and

$$\begin{aligned} \left|\frac{dx_i}{dx_j}\right|\left|\frac{dx_j}{dx_i}\right| \leq& \frac{h_2(x_i,x_j) - h(x_i,x_j)/(2x_j)}{h_1(x_i,x_j)} \cdot \frac{h_2(x_j,x_i) - h(x_j,x_i)/(2x_i)}{h_1(x_j,x_i)} \\ =& \frac{f(x_i,x_j)f(x_j,x_i)}{4\alpha^2(1-\alpha)^2(x_i+x_i^{2\alpha})(x_j+x_j^{2\alpha})(1-x_ix_j)^2} \end{aligned}$$

where

$$f(x_i,x_j) = -x_i(1-\alpha) + x_i^{1+\alpha}(1-x_j) - \alpha x_i^{2\alpha} + x_i^{1+2\alpha}x_j(1-\alpha) + \alpha x_j x_i^2$$

To prove uniqueness, we it suffices to show that

$$f(x_i,x_j) \leq 2\alpha(1-\alpha)(x+x^{2\alpha})(1-x_ix_j)$$

Note that f is a linear function of the second argument. It is straightforward to establish the inequality for $x_j = 0, 1$ and therefore it holds for all $x_j \in [0,1]$. This completes the proof of uniqueness. To see part (ii), note that x_i is increasing in K and hence $|y_i|$ is increasing in k_i. Moreover, x_i is decreasing in x_j and hence $|y_i|$ is decreasing in $|y_j|$. This proves part (ii) of the proposition. □

Proof of Proposition 7 From the first order condition it follows that x_j stays bounded away from zero along any sequence where K_j stays bounded away from zero. Since x_j remains bounded away from zero, the first order condition for x_i implies that x_i converges to zero as K_i converges to zero. Since x_i goes to zero the first order condition for x_j implies that x_j converges to 1 as K_i converges to zero. □

13. Appendix F: Asymmetric Information

Define $V_t^i(\omega)$ to be the continuation payoff of type $i = 0, 1$ in a monotone equilibrium in period t after history $(X_\tau(\omega))_{\tau \leq t}$.

Lemma F1: *(i) $V_t^0(\omega) > 0$ for $p_t(\omega) > p(z^*)$. (ii) There is $\delta > 0$ such that $V_t^1(\omega) > 0$ for $p_t(\omega) > p(z^*) - \delta$ for some $\epsilon > 0$.*

Proof: A lower bound to type 0's payoff can be computed by assuming the lower bound on $p_\tau(\omega)$ is attained for all $\tau \geq t$ and type 0 quits whenever $p_\tau = p(z^*)$. Let z be such that

$$p_t(\omega) = \frac{1}{1 + e^{-z}}$$

and assume (wlog) $p_t(\omega) < 1/2$. For $z_1 \leq 0 \leq z_2$, $P(z_1, z_2)$ is the probability that a Brownian motion X_t with drift $-1/2$ and variance 1 hits z_2 before it hits z_1 given that $X_0 = 0$ and $T(z_1, z_2)$ be the expected time a Brownian motion with drift $-1/2$ spends until it hits either z_1 or z_2 given that $X_t = 0$ (where $z_1 \leq 0 \leq z_2$). Then,

$$V_t^0(\omega) \geq P(z^* - z, -z) - k_1 T(z^* - z, -z)$$

Substituting the expressions for P and T it is straightforward to show that $V_t^0(\omega) > 0$ for $z > z^*$.

The statement for V^1 follows from an analogous argument where $\mu = 1/2$ is substituted to obtain the corresponding expression for V_t^1. □

Lemma F2: *Let $p_t(\omega) = p(z^*)$. Then $V_t^0(\omega) = 0$.*

Proof: It suffices to show that $V_t^0(\omega) \leq 0$. Assume that type 0 never quits but incurs the cost of information only if $p_t \in (p(z^*), 1/2]$. Clearly, this strategy provides an upper bound to the payoff of type 0. Denote the corresponding payoff by W^*. Note that when $p_t \in (p(z^*), 1/2]$ neither type quits with positive probability and hence p_t is continuous in this range.

We can compute an upper bound to W^* by assuming that when $p(z^*)$ is reached then information is provided at no cost to the players until a $p_t = \frac{1}{1+e^{-(z^*+\delta)}}$ is reached for some $\delta > 0$. For $z \in (z^*, 0)$ let

$$\Pi(x, z) = P(z - x, -x) - \frac{k}{2}T(z - x, -x)$$

and note that $\Pi(x, \cdot)$ has a unique maximum at z^* (independent of x). Clearly

$$V_t^0(\omega) \leq \frac{1}{P(-\delta, -z^* - \delta)}\Pi(z^* + \delta, z^*)$$

for all $\delta > 0$. We will show that

$$\lim_{\delta \to 0} \frac{1}{P(-\delta, -z^* - \delta)}\Pi(z^* + \delta, z^*) = \frac{1 - e^{-z^*}}{1 - e^{-\delta}}\Pi(z^* + \delta, z^*) = 0$$

Clearly $\Pi(x, x) = 0$ for all x and therefore

$$\lim_{\delta \to 0} \frac{\Pi(z^* + \delta, z^*)}{1 - e^{-\delta}} = \lim_{\delta \to 0} \frac{\Pi(z^* + \delta, z^*) - \Pi(z^* + \delta, z^* + \delta)}{1 - e^{-\delta}} = 0$$

where the last equality follows since $\partial \Pi^0(x, z^*)/\partial z = 0$ uniformly for all x by the optimality of z^*. □

Lemma F3: *In any monotone equilibrium, if $Q_t^1 < 1$ then (i) $p_t \geq p(z^*)$ and (ii) p_t is continuous.*

Proof: We first note two properties of p_t:

(i) p_t is continuous at t for ω with $p_t(\omega) > p(z^*)$ since both types quit with probability 0 when $p_t > p(z^*)$.

(ii) $p_t(\omega)$ is right-continuous since $Q_t^i(\omega)$ is right-continuous.

Assume $0 < p_t(\omega) < p(z^*)$ for some t, ω. Let T be a stopping time with the following property: T is the first time such that either $Q^0_{t+T}(\omega) = 1$ or $p_{t+T}(\omega) = p(z^*)$. If neither or these occurs along a sample path then $T = \infty$. Notice that if $T = \infty$ then $p_s(\omega) < p(z^*)$ for all $s > t$. This follows from properties (i) and (ii). Therefore, along a sample path with $T = \infty$ the payoff is $-\infty$. If T is finite then $V^0_{t+T}(\omega) = 0$. It follows that the expected payoff of type 0 is $-\frac{k}{2}\int(TdP^0(\cdot|\,\omega_t)$. If $Q^0_t < 1$ then $T > 0$ along the sample path since Q is right-continuous. Therefore the payoff of type zero is strictly negative and we conclude that $Q^0_t(\omega) = 1$ for any best response. Since $Q^1_t < 1$ this in turn implies that $p_t(\omega) = 1$. We conclude that $p_t(\omega) \geq p(z^*)$ for all ω, t with $Q^1_t < 1$.

Continuity of p_t now follows from the fact that neither type quits when $p_t > p(z^*)$.

□

Proof of Proposition 8: We first prove the result for $\pi \geq p(z^*)$. We claim that along any sample path $Q^1_t = 0$ (type 1 never quits). Let $\tau = \inf\{t : Q^1_t(\omega) = 1\}$. If $\tau = \infty$ we have $Q^1_t < 1$ for all t and therefore by Lemma F3 $p_t \geq p(z^*)$ for all t. Lemma F1 then implies $Q^1_t = 0$ for all t as desired. Therefore, assume $\tau < \infty$. First, we show that $Q^1_\tau = 0$. If $\tau = 0$ then this follows from $\pi \geq p(z^*)$ and Lemma F1. If $\tau > 0$ then $p_t \geq p(z^*)$ for all $t < \tau$ and, by monotonicity $p_\tau \geq p(z^*)$ if type 1 does not quit at time τ (and therefore the game is running at τ). It follows that $Q^1_\tau = 0$. Since Q^1_t is right-continuous we conclude that $Q^1_s < 1$ for some $s > \tau$ establishing that τ finite is impossible. This proves that $Q^1_t = 0$ in any monotone equilibrium.

Let $Y_t = \inf_{\tau<t} X_\tau$. If $Y_t \geq z^*$ then by Lemma F1 and the right-continuity of Q^i_t we have $Q^0_t = Q^1_t = 0$ and

$$p_t = \frac{1}{1 + e^{-X_t}}$$

For $Y_t < z^*$, let $s \leq t$ be such that $X_s = Y_t$. We claim that $p_s = p(z^*)$. Since neither type quits when $p_t > p(z^*)$ it follows that $p_r = p(z^*)$ for some $r \leq s$. Let $r^* = \max\{r : p_r = p(z^*), r \leq s\}$. Note that r^* is well-defined since p_t is continuous. Therefore,

$$p_s = \frac{1}{1 + \frac{1-p(z^*)}{p(z^*)} e^{-X_s + X_{r^*}}}$$

Since $X_s \leq X_r$ for all $r \leq s$ this implies that $r^* = s$ and $p_s = p(z^*)$. Since $p_s = p(z^*)$ it follows from Lemma F3 that $p_t > p(z^*)$ for $t > s$ and therefore

$$p_t = \frac{1}{1 + \frac{1-p(z^*)}{p(z^*)} e^{-X_t + Y_t}} = \frac{1}{1 + e^{-X_t + Y_t - z^*}} \qquad (*)$$

Equation $(*)$ implies that the belief process p_t has the desired form. Since $Q_t^1 = 0$ for all t by Lemma F1 and Lemma F3 we can use $(*)$ to solve for Q_t^0. This yields $Q_t^0 = Q_t^{z^*}$, as desired.

It remains to extend the proposition to initial conditions with $\pi = p_0 < p(z^*)$. Let $\underline{V}_0^1$ be the payoff of type 1 at the beginning of the game when $\pi = p(z^*)$. By Lemma F1, $\underline{V}_0^1 > 0$. Let $\underline{p} \leq p(z^*)$ be such that $Q_0^1 < 1$ in any equilibrium when $\pi \geq \underline{p}$. We will show that if $\underline{p} > 0$ there is some $\epsilon > 0$ (independent of $\underline{p}$) such that $Q_0^1 = 0$ when $\pi > \min\{0, \underline{p} - \epsilon\}$. By Lemma F3, this implies that $p_0 \geq p(z^*)$ and therefore the argument above applies.

If $\pi \geq \underline{p}$ then $Q_0^1 < 1$ (by assumption) and therefore $p_0 \geq p(z^*)$ by Lemma F3. This in turn implies that the payoff of type 0 at the beginning of the game is at least $\underline{V}_0^1 > 0$. There is some $\epsilon > 0$ (independent of $\underline{p}$) such that for $\pi \geq \underline{p} - \epsilon$, type 1 has a strict incentive not to quit. To see this, let $\pi/\underline{p} = e^{-\delta}$ and let $Z_t = \sup_{\tau \leq t} X_\tau$. Since beliefs are monotone, the payoff of type 1 from not quitting is bounded below by

$$\Pr\{Z_\delta > \delta | \mu = 1/2\} \cdot \underline{V}_0^1 - \frac{k}{2}\delta$$

Since $\Pr\{Z_\delta > \delta | \mu = 1/2\} \to 1$ as $\delta \to 0$ the claim follows. We conclude that type 1 quits with probability 0 at time 0 for any $\pi > 0$. By Lemma F3 it follows that $p_0 \geq p(z*)$ and hence the argument above applies. □

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