Modelling the influence of campaign contribution and advertising on US Presidential Elections

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Abstract

We provide a stochastic electoral model of the US Presidential election where candidates use the campaign contributions they receive from special interest groups (SIGs) to run their electoral campaign. Prior to the election, candidates announce their policy platforms and advertising (ad) campaigns and use the contributions of SIGs to generate two SIG valences to enhance their electoral valence. Voters have preferences over candidates' policies relative to their ideal policy and over candidates' ad messages relative to their ideal message frequency, their *campaign tolerance level*. Voters' choices are also influenced by two endogenous SIG valences: one associated with candidate's policy platform, the other with the candidate's ad campaign. Citizens' votes also depend on their sociodemographic characteristics and candidates' traits and competences. Candidates' critical policy and ad campaign are a weak local Nash equilibrium (LNE) of the election if the expected vote shares of all candidates are greater than their corresponding pivotal vote shares. If the expected vote share of at least one candidate is lower than its pivotal vote share, then the electoral mean is not a LNE of the election. In LNE, candidates' policy and ad campaign balance the electoral and SIG pulls.

Key words: stochastic votes, policies, advertising campaign, valences, local Nash equilibrium, pivotal probabilities, pivotal vote shares.

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1 Introduction

Money matters in US Presidential elections through the influence interest groups exert on policy making through their campaign contributions and through candidates use of these resources to fund the advertising (or ad) campaigns they run to convince voters to vote for them. Campaign contributions substantially increased in recent elections giving special interest groups (SIGs) more influence over candidates' policy platforms.[1]

Moreover, the emergence of smart phones and social media coupled with large data sets on voters' personal information[2] and public and private pre-election polls has substantially changed campaign advertising and U.S. Presidential elections as new media and new technologies give voice to new candidates and voters. The congruence of these factors transformed electoral campaigns as candidates now tailor their message to voters taking their sociodemographic characteristics and political preferences into account. Candidates have now greater control over their message than when using traditional media (e.g., TV and radio).

In 2008, Obama became the first Presidential candidate to use social media. In 2016, Trump took the political social media game to unprecedented levels. He stormed the mainstream media, constantly trended on Facebook and Twitter, used Twitter to speak freely about his platform and captured the attention of mainstream media with his flamboyant remarks which bought him free publicity through traditional media channels (Fernando, 2016). When talking to *60 Minutes*, Trump stated that "I really believe that the fact that I have such power in terms of numbers with Facebook, Twitter, Instagram, etc... I think it helped me win all of these races where they're spending much more money than I spent" (Fernando, 2016). Trump used his advantage over Clinton on social media, in particular on Twitter, to build relationships with his supporters and to create a buzz for his brand. As Rivero (2016) states "Trump marketed to voters based on their wants, social class, income, ethnicity, location, opinions, values and lifestyles and held events attended by those market segments." Money spent on traditional advertising is then only part of the advertising story.

Using his unorthodox campaign, promises to make "America Great Again" and "Common Sense" policy platform, Trump capitalized on voters' dissatisfaction with the status quo, targeting voters who felt disenfranchised and were angry at the Washington establishment. Arguing that the US was in economic and societal trouble, Trump's heresthetics influenced voters' perceptions of his opponents, during the primaries and the electoral campaign, posi-

[1] Ansolabehere et al (2003) estimate that the marginal impact of $100,000 spent in a House race, ceteris paribus, is about 1% gain in vote. Thus, highlighting the powerful influence of ad campaigns on elections.

[2] Data on voters' information contains their sociodemographic characteristics (age, education, gender, financial situation, etc.), contact information (home address, emails, phone numbers, social media and twitter accounts, etc.) and answers to past opinion polls on how they would vote where an election held that day.

tioning himself as the antiestablishment candidate who would solve these problems.

While, widely recognized that Trump's social media campaign and the great dissatisfaction of many voters with the status quo won him the election, the role of money and the influence of Special Interest Groups (SIGs) on Trump's campaign is still currently debated.[3] The influence of domestic SIGs on Trump's policies implicitly behind the scene can also be seen through his post-election appointed cabinet, all extremely rich people highly connected SIGs. A special prosecutor was appinted to investigate the Russian intervention favouring Trump in the election.

The objective of this paper is to examine the influence money has in the US Presidential elections by incorporating the effect that campaign contributions have on candidates' policy positions and ad campaigns through the effect that candidates use of these resources have voters' choices and on the electoral outcome. Campaign contributions and ad campaigns are then non-policy dimensions along which candidates compete. To understand how candidates choose their policy platforms and advertising campaigns, we must take into account the existence of two forces pulling candidates in opposite directions: the *SIG pull* pressuring candidates towards the SIGs' preferred policies and the *electoral pull* exerted by candidates' supporters through their ballot.

Candidates use the campaign contributions made by SIGs to run their electoral campaign in an effort to present themselves more effectively to the electorate. Since SIGs have more radical preferences than the average voter, candidates face a dilemma. By accommodating the political demands of SIGs, candidates receive campaign contributions that they then use to run their electoral campaigns. However, by moving towards the prefered policies of the SIGs candidates appear more extreme to voters and lose some electoral support. When selecting their electoral campaign, candidates must then *balance* the SIG and electoral pulls.

Huber and Arceneaux (2007) argue that campaign advertising is there to inform, mobilize and persuade voters.[4] With everyone voting in our model and policy platforms commonly known, ad campaigns are not there to mobilize voters or to serve as an information mechanism, rather they are there to persuade voters to vote for the candidate, i.e., to give voters a further impetus to vote for the candidate.

Our model extends Schofield's (2006) model by allowing candidates to run advertising

[3]Bonica (2013) and others point out that Political Action Committees (PACs) and individual donors have preferences over the policies they would like elected politicians to implement and seek to influence candidates' policies platforms and voters through the PACs own ad campaigns.

[4]Huber and Arceneaux (2007) examine the role of these three effects on the 2000 US Presidential election. They study the effect of ad campaigns on voters in non-battleground states adjacent to swing states as media markets reach beyond the boarders of swing states. Using the National Annenberg Election Survey and data on local advertising, they find strong evidence that in the 2000 presidential campaign, advertising did not mobilize or inform citizens, but instead had a strong persuasive effect.

campaigns. In this model, candidates have two electoral campaign instruments – policy and advertising – and use the campaign contributions of SIGs to generate two positive SIGs valences. We examine how policy and advertising campaigns are shaped by voters' preferences along these two dimensions and allow voters to have preference for candidates' policy and advertising campaigns. In our model, voters are identified by their ideal policy, *campaign tolerance level*,[5] their sociodemographic characteristics and perceptions of candidates' traits and competencies. Differences in campaign tolerance levels across voters capture differences in how often they want to be contacted by candidates. Policy and campaign tolerance preferences vary across voters with the mean sociodemographic, trait and competence valences being exogenously given and constant across voters. Voters' valences have an idiosyncratic component unobserved by candidates. Since the private component of voters' valences are drawn from Type I Extreme Value Distributions, at the campaign selection stage candidates view voters' utility as random and cannot perfectly anticipate how each individual votes but can estimate their expected vote shares.

Candidates use their policy, ad campaign and SIG valences to convince voters to vote for them, i.e., choose the policy and ad campaign to maximize their expected vote share. The model studies how voters' preferences affect candidates' policy and ad campaigns choices and how candidates' campaigns affect their electoral prospects.

We show that in the weak local Nash equilibrium (LNE) of the election, candidates do *not* adopt the electoral mean[6] as their campaign strategy in contrast to Gallego and Schofield (2016) where there are no SIGs valences. Rather candidates' policy and advertising campaigns are determined by two components. The first is associated with the influence voters exert on candidates' campaign and given by candidates' weighted electoral mean,[7] where the endogenous weight is that given by candidates to each voter in their campaigns. Since candidates weight voters differently, giving more weight to undecided than to committed voters, policies and advertising campaigns are more reflective of the ideal policies and campaign tolerance levels of uncommitted voters. The second component is associated with the influence of the SIGs on candidates' campaign and is given by the marginal effect that the SIGs valences have on voters' utilities weighted by the relative importance voters give to the SIGs valences. Since these two components are respectively associated with the electoral and SIG pulls, in equilibrium candidates campaigns *balance* the electoral and SIG pulls.

Our results also show that the necessary condition for candidate j to adopt its *critical*[8]

[5] Voters' campaign tolerance level gives the ideal number of mesasages they want to receive from any candidate (see formal definition in Section 2).

[6] The electoral mean is the mean of voters' ideal policies and campaign tolerance levels.

[7] The weighted electoral mean is the weighted mean of voters' ideal policies and campaign tolerance levels.

[8] The critical campaign are those satissfying the fist order conditions for candidates to be maximizing their

campaigns is that j's expected vote share is *high enough*, i.e., higher than j's pivotal vote share. When this condition is met for all candidates, the critical campaigns correspond to a weak local Nash equlibrium (LNE) of the election. If for at least one candidate, j's expected vote share is *too low*, i.e., lower than j's pivotal vote share, then the critical campaigns are not a LNE of the election.

The preliminaries of the model are described in Section 2 and the stochastic election model in Section 3. The equilibrium concept used in the analysis is defined in Section 4 with candidates' best response functions derived in Section 5. Section 6 shows that candidates do not converge to a common electoral campaign. The conditions for convergence to the local Nash equilibrium are derived in Section 7. Our contributions to the the literature are discussed in Section 8 and final comments in Section 9. The second order conditions for convergence to the critical campaigns are derived in the Appendix in Section 10.

2 Preliminaries

We model a stochastic electoral model in which at least two candidates compete in the election (e.g., Obama, McCain and Nader ran in 2000). Candidates' *electoral campaigns* consists of two instruments: their policy platforms and advertising campaigns messages.[9] In addition, candidates use the campaign contributions made by Special Interst groups (SIGs) to sway voters in their favour and do so through two *SIGs valences* that influence citizens' voting decissions. After observing candidates' electoral campaign and the SIG valences, citizens vote with their preferences depending on candidates' policy platforms, advertising campaigns, the SIGs valences and voters' non-policy evaluation of candidates or *voters' valences*. We study candidates' electoral game in response to the anticipated electoral outcome.

Candidates use their policy platforms, ad campaigns and the SIGs valences to persuade voters that they matter to them, i.e., use these instruments to give voters an impetus to vote for them. The advertising portion of the campaign consists of messages candidates send to voters with campaign advertising costs rising as message frequency increases, regardless of the effectiveness of the ad campaigns.

Let $\mathcal{C} = \{1, .., j, .., c\}$ be the set of candidates competing in the election. Prior to the election, candidates simultaneously announce their electoral campaigns and SIGs valences. Candidate j's *electoral campaign*, for all $j \in \mathcal{C}$, consists of a policy, $z_j \in \mathcal{Z} \subseteq \mathbb{R}$ and an ad message, $a_j \in \mathcal{A} \subseteq \mathbb{R}$. Denote by $\mathcal{K}_j = \mathcal{Z} \times \mathcal{A}$ each candidate's electoral campaign space

expected vote shares. These critical campaigns may not satidy the second order conditions for a maximum.

[9] Messages are measured in continuous, rather than discrete, terms so that messages of different lengths and frequencies can be compared. If message 1 is twice as long as message 2, then message 1 is assigned a number twice as large as message 2. In this paper, we assume messages are one-dimensional.

and by $\mathcal{K} = \prod_{j \in \mathcal{C}} \mathcal{K}_j$ the campaign space of all candidates.

Denote by $\mathbf{z}$ and $\mathbf{a}$ the policy and ad message profile of *all* candidates, i.e.,

$$\mathbf{z} \equiv (z_1, ..., z_j, ..., z_c) \qquad \text{and} \qquad \mathbf{a} \equiv (a_1..., a_j, ..., a_c).$$

The vector $\mathbf{z}_{-j}$ (respectively $\mathbf{a}_{-j}$) denotes the profile where all policies (*ad messages*) are held constant *except for* j's policy (*ad message*).

Denote by n and $\mathcal{N} = \{1, .., i, .., n\}$ the number and the set of voters in the country. Voter i's preferences are identified by the vector of utilities i derives from each candidate,

$$\mathbf{U}_i(\mathbf{z}, \mathbf{a}) = (u_{i1}(z_1, a_1), .., u_{ij}(z_j, a_j), ..., u_{ic}(z_c, a_c))$$

where the utility i derives from candidate j, $u_{ij}(z_j, a_j)$, is given by

$$\begin{aligned} u_{ij}(z_j, a_j) &= -\frac{b}{2}(x_i - z_j)^2 - \frac{e}{2}(t_i - a_j)^2 + \sigma_j(z_j) + \alpha_j(a_j) + (\bar{\mathfrak{s}}_j + \mathfrak{s}_{ij}) + (\bar{\tau}_j + \tau_{ij}) + (\bar{\kappa}_j + \kappa_{ij}) \\ &= u^O_{ij}(z_j, a_j) + \mathfrak{s}_{ij} + \tau_{ij} + \kappa_{ij}. \end{aligned} \quad (1)$$

Voter i is characterized by her ideal policy, $x_i \in \mathcal{Z}$, and the ideal number of messages she wants to receive from any candidate, her *campaign tolerance level*, $t_i \in \mathcal{A}$ with the (x_i, t_i) pair drawn from a joint distribution $(\mathcal{D}_x \times D_a)$. We take it as understood that u_{ij} depends on (x_i, t_i) with $u^O_{ij}(z_j, a_j)$ in (1) denoting the observable component of i's utility function u_{ij}. The coefficients b and e in (1) are positive, the same for all voters, known by all players and measure the importance voters give to differences with j's policy and ad campaign. The *direct* effect that j's policy has on i's preferences is modelled by $-b(x_i - z_j)^2$ in (1) implying that the farther j's policy z_j is from i's ideal policy x_i, the lower the utility i derives from j.

Since candidates run ad campaigns, we endow voters with preferences over how often they want to be contacted by candidates, i.e., voters are also are characterized by their *campaign tolerance level.* Voters care about the frequency of messages they receive from candidates relative to their campaign tolerance level and depending on candidates' ad campaign view candidates as engaging in too much or too little advertising. The *direct* ad campaign effect on voter i's utility is modelled giving i negative quadratic preferences over j's ad campaign, i.e., is modelled by $-e(t_i - a_j)^2$ in (1), which depends on the importance voters give to candidates' ad campaigns, e, and on the messages j sends a_j relative to i's campaign tolerance t_i. When $a_j > t_i$, i believes j engaged in too much advertising, an irritant to voter i that leads to campaign fatigue[10] and if $a_j < t_i$, i believes that j's ad campaign reflects j's lack of concern

[10] Which may occur when candidates use robo-calls or too many Tweets. The preferences for being contacted by candidates (e.g., through visits from volunteers or Tweets) varies across individuals. Some gladly

for voters. The direct persuasiveness effect of j's ad campaign varies across voters.

Candidates use the campaign contributions of SIGs to generate two valences that also influence voters' decissions. We assume that associated with candidate j, for all $j \in \mathcal{C}$, are two SIGs, SIG_j^z and SIG_j^a, whose campaign contributions are tied to j adopting a policy platform and an ad campaign more preferred by these SIGs.[11] Candidates use these resources to present themselves more effectively to the electorate which increase their electoral valence. The SIG policy and advertising valences, $\sigma_j(z_j)$ and $\alpha_j(a_j)$ in (1), are generated by the campaign contributions made by these SIGs[12] and since they respectively depend on j's policy platform and ad campaign, they are endogenously determined. Candidate j's SIGs valences, $\sigma_j(z_j) \in \mathfrak{S} \subseteq \mathbb{R}$ and $\alpha_j(a_j) \in \mathfrak{A} \subseteq \mathbb{R}$ respectively depend on j's policy platform and ad campaign. The SIG valence functions are assumed to be concave, so that they exert a positive impact on voters' utilities with their marginal impact decreasing as the policy or ad campaign increase and have no cross partial effect, i.e.,

$$\begin{aligned}
\frac{\partial \sigma_j}{\partial z_j}(z_j) &> 0, \quad \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j) < 0 \quad \text{and} \quad \frac{\partial^2 \sigma_j}{\partial (z_j) \partial a_j}(a_j) = 0 \\
\frac{\partial \alpha_j}{\partial a_j}(a_j) &> 0 \quad \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j) < 0 \quad \text{and} \quad \frac{\partial^2 \alpha_j}{\partial a_j \partial (z_j)}(z_j) = 0.
\end{aligned}$$

Voters' sociodemographic characteristics affect their choice of candidate. Voter i's *sociodemographic valence* for candidate j is given by $(\bar{\mathfrak{s}}_j + \mathfrak{s}_{ij})$ in (1). Candidate j's *mean* sociodemographic valence, $\bar{\mathfrak{s}}_j$, captures the idea that voters with similar sociodemographic characteristics (gender, age, class, education, financial situation, etc.) share a common evaluation or bias for j. Voters are identified by an s-vector $\mathbf{s}_i$ denoting their individual sociodemographic characteristics with mean $\mathbf{s} = \frac{1}{n} \sum_{i \in \mathcal{N}} \mathbf{s}_i$. The importance voters give to their sociodemographic characteristics on their utility from j are modelled by an s-vector $\boldsymbol{\gamma}_j$ *common* to all voters. The composition $\bar{\mathfrak{s}}_j = \boldsymbol{\gamma}_j \cdot \mathbf{s}$ measures the mean sociodemographic valence for j. The effect that i's sociodemographic characteristics have on i's choice of candidate has an idiosyncratic component, $\mathfrak{s}_{ij}$, that varies around $\bar{\mathfrak{s}}_j$ according to a Type-I extreme value distribution, $\mathcal{S}(0, \pi/6)$, with mean zero and variance $\pi/6$.

The traits valence, $(\bar{\tau}_j + \tau_{ij})$ in (1), measures the effect that candidate j's traits (race,

talk to volunteers or read and re-Tweet the messages they receive, others just close their doors on volunteers or delete the messages they receive.

[11] For example, in the 2016 elections the Russians send messages to voters that were aimed at convincing the voters to vote for Trump. The messages had no policy content. We model this effect as being directly generated by candidates in this model through the SIG ad valence.

[12] In this model, candidates know with certainty the effects that their policy and ad campaigns have on voters' utility and implement their policy platform if elected. The role of SIGs in this model is not to inform candidates as to the state of the world as in Dellis and Oak (and others).

gender, age, education, charisma, experience in public life, ...) have on voter i's utility. The commonly known *mean trait valence* for j, $\bar{\tau}_j$, measures the mean influence j's traits have on voters' choice with the effect on voter i also depending on an idiosyncratic component, τ_{ij}. Candidates are identified by a t-vector $\mathbf{t}_j$ capturing their individual traits. The average importance voters give to these traits is given by a t-vector $\bar{\boldsymbol{\omega}}_j$ *common* to all voters. The composition $\bar{\tau}_j = \mathbf{t}_j \cdot \bar{\boldsymbol{\omega}}_j$ measures the mean effect that j's traits have on voters. Voter i's private component, τ_{ij}, is unobserved by candidates who know it varies around the mean traits valence, $\bar{\tau}_j$, and is drawn from a Type-I extreme value distribution, $\mathcal{T}(0, \pi/6)$, common to all voters with zero mean and variance $\pi/6$.

The term $(\bar{\kappa}_j + \kappa_{ij})$ in (1) represents i's belief on candidate j's competence or ability to govern. The mean competence valence, $\bar{\kappa}_j$, measures the *common belief* voters have on j's ability to govern. The private portion of i's competence signal, κ_{ij}, unobserved by candidates, varies around $\bar{\kappa}_j$ according to a Type-I extreme value distribution, $\mathfrak{K}(0, \pi/6)$, common to all voters with zero mean and variance $\pi/6$.

The three idiosyncratic valence components in voter i's utility are drawn independent of each other. Train (2003) shows that the sum of independent Type-I Extreme Value Distributions with identical means and variances also has a Type I Extreme Value Distribution with the same mean and variance. Mathematically, if $\mathfrak{s}_{ij} \sim \mathcal{S}(0, \pi/6)$, $\tau_{ij} \sim \mathcal{T}(0, \pi/6)$ and $\kappa_{ij} \sim \mathfrak{K}(0, \pi/6)$ then their sum $\varepsilon_{ij} \equiv \mathfrak{s}_{ij} + \tau_{ij} + \kappa_{ij} \sim \Xi(0, \pi/6)$ also has a Type-I Extreme value Distribution. Using this sum, we can re-write voter i's utility in (1) as follows:

$$\begin{aligned} u_{ij}(z_j, a_j) &= -\frac{b}{2}(x_i - z_j)^2 - \frac{e}{2}(t_i - a_j)^2 + \sigma_j(z_j) + \alpha_j(a_j) + \bar{\mathfrak{s}}_j + \bar{\tau}_j + \bar{\kappa}_j + \varepsilon_{ij} \\ &= u^O_{ij}(z_j, a_j) + \varepsilon_{ij}. \end{aligned} \quad (2)$$

Since candidates do not observe voters' idiosyncratic valence components but know that they are drawn from Type-I Extreme Value Distributions, at the electoral campaign selection stage, candidates view voters' utilities as stochastic. Given any electoral campaign, (z_j, a_j), the SIGs valences, $(\sigma_j(z_j), \alpha_j(a_j))$, voters' utility in (1) or (2) and the distributions of voters' idiosyncratic valence components, $\mathcal{S}$,$\mathcal{T}$,$\mathfrak{K}$ and Ξ, at the campaign selection stage, candidates' view the probability that voter i chooses candidate j as given by

$$p_{ij}(\mathbf{z}, \mathbf{a}) = \Pr[u_{ij}(z_j, a_j) > u_{ih}(z_h, a_h), \text{ for all } h \neq j \in \mathcal{C}]$$

where Pr is the probability operator generated by the distribution assumptions on $\mathcal{S}$,$\mathcal{T}$,$\mathfrak{K}$ and Ξ, with $u_{ij}(z_j, a_j)$ given by (1) or (2). The probability that i votes for j is given by the probability that $u_{ij}(z_j, a_j) > u_{ih}(z_h, a_h)$ for all $h, j \in \mathcal{C}$, i.e., is given by the probability that

i gets a higher utility from j than from any other candidate.

With voters' idiosyncratic valences and their sum following Type-I extreme value distributions, the probability that i votes for j has a logit specification, i.e.,

$$p_{ij} \equiv p_{ij}(\mathbf{z}, \mathbf{a}) \equiv \frac{\exp[u_{ij}^O(z_j, a_j)]}{\sum_{h=1}^{c} \exp[u_{ih}^O(z_h, a_h)]} = \left[\sum_{h=1}^{c} \exp[u_{ih}^O(z_h, a_h) - u_{ij}^O(z_j, a_j)] \right]^{-1} \quad (3)$$

for all $i \in \mathcal{N}$ and $j \in \mathcal{C}$ with the dependence of p_{ij} on $(\mathbf{z}, \mathbf{a})$ sometimes omitted.

Candidate j's expected vote share is the average of the voting probabilities across all voters in the country, i.e.,

$$V_j(\mathbf{z}, \mathbf{a}) \equiv \frac{1}{n} \sum\nolimits_{i \in \mathcal{N}} p_{ij}(\mathbf{z}, \mathbf{a}) \quad (4)$$

where $p_{ij}(\mathbf{z}, \mathbf{a})$ is given by (3) with expected vote shares adding to 1, $\sum_{j \in \mathcal{C}} V_j(\mathbf{z}, \mathbf{a}) = 1$.

Since at the campaign selection stage candidates do not know voters' idiosyncratic valences, they view voters' decisions as stochastic and do not know how individuals will vote but can estimate their expected vote shares. Candidate j chooses its policy and ad campaign, (z_j, a_j), to maximize its expected vote share, taking the campaigns of other candidates, $(\mathbf{z}_{-j}, \mathbf{a}_{-j})$, as given, i.e., j's objective is to

$$\max_{z_j \in \mathcal{Z},\ a_j \in \mathcal{A}} V_j(\mathbf{z}, \mathbf{a}) \ \text{ for all } \ j \in \mathcal{C}. \quad (5)$$

3 The Stochastic Electoral Model

This stochastic election model is characterized by the following timing of events:

1. Candidates simultaneously announce their policy platforms, advertising campaigns and their SIGs valences.

2. After observing candidates' electoral campaign and SIGs valences, each voter learns the idiosyncratic components of their sociodemographic, traits and competence valences.

3. The election takes place.

4. The President elect implements the announced policy platform.

We now define in more formal terms this stochastic electoral campaign model.

Definition 1 *The stochastic electoral campaign model can be represented by a game in normal form* $\mathcal{G}\langle \mathcal{C}, \mathcal{K}, \sigma_j(\cdot), \alpha_j(\cdot), D_x \times D_a, \mathbf{V} \rangle$, *where*

1. ***Players***: $\mathcal{C} = \{1, .., j, .., c\}$ *is the set of candidates competing in the election.*

2. ***Strategies***: *Candidates choose their policy platform and ad campaign* $(z_j, a_j) \in \mathcal{K}_j$ *where j's campaign space is given by* $\mathcal{K}_j = \mathcal{Z} \times \mathcal{A}$ *with* $\mathcal{Z}$ *and* $\mathcal{A}$ *denoting j's policy and ad messaging spaces, and where the electoral campaign space of all candidates is given by* $\mathcal{K} = \prod_{j \in \mathcal{C}} (\mathcal{K}_j)$*. Candidates also use two SIGs valence,* $\sigma_j(z_j)$ *and* $\alpha_j(a_j)$*, generated by the campaign contributions of the SIGs to sway voters in their favour.*

3. ***Voters***: *Voters are characterised by their ideal policy and campaign tolerance level,* $(x_i, t_i) \in \mathcal{Z} \times \mathcal{A}$ *for all* $i \in \mathcal{N}$*, by the importance they give to the direct effect that candidates' policy platform and ad campaign have on their utilities,* (b, e)*, and by the mean sociodemographic, traits and competence valences,* $(\bar{\mathfrak{s}}_j, \bar{\tau}_j, \bar{\kappa}_j)$ *with the distribution of voters' ideal policies and campaign tolerance levels drawn from* $\mathcal{D}_x \times \mathcal{D}_a$*. Voters' preferences for each candidate are also influenced by that candidate's SIGs valences,* $\sigma_j(z_j)$ *and* $\alpha_j(a_j)$ *for* $j \in \mathcal{C}$*. The utility voter i derives from candidate j is given by (1) or (2) is subject to three idiosyncratic components drawn from Type I Extreme Value Distributions,* $\mathcal{S}(0, \pi/6)$*,* $\mathcal{T}(0, \pi/6)$ *and* $\mathfrak{K}(0, \pi/6)$ *whose sum* $\Xi(0, \pi/6)$ *also has a Type I Extreme value distribution.*

4. ***Candidates' Payoff Functions***: *Candidate j's payoff, j's expected vote share function, is given by* $V_j : \Pi_{j \in \mathcal{C}} \mathcal{K}_j \rightarrow [0, 1]$ *in (4) and the vector* $\mathbf{V} = (V_j$ *for all* $j \in \mathcal{C})$ *of vote shares represents the payoff profile of all candidates.*

Let us find the *local Nash equilibrium* (*LNE*) of this stochastic electoral campaign model.

4 The equilibrium of the stochastic election model

A weak local Nash equilibrium (*LNE*) of this stochastic electoral campaign model is such that no candidate may shift its policy or ad messaging level by a small amount – ceteris paribus – to increase its expected vote share. The electoral campaign profile $(\mathbf{z}^*, \mathbf{a}^*)$ is a LNE, iff when candidates choose their policies in $\mathbf{z}^*$ and advertising campaigns in $\mathbf{a}^*$, each candidate's expected vote share function is at a local maximum at $(\mathbf{z}^*, \mathbf{a}^*)$. Formally,

Definition 2 *A weak local Nash equilibrium of the stochastic electoral campaign model* $\mathcal{G}\langle \mathcal{C}, \mathcal{K}, \sigma_j(\cdot), \alpha_j(\cdot), D_x \times D_a, \mathbf{V} \rangle$ *is a vector of candidate policies and advertising campaign levels,* $(\mathbf{z}^*, \mathbf{a}^*)$ *such that for each candidate* $j \in \mathcal{C}$

1. *there exists a small neighborhood of* z_j^**,* $B_j(z_j^*) \subseteq \mathcal{Z}$*, such that*

$$V_j(\mathbf{z}^*, \mathbf{a}^*) \geqslant V_j(\mathbf{z}^*_{-j}, z'_j, \mathbf{a}^*) \text{ for all } z'_j \in B_j(z_j^*) - \{z_j^*\} \quad \text{and}$$

2. there exists a small neighborhood of a_j^*, $B_j(a_j^*) \subseteq \mathcal{A}$, such that

$$V_j(\mathbf{z}^*, \mathbf{a}^*) \geqslant V_j(\mathbf{z}^*, a_j', \mathbf{a}_{-j}^*) \text{ for all } a_j' \in B_j(a_j^*) - \{a_j^*\}.$$

Since the distributions of voters' ideal policies and campaign tolerance levels, $\mathcal{D}_x \times \mathcal{D}_a$ and the observable component of voter i's utility for candidate j, $u_{ij}^O(z_j, a_j)$ in (1) or (2) for all $i \in \mathcal{N}$, are continuously differentiable, then so are $p_{ij}(\mathbf{z}, \mathbf{a})$ in (3) and $V_j(\mathbf{z}, \mathbf{a})$ in (4). It is then easy to calculate how candidate j's expected vote share changes if j marginally adjusts its policy or ad campaign, *ceteris paribus.*

Remark 1 *If in definition 2 we can substitute $\mathcal{Z}$ for $B_j(z_j^*)$ and $\mathcal{A}$ for $B_j(a_j^*)$, then a LNE is also a pure strategy Nash equilibrium (PNE) of the election.*

Remark 2 *In these models, the equilibrium may be characterized by positive eigenvalues of the Hessian of one of the candidates. As a consequence, the expected vote share function of such a candidate fails concavity, so that none of the usual fixed point arguments can be used to assert existence of a "global" pure Nash equilibrium (PNE) of the election. For this reason, we use the concept of a "critical Nash equilibrium" (CNE), a vector of strategies which satisfies the first-order condition for a local maximum of candidates' expected vote share functions. Standard arguments based on the index, together with transversality arguments can be used to show that a CNE exists and that, generically, it will be isolated.[13] A local Nash equilibrium (LNE) satisfies the first-order condition, together with the necessary second-order conditions that the Hessians of all candidates are negative semi-definite at the CNE. Clearly, the set of LNE of the election contains the PNE.*

Given the parameters, $(b, e, \bar{\mathfrak{s}}_j, \bar{\tau}_j, \bar{\kappa}_j)$ for all $j \in \mathcal{C}$, in voter i's utility function in (1), j's SIGs valences $(\sigma_j(\cdot), \alpha_j(\cdot))$ for all $j \in \mathcal{C}$, and the distributions $\mathcal{D}_x \times \mathcal{D}_a$ of voters' ideal policies and campaign tolerance levels, (x_i, t_i) for all $i \in \mathcal{N}$, any profile of policies and advertising messages, $(\mathbf{z}, \mathbf{a})$, can be mapped to a vector of expected vote share functions, i.e., the following vector of expected voters shares can be estimated

$$\mathbf{V} = (V_j(\mathbf{z}, \mathbf{a}) \text{ for all } j \in \mathcal{C}).$$

Candidate j's expected vote share function is at a local maximum if the following first and second order conditions are satisfied. Candidate j's first order necessary conditions (FONC) determine j's best response policy and advertising functions. With two campaign

[13] Ansolabehere & Snyder (2000) show equilibrium existence in multidimensional policy spaces with valence.

instruments (policy and advertising) and c candidates, the game generates $2 \times c$ best response functions and their associated $2 \times c$ *critical*[14] values for each candidate.

To determine whether j's vote share function, for all $j \in \mathcal{C}$, is at a maximum, minimum or saddle point at these critical values, i.e., to determine the local curvature of j's vote share function at these critical values, we use the Hessians matrix of second order partial derivatives of the vote share functions evaluated at these critical values. The *necessary* second order condition (SOC) for j's expected vote share function to be at a maximum at the critical values is that the Hessian be negative semi-definite at these critical values which occurs only when the eigenvalues of the Hessian of the vote shares are non-positive at these critical values.

The critical (C) policy and ad campaign profile of all candidates are then given by

$$\mathbf{z}^C \equiv (z_1^C, ..., z_j^C, ..., z_c^C) \qquad \text{and} \qquad \mathbf{a}^C \equiv (a_1^C ..., a_j^C, ..., a_c^C). \tag{6}$$

The profile $(\mathbf{z}^C, \mathbf{a}^C)$ represents a *prospective*[15] LNE of the election.

5 Candidates' best response functions

We first determine the impact of changes in j's policy or ad messages on voters' choices, then their effects on j's expected vote share, and from these derive j's best response functions.

5.1 Marginal effects on voters' decisions

The *marginal* impact of a change in j's policy or message (z_j or a_j) on the probability that i votes for j is given by *partial* derivative of p_{ij} in (3) *with respect to* (*wrt*) to each choice variable, *ceteris paribus*, i.e.,

$$\mathbf{D}p_{ij}(z_j, a_j) \equiv \begin{pmatrix} \frac{\partial p_{ij}}{\partial z_j} \\ \frac{\partial p_{ij}}{\partial a_j} \end{pmatrix} = p_{ij}(1 - p_{ij}) \begin{pmatrix} b(x_i - z_j) + \frac{\partial \sigma_j}{\partial z_j}(z_j) \\ e(t_i - a_j) + \frac{\partial \alpha_j}{\partial a_j}(a_j) \end{pmatrix}. \tag{7}$$

So that the impact of a marginal change in one of candidate j's choice variables on p_{ij} depends on the *endogenous* probability voter i votes for j, p_{ij} in (3), and for any other candidate, $(1 - p_{ij})$; respectively on the importance voters give to each dimension, b or e and on how far j's corresponding choice variable z_j or a_j is from i's ideal policy or tolerance level, x_i or t_i; and respectively on the marginal impact that changes in j's policy or ad campaign

[14] Called *critical* as at these values the expected vote shares functions may not be at a maximum.
[15] *Prospective* as it is not yet known if the critical choices satisfy the SOC for a maximum.

has on the SIGs donations which in turn affect j's SIG policy or advertising valences, $\sigma_j(z_j)$ and $\alpha_j(a_j)$, as marginal changes in these valences affect i's utility and choice of candidate.

5.2 Candidates' best response functions

Before proceeding we define the following terms used later on.

Definition 3 *Denote by* (z_j^w, a_j^w) *j's weighted electoral mean given by*

$$\begin{pmatrix} z_j^w \\ a_j^w \end{pmatrix} \equiv \sum\nolimits_{i\in\mathcal{N}} w_{ij}^C \begin{pmatrix} x_i \\ t_i \end{pmatrix} \tag{8}$$

$$\text{where} \quad w_{ij}^C \equiv \frac{p_{ij}^C(1-p_{ij}^C)}{\sum_{i\in\mathcal{N}} p_{ij}^C(1-p_{ij}^C)} = \frac{q_{ij}^C}{\sum_{i\in\mathcal{N}} q_{ij}^{kC}}, \tag{9}$$

$$\text{with} \quad p_{ij}^C \equiv p_{ij}(z_j^C, a_j^C) \tag{10}$$

$$\text{and} \quad q_{ij}^C \equiv p_{ij}^C(1-p_{ij}^C). \tag{11}$$

The term w_{ij}^C *in (8) defined in (9) represents the weight* j *gives voter* i *in* j*'s policy and ad campaign and depends on how likely is* i *to vote for* j*,* p_{ij}^C *and to vote for any other candidate,* $(1-p_{ij}^C)$*, relative to all voters.*[16] *Voter* i*'s probability of voting for* j *at* (z_j^C, a_j^C) *is given by* p_{ij}^C *in (9) and (10) and corresponds to* p_{ij} *in (3) when evaluated at* $(\mathbf{z}^C, \mathbf{a}^C)$*. The term* q_{ij}^C *in (9) and (10) gives the probability that* i *votes for* j *or for any other candidate. Moreover, since* i*'s utility in (1) depends on* j*'s critical campaigns,* (z_j^C, a_j^C)*, and since* p_{ij}^C *depends on candidates' policies and ad campaigns,* $(\mathbf{z}^C, \mathbf{a}^C)$*, so do* w_{ij}^C *and* q_{ij}^C*, i.e.,* $w_{ij}^C \equiv w_{ij}(\mathbf{z}^C, \mathbf{a}^C)$*, and* $q_{ij}^C \equiv q_{ij}(\mathbf{z}^C, \mathbf{a}^C)$*.*[17]

Candidate j chooses its policy and ad campaign to maximize its expected vote share $V_j(\mathbf{z}, \mathbf{a})$ in (4) holding the policies and ad campains of all other candidates at their critical values, i.e., holding $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ constant. To find j's best response function take the partial derivative of (4) wrt z_j and a_j and set then equal to zero, i.e.,

$$\mathbf{D}V_j(z_j, a_j) = \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \mathbf{D}p_{ij} = \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} p_{ij}(1-p_{ij}) \begin{pmatrix} b(x_i - z_j) + \frac{\partial\sigma_j}{\partial z_j}(z_j) \\ e(t_i - a_j) + \frac{\partial\alpha_j}{\partial a_j}(a_j) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12}$$

[16] Suppose voters vote with equal probability for j, i.e., vote with probability $p_{ij} = \upsilon$. The weight j gives each voter is $w_{ij} = 1/n$, i.e., j weights all voters equally and according to the inverse of the voting population.

[17] To highlight the dependence of these terms on j's policy and ad campaign, we sometimes write $w_{ij}^C \equiv w_{ij}(z_j^C, a_j^C)$, $p_{ij}^C \equiv p_{ij}(z_j^C, a_j^C)$ and $q_{ij}^C \equiv q_{ij}(z_j^C, a_j^C)$ taking as understood that these terms also depend on the critical campaigns of all other candidates, i.e., depend on $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$.

where the last term follows from (7) and p_{ij} is given by (3). This FONC reduces to

$$\sum_{i\in\mathcal{N}} p_{ij}(1-p_{ij})\begin{pmatrix}(x_i - z_j) + \frac{1}{b}\frac{\partial\sigma_j}{\partial z_j}(z_j)\\ (t_i - a_j) + \frac{1}{e}\frac{\partial\alpha_j}{\partial a_j}(a_j)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Since the importance voters give to the direct effect of candidates' policy and ad campaigns, b and e, in their utility function in (1) and the SIGs valences, $\sigma_j(z_j)$ and $\alpha_j(a_j)$, are constant across voters, after some manipulation, dividing by $\sum_{i\in\mathcal{N}} p_{ij}(1-p_{ij})$ and isolating z_j and a_j, j's *best response functions*[18] are given by

$$\begin{pmatrix}z_j^C\\ a_j^C\end{pmatrix} \equiv \begin{pmatrix}z_j^C(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)\\ a_j^C(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)\end{pmatrix} = \sum_{i\in\mathcal{N}} w_{ij}^C \begin{pmatrix}x_i\\ t_i\end{pmatrix} + \begin{pmatrix}\frac{1}{b}\frac{\partial\sigma_j}{\partial z_j}(z_j^C)\\ \frac{1}{e}\frac{\partial\alpha_j}{\partial a_j}(a_j^C)\end{pmatrix} \tag{13}$$

$$= \begin{pmatrix}z_j^w\\ a_j^w\end{pmatrix} + \begin{pmatrix}\frac{1}{b}\frac{\partial\sigma_j}{\partial z_j}(z_j^C)\\ \frac{1}{e}\frac{\partial\alpha_j}{\partial a_j}(a_j^C)\end{pmatrix}. \tag{14}$$

where w_{ij}^C is given in (9) and where the last line follows after substituting in j's weighted electoral mean in (8).

These best reponse functions implicitly define j's critical policy and ad campaign since w_{ij}^C depends on (z_j^C, a_j^C) as do σ_j and α_j. Moreover, since w_{ij}^C depends on the profile of all candidates, $(\mathbf{z}^C, \mathbf{a}^C)$, j's best response fuctions depend on the policy and ad campaign of all *other* candidates, i.e., depend on $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ as shown in (13).

The best response functions, (z_j^C, a_j^C) in (13), depend on two components. The first component on the right hand side (*RHS*) of (13) is a weighted average of the ideal policies or campaign tolerance levels of voters where the weight candidate j gives voter i along the policy and advertising dimensions is the same and given by w_{ij}^C in (9). Because this component depends exclusively on voters' ideal policies and campaign tolerance levels, it gives a measure of the *electoral pull* voters exert on candidates' policy platforms and ad campaigns.

The weight w_{ij}^C in (9) is non-monotonic in the probability that i votes for j, p_{ij}^C. To see this, recall that the numerator of w_{ij}^C in (9) is given by $q_{ij}^C \equiv p_{ij}^C(1-p_{ij}^C)$ in (11). When i votes for j with probability one (so that $p_{ij}^C = 1$ as i is a *core* supporter) or when i votes for j with probability zero ($p_{ij}^C = 0$), then $q_{ij}^C = 0$ and j gives these two voters zero weight in its campaign choices (i.e., $w_{ij}^C = 0$). When i votes for j with probability $p_{ij}^C = \frac{1}{2}$, i is *undecided*, and since $q_{ij}^C = \frac{1}{4}$, j gives i the highest weight in its campaign choices. Even though the one-person-one-vote principle applies in this model, candidates weight voters differently, e.g., $w_{ij}^C(p_{ij}^C = 0) = w_{ij}^C(p_{ij}^C = 1) = 0 < w_{ij}^C(p_{ij}^C = \frac{1}{2})$. In particular, j caters to undecided voters

[18] All functions and values evaluated at the critical campaigns are identified by superscript C.

by giving them a higher weight in its best response choice functions.

The second component on the RHS of (13) shows the effect that changes in j's policy or ad message have on j's best response functions through j's SIG valences. Since these valences depend on the campaign contributions of the SIGs, they depend on j's response to the demands that the SIGs place on j. That is, these valences measure the effect that the SIGs exert on j's policy or ad campaign and thus represent the *SIG pull* on candidate j's best response policy or ad campaign. The magnitude of the effect of the SIGs valences on each best response function depends on the marginal impact that the SIG valence has on voters' utility at (z_j^C, a_j^C), either $\frac{\partial \sigma_j}{\partial z_j}(z_j^C)$ or $\frac{\partial \alpha_j}{\partial a_j}(a_j^C)$, and on the inverse of the importance voters gives to the policy or ad campaign, $\frac{1}{b}$ or $\frac{1}{e}$, which effectively measures the importance voters give to the SIG valences.

Using (14), we can re-write (13) as

$$\begin{pmatrix} z_j^w - z_j^C \\ a_j^w - a_j^C \end{pmatrix} + \begin{pmatrix} \frac{1}{b}\frac{\partial \sigma_j}{\partial z_j}(z_j^C) \\ \frac{1}{e}\frac{\partial \alpha_j}{\partial a_j}(a_j^C) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{15}$$

where the first term on the left hand side (LHS) of (15) measures the *marginal electoral pul*l that voters exert on candidate j at (z_j^C, a_j^C) and the second term the *marginal SIG pull* that SIGs exert on j at (z_j^C, a_j^C). In essence, (15) shows that j's best response policy and ad campaign are determined by the relative magnitude of the electoral and SIG pulls, i.e., that j sets its policy and ad campaign in such a way as to *balance* the SIG and electoral pulls.

We now examine if candidates adopt a common campaign.

6 Do candidates adopt a common campaign?

Our objective is to find candidates' local Nash equilibrium (LNE) policies and advertising campaigns of this electoral campaign game. After the following definitions, we examine if candidates *converge to* or *adopt* a *common* campaign.

Definition 4 *Let (z_m, a_m) represent voters' electoral mean, i.e., the mean of voters' ideal policies and campaign tolerance levels, given by*

$$\begin{pmatrix} z_m \\ a_m \end{pmatrix} \equiv \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \begin{pmatrix} x_i \\ t_i \end{pmatrix}. \tag{16}$$

Lemma 1 *Generally, there is no convergence to common electoral campaign.*

Proof. *By contradiction.* The gist of the proof is as follows. Assume candidates adopt identical campaigns. In this case, the utility difference that voters experience between any two candidates is independent of their ideal policy and campaign tolerance but depends on the SIG valences as well as on voters' mean sociodemographic, traits and competency valences. Since the probability of voting for each candidate is the same across voters, candidates weight voters equaly in their campaigns. When the effect of the SIG valences on voters vary across candidates so do candidates' campaigns. Citizens then vote with different probabilities for each candidate which futher induces differences across candidates' campaigns. Contradicting the assumtions of identical campaigns.

Suppose candidates adopt the same (s) campaign, i.e., $(z_j, a_j) = (z^s, a^s)$ for all $j \in \mathcal{C}$. The probability that i chooses j, p_{ij} in (3), at (z^s, a^s) depends on the *difference* between the *observable* components of i's utility from candidates h and j, so that using (1) is given by

$$\begin{aligned} u_{ih}^{sO} - u_{ij}^{sO} &\equiv u_{ih}^{O}(z^s, a^s) - u_{ij}^{O}(z^s, a^s) \qquad (17)\\ &= [\sigma_h(z^s) - \sigma_j(z^s)] + [\alpha_h(a^s) - \alpha_j(a^s)] + (\bar{\mathfrak{s}}_h - \bar{\mathfrak{s}}_j) + (\bar{\tau}_h - \bar{\tau}_j) + (\bar{\kappa}_h - \bar{\kappa}_j) \end{aligned}$$

since the negative quadratic policy and ad campaign components of voter i's utility functions in (1) cancel out implying that this utility difference does not depend on voters' ideal policies or campaign tolerance levels and is thus the same across voters. This utility difference depends on the difference between candidates' SIGs policy and ad campaign valences and on the diferences between candidates' mean sociodemographic, traits and competence valences which are exogenously given. With candidates adopting the same policy, z^s, the SIG policy valence difference depends only on the difference between the two SIG policy valence functions, similarly for the SIG advertising valence difference.

Using (17), the probability that voter i chooses candidate j is the *same* for all voters as p_{ij} in (3) when evaluated at (z^s, a^s) reduces to

$$\begin{aligned} p_j^{sO} &= \left[\sum_{h\neq j, h=1}^{c} \exp[u_{ih}^{sO} - u_{ij}^{sO}]\right]^{-1} \qquad (18)\\ &= \left[\sum_{h\neq j, h=1}^{c} \exp[\sigma_h(z^s) - \sigma_j(z^s) + \alpha_h(a^s) - \alpha_j(a^s) + (\bar{\mathfrak{s}}_h - \bar{\mathfrak{s}}_j) + (\bar{\tau}_h - \bar{\tau}_j) + (\bar{\kappa}_h - \bar{\kappa}_j)]\right]^{-1}, \end{aligned}$$

i.e., ρ_j^{sO} depends only on the difference between the SIG policy and ad campaigns valence functions of the two candidates and on the difference between the mean sociodemographic, traits and competence valences of the two candidates.

Since p_j^{sO} is independent of voter i's ideal policy or campain tolerance level, everyone votes with the same probability for candidate j, i.e., $p_{ij} = p_j^{sO}$ for all $i \in \mathcal{N}$. Moreover, j weights all voters equally its electoral campaign as w_{ij}^C in (9) reduces to

$$w^{sO} \equiv \frac{p_j^{sO}(1-p_j^{sO})}{\sum_{i\in\mathcal{N}} p_j^{sO}(1-p_j^{sO})} = \frac{1}{n}, \tag{19}$$

i.e., j weights voters according to the inverse of the voting population.

In this case, j's critical *choices,* (z_j^C, a_j^C) in (13) using (19), are given by

$$\begin{aligned}\begin{pmatrix} z_j^C \\ a_j^C \end{pmatrix} &= \sum_{i\in\mathcal{N}} w^{sO}\begin{pmatrix} x_i \\ t_i \end{pmatrix} + \begin{pmatrix} \frac{1}{b}\frac{\partial\sigma_j}{\partial z_j}(z^s) \\ \frac{1}{e}\frac{\partial\alpha_j}{\partial a_j}(a^s) \end{pmatrix} = \frac{1}{n}\sum_{i\in\mathcal{N}}\begin{pmatrix} x_i \\ t_i \end{pmatrix} + \begin{pmatrix} \frac{1}{b}\frac{\partial\sigma_j}{\partial z_j}(z^s) \\ \frac{1}{e}\frac{\partial\alpha_j}{\partial a_j}(a^s) \end{pmatrix} \\ &= \begin{pmatrix} z_m \\ a_m \end{pmatrix} + \begin{pmatrix} \frac{1}{b}\frac{\partial\sigma_j}{\partial z_j}(z^s) \\ \frac{1}{e}\frac{\partial\alpha_j}{\partial a_j}(a^s) \end{pmatrix}\end{aligned} \tag{20}$$

where in the last line (z_m, a_m) represents *electoral mean* given in (16).

From (20) it is evident that candidates adopt the same campaign, (z^s, a^s), only when the SIG policy ad valence functions are *identical* across candidates, i.e., when $\sigma_j = \sigma$ and $\alpha_j = \alpha$ for all $j \in \mathcal{C}$. In this case, p_j^{sO} in (18) reduces to

$$p_j^{sO} = \left[\sum_{h\neq j, h=1}^{c} \exp[u_{ih}^{sO} - u_{ij}^{sO}]\right]^{-1} = \left[\sum_{h\neq j, h=1}^{c} \exp[(\bar{\mathfrak{s}}_h - \bar{\mathfrak{s}}_j) + (\bar{\tau}_h - \bar{\tau}_j) + (\bar{\kappa}_h - \bar{\kappa}_j)]\right]$$

i.e., depends only on the differences between voters' sociodemographic, traits and compentence valences since the SIG valence differences in (17) cancel out. In this case, using (20), candidates' policy platform and ad campaign is then given by

$$\begin{pmatrix} z_j^C \\ a_j^C \end{pmatrix} = \begin{pmatrix} z_m \\ a_m \end{pmatrix} + \begin{pmatrix} \frac{1}{b}\frac{\partial\sigma}{\partial z_j}(z^s) \\ \frac{1}{e}\frac{\partial\alpha}{\partial a_j}(a^s) \end{pmatrix} \equiv \begin{pmatrix} z^s \\ a^s \end{pmatrix}. \tag{21}$$

When $\sigma_j = \sigma \equiv 0$ and $\alpha_j = \alpha \equiv 0$ for all $j \in \mathcal{C}$, either because the SIGs have no influence on candidates electoral campaing or because the SIG valences exert no influence on voters' choices, (21) shows that candidates adopt the electoral mean, (z_m, a_m), as their campaign strategy. These are the cases examined in Gallego and Schofield (2016).

When candidates and voters care about the SIG valences and candidates face the same SIG valences, i.e., when $\sigma_j = \sigma \neq 0$ and $\alpha_j = \alpha \neq 0$ for all $j \in \mathcal{C}$, then (21) shows candidates adopt the *same* campaign even though they do *not* adopt the electoral mean (z_m, a_m) as their campaign strategy. In terms of policy, all candidates locate on the same side of z_m, a fact not borne out by the empirical evidence.

The general case is one where the SIG valence functions differ across candidates, i.e., where $\sigma_j \neq \sigma_h$ and $\alpha_j \neq \alpha_h$ for all $h \neq j \in \mathcal{C}$. In this case, (20) shows that even if candidates weight voters equally, which is not the case, the differences in SIGs valences would lead them to adopt different policy platforms and ad campaigns. Moreover, since candidates adopt different campaigns, the difference in voter i's utility from candidates h and j is *not* given by (17) as the quadratic components of i utility from h's and j's policy and ad campaign no longer cancel each other out implying that the probability that i votes for j and h differ, i.e., that $p_{ij} \neq p_{ih}$ for all $h \neq j \in \mathcal{C}$ and $i \in \mathcal{N}$. Candidates give then voters different weights in their campaigns, i.e. $w_{ij} \neq w_{kj}$ in (9) for all $k \neq i \in \mathcal{N}$. From (13), it is clear that j's electoral pull is *not* towards voters' electoral mean (z_m, a_m) in (16), but rather towards j' *weighted electoral mean*, (z_j^w, a_j^w) in (8). Note that since the weights w_{ij} vary across voters and candidates, this weighted mean varies across candidates.

In conclusion, when the SIGs valence functions differ across candidates, this difference and the different weights candidates give voters in their campaign lead candidates to adopt different campaigns, contrary to the assumption made at the beginig of this proof. ■

Let us now determine the conditions under which candidates adopt their *critical* campaigns as their campaign strategy.

7 Convergence to (z_j^C, a_j^C)

We now derive the conditions under which j *adopts* its critical campaign, (z_j^C, a_j^C) assuming that all other candidates adopt their critical campaigns, i.e., holding $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$ constant.

The Appendix shows that the weak LNE of the election is characterized by a pivotal probability for candidate j for each voter from which j's pivotal vote share is derived identifying the *necessary* condition under which j adopts (z_j^C, a_j^C) as its campaign. Before summarizing the results presented in the Appendix in Section 10.2, we give the following definitions that facilitate the discussion of these results.

Definition 5 *Denote voter i's necessary pivotal probability for candidate j at (z_j^C, a_j^C) by*

$$\varphi_{ij}^C \equiv \varphi_{ij}(\mathbf{z}^C, \mathbf{a}^C) \equiv \frac{1}{2} - \frac{1}{4}\frac{Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)}{Tr\left(\boldsymbol{\Delta}_{ij}^C\right)} = \frac{1}{2} - \frac{1}{4}\frac{\left[b - \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C) + e - \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C)\right]}{b(x_i - z_j^w)^2 b + e(t_i - a_j^w)^2 e} \quad (22)$$

where the matrix $\boldsymbol{\iota}$ denotes the importance voters give to each dimension and $\mathbf{D}^2 v_j^C$ gives the rate at which the SIG valence functions impact i's utility from j at (z_j^C, a_j^C) which are negative for both SIG valences since $\frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j) < 0$ for all $z_j \in \mathcal{Z}$ and $\frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j) < 0$ for all

$a_j \in \mathcal{A}$ and where the SIG valences exert no cross partial effects on voters' utility in (1), i.e., where $\frac{\partial^2 \sigma_j}{\partial z_j \partial a_j} \equiv 0$ and $\frac{\partial^2 \alpha_j}{\partial a_j \partial z_j} \equiv 0$. So that, $\boldsymbol{\iota}$ and $\mathbf{D}^2 v_j^C$ are given by

$$\boldsymbol{\iota} \equiv \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix} \quad \textit{and} \quad \mathbf{D}^2 v_j^C \equiv \begin{pmatrix} \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C) & 0 \\ 0 & \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C) \end{pmatrix}. \tag{23}$$

The rate at which voter i's marginal utility changes in z_j and a_j is given by the matrix $\left(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C\right)$ in (22) and defined by

$$\boldsymbol{\iota} - \mathbf{D}^2 v_j^C \equiv \begin{pmatrix} b - \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C) & 0 \\ 0 & e - \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C) \end{pmatrix} \tag{24}$$

with the trace of $\left(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C\right)$ being always positive and given by

$$Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C) \equiv \left[b - \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C)\right] + \left[e - \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C)\right] > 0. \tag{25}$$

Using $\boldsymbol{\iota}$ in (23), the $\boldsymbol{\Delta}_{ij}^C$ in (22) is the $\boldsymbol{\iota}-$weighted variance-covariance matrix of i's preferences around j's weighted electoral mean (z_j^w, a_j^w) in (8) defined by

$$\boldsymbol{\Delta}_{ij}^C \equiv \begin{pmatrix} b(x_i - z_j^w)^2 b & b(x_i - z_j^w)(t_i - a_j^w)e \\ b(x_i - z_j^w)(t_i - a_j^w)e & e(t_i - a_j^w)^2 e \end{pmatrix}, \tag{26}$$

Note that the SIG valences, $\sigma_j(z_j^C)$ and $\alpha_j(a_j^C)$, affect (26) through (z_j^w, a_j^w) as they depend on the weight j gives i, w_{ij}^C in (9) which depends on p_{ij}^C in (10) and thus depends on voter i's utility in (1) that depend on the SIG valences. The trace $\boldsymbol{\Delta}_{ij}^C$ measuring the aggregate variance of i's preferences around j's weighted electoral mean is always positive and given by

$$Tr(\boldsymbol{\Delta}_{ij}^C) = b(x_i - z_j^w)^2 b + e(t_i - a_j^w)^2 e > 0. \tag{27}$$

Candidate j's expected vote share in (4) evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$ is given by

$$V_j^C \equiv \frac{1}{n} \sum\nolimits_{i \in \mathcal{N}} p_{ij}^C. \tag{28}$$

Define j's necessary pivotal vote share[19] *at* $(\mathbf{z}^C, \mathbf{a}^C)$, *as*

$$\Phi_j^C \equiv \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \varphi_{ij}^C \equiv \frac{1}{2} - \frac{1}{4}\frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \frac{Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)}{Tr\left(\boldsymbol{\Delta}_{ij}^C\right)} \tag{29}$$

where $Tr\left(\boldsymbol{\Delta}_{ij}^C\right)$ *by (27) and* $Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)$ *by (25).*

The following proposition gives the necessary condition for the critical campaign vector, $(\mathbf{z}^C, \mathbf{a}^C)$, to be a weak local Nash equilibrium (LNE) of the electoral game.

Proposition 1 ***Convergence to the critical campaigns***
The weak local Nash equilibrium of the electoral campaign game, $\mathcal{G}\langle\mathcal{C},\mathcal{K},\sigma_j(\cdot),\alpha_j(\cdot),D_x\times D_a,\mathbf{V}\rangle$ *is characterized by the following conditions.*

- *If* $\Phi_j^C \leqslant V_j^C$ *for all* $j \in \mathcal{C}$, *then the necessary condition for convergence to* $(\mathbf{z}^C, \mathbf{a}^C)$ *has been met by all candideates. Candidates' critical campaigns,* $(\mathbf{z}^C, \mathbf{a}^C)$, *constitute a weak LNE of the election.*

- *If* $V_j^C < \Phi_j^C$ *for some* $j \in \mathcal{C}$, *the necessary condition for convergence to* (z_j^C, a_j^C) *has* ***not*** *been met by j. Candidate j has an incentive to move away from* (z_j^C, a_j^C) *to increase its vote share as may other candidates. Candidates' critical campaigns,* $(\mathbf{z}^C, \mathbf{a}^C)$, *are* ***not*** *a LNE of the election.*

Proof*: See the Appendix in Section 10.2.*

The intuition behind Proposition 1 follows the proofs provided in the Appendix assuming that all other candidates adopt their critical campaigns, i.e., given $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$.

The *necessary* convergence condition for candidate j to adopt (z_j^C, a_j^C) as its campaign – also given in the Appendix in (56) – is that

$$\Phi_j^C < V_j^C \tag{30}$$

where V_j^C is given by (28) and Φ_j^C by (29). In words, (30) says that the necessary condition for j to adopt (z_j^C, a_j^C) as its campaign is that j's expected vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C, be *high* enough, i.e., is greater than j's necessary pivotal vote share, Φ_j^C.

Condition (**??**) is met when there are *enough* voters for whom the *necessary* condition *on voter i* for candidate j is satisfied, i.e. when there are enough voters for whom the probability that i votes j be *higher* than i's necessary pivotal probability, φ_{ij}^C in (22), i.e., that

$$\varphi_{ij}^C < p_{ij}^C. \tag{31}$$

[19] Since Φ_j^C depends on j's critical campaign, (z_j^C, a_j^C), it varies across candidates.

Averaging the LHS of (31) over voters gives j's *necessary pivotal vote share*, Φ_{ij}^C in (29) and averaging the RHS of (31) over voters j's expected vote share, V_j^C in (28).

Thus, when $\varphi_{ij}^C < p_{ij}^C$ in (31) is satisfied for *enough* voters so that $\Phi_j^C < V_j^C$ in (**??**) is satisfied, then j adopts (z_j^C, a_j^C) as it campaign. When $\Phi_j^C < V_j^C$ in (**??**) is satisfied for all $j \in \mathcal{C}$, candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, consitute a *weak* LNE of the election. These conditions do not have to be satisfied for the same voters for all candidates.

When $V_j^C < \Phi_j^C$, j's expected vote share does *not* meet the necessary convergence condition in (**??**), j moves away from (z_j^C, a_j^C) to increase its votes since j's vote share is at a minimum or at a saddle-point at $(\mathbf{z}^C, \mathbf{a}^C)$. If j moves away from (z_j^C, a_j^C) to increase its expected vote share, other candidates may also find it in their interest to change their electoral campaigns. Candidates' critical campaigns, $(\mathbf{z}^C, \mathbf{a}^C)$, are then *not* a LNE of the election.

8 The literature

SIG matter in US elections because politicians use their contributions to influence voters' electoral choices as shown, for example, by Bartels (2008) for the primaries and for Presidential elections. Kenski et al (2010) chart the influence of money and thus advertising on the contest between Obama and McCain. Schofield and Schnidman (2011) show that the relative popularity of Obama and Clinton in the 2008 primary races was largely determined by their campaign expenditures.

Some study the effects of campaign contributions or expenditures on US elections ignoring differences across states. In Coate (2004) SIG provide contributions to like-minded candidates to finance ad campaigns that provide information to voters about candidates' ideologies and shows that contribution limits redistribute welfare from voters to interest groups. In Meirowitz (2008) candidates select their campaign effort and shows that marginal asymmetries in costs or technology can explain incumbency advantage. Herrera et al (2008) find that greater volatility in voters' preferences forces candidates to spend more on the election. In Ashworth and Bueno de Mesquita (2009) candidates buy valence to increase their electoral chances. Snyder and Ting (2008) model contracts between SIG and politicians. Ansolabehere et al (2003) study Congressional and Presidential election campaign contributions up to 2000 with candidates, parties and organizations raising and spending about \$3 billion in the 1999-2000 election cycle, a small amount compared to the \$2 trillion spent by the federal government. They argue that the reason so little is spent on campaigns is that contributions are a consumption, rather than an investment, good but note that the electoral motive is not insignificant as the marginal impact of \$100,000 spent in a House race, ceteris paribus, is about 1% gain in vote. Thus, highlighting the powerful influence of ad campaigns on elections.

Others develop models to study the effect that advertising campaigns have on US Presidential elections at the state level ignoring the policy side of the campaign. Brams and Davis (1973) examine the allocation of campaign resources across states in a probabilistic voting model where candidates are perfectly symmetric, have no political leanings, have the same "electoral production function" across states and either maximize their expected electoral College votes (ECVs) or the expected popular vote. Under direct popular vote, they find that resources are allocated in proportion to the number of uncommitted voters in each state. While under the ECV there is no pure strategy Nash equilibrium allocation of resources, an equilibrium exists if candidates with identical resources match each other's resource expenditures with candidates spending the majority of their resources in states with large ECVs relative to their population. Colantoni et al (1975) find that a proportional rule that accounts for the ex-post closeness of the election was a better predictor of campaign allocations across states than those predicted using Brams and Davis' model.

Snyder (1989) develops two-party legislative election models where parties allocate their campaign re-sources across districts to maximize either the expected number of seats or the probability of winning a majority of seats. Under seat maximization, spending in a district is higher the closer that district's race. When aiming for a majority of seats, if party D has more safe districts than R and the races in all safe districts are equally competitive, then more resources are spent in D.s safe districts than in R.s safe districts. Even if the races of R.s safe districts are closer, spending may be higher in D.s safe districts.

Some develop two party probabilistic voting models where candidates choose redistributive policies (taxes, public spending, transfers, rents) to allocate between groups of voters or districts under different electoral systems ignoring campaign advertising. In these models, parties are uncertain about the voters' preferences when announcing their platforms. Lindbeck and Weibull (1987) study balanced-budget redistributions and find that if voters have identical consumption preferences but differ in their known party preferences, redistribution by both parties favor marginal voters who have weak party preferences. Person and Tabellini (1999) model three states having one Electoral College vote each and find that majoritarian rather than proportional elections lead parties to concentrate their resources in key marginal districts and to produce less public goods, fewer rents for politicians, more redistribution and larger governments. In Lizzeri and Persico (2001) less public good is provided in winner-take-all (relative to proportional representation) systems as benefits are widely spread compared to pork-barrel projects targeted to groups of voters. In contrast to Person and Tabellini (1999), Milesi-Ferretti et al (2002) find that government transfers (goods and services) is higher in proportional (majoritarian) systems with the effect of the electoral system on government spending depending on the share of transfers in total spending and thus on the distribution of voters' preferences.

Others examine how Special Interest Groups (SIG) influence electoral outcomes at the national level, ignoring the advertising side of the campaign. Grossman and Helpman (GH, 1994, 1996, 2001) model how SIG promote the electoral chances of a preferred candidate and/or influence candidates' policy platforms in elections. Studying only national outcomes, they show how SIG: use their superior knowledge of policy environments to influence candidates' policies; educate voters; influence legislative policy choices when members have diverse objectives; and influence party's positions. In their 1996 paper, they define two distinct motives for SIG: "Contributors with an electoral motive intend to promote the electoral prospects of preferred candidates. Those with an influence motive aim to influence the politicians' policy pronouncements."

Levebvre and Martimort (2017) develop a model in which two SIGs with conflicting interest influence a decision-maker's policy ignoring elections and differences across states. They model the influence of SIGs as a common agency game in which individuals must decide whether to form a SIG. When individuals' preferences are common knowledge, there is efficient group formation and competition across SIG perfectly aggregates preferences to achieve the right policy though contingent payments offered to candidates. When preferences are private information, the free riding problem in collective action arises within groups. In this case, when groups form they choose lobbyists with moderate preferences and lobbying competition aggregates preferences imperfectly and show that a strong coalition has a greater impact on the decision-maker's policy.

This literature review indicates that campaign contributions and advertising have been studied separately mainly because as Coate (2004, 774) states: .incorporating campaign contributions into the theory of electoral competition has proved a major challenge for political theorists. First, it is necessary to take a stand on how and why campaign spending impacts voter behavior and why contributors give to candidates. These are issues on which the empirical literature offers no unambiguous guidance. Then one faces the analytical challenge of incorporating the behavior of campaign contributors with that of political parties and voters.

9 Conclusion

This paper generalizes Schofield (2007) in several directions. We allow candidates to use two electoral campaign instruments — policy and advertising — in their bid for office as well as to use the campaign contributions of SIGs to generate two SIGs valences to enhance their electoral valences and their electoral prospects in a world where voters' preferences depend on policies, advertising campaigns, the SIGs valences and on the mean sociodemographic, traits and competency valences that are subject to three idiosyncratic shocks. We extend Schofield's (2006) model by allowing candidates to run ad campaigns and extend Gallego and Schofield's (2016) policy and advertising campaign model by allowing the SIGs valences to influence voters' decissions.

The introduction of campaign messages and voters' campaign tolerance levels allows us to show that candidates must not only find an optimal policy with which to run in the election that balances the electoral and SIGs pulls but also that they must find the optimal message frequency in order not to alienate to many voters or the SIGs. It shows that candidates may not necessarily adjust their policy platforms but that they can instead adjust their ad messages to increase their expected vote shares.

Our results show even though the one-person-one-vote principle applies in this model, candidates weight voters differently, giving more weight to undecided voters than to committed voters.

We derive the necessary condition for candidates to adopt their critical campaigns and characterize these conditions in very intuitive terms showing that the weak local Nash equilibrium is characterized by a pivotal vote share for every voter from which candidates' pivotal vote shares are derived for each candidate. Proposition 1 summarizes the necessary condition for candidates to adopt their critical campaigns expressed in terms of pivotal vote shares. This proposition shows that candidate j adopts (z_j^C, a_j^C) as its critical campaign when there must be enough voters voting for candidate j with high enough probability. When the necessary condition is satisfied for all candidates, $(\mathbf{z}^C, \mathbf{a}^C)$ is a weak LNE of the election. When it is *not*, meaning that the necessary condition is *not* satisfied for at least one candidate, $(\mathbf{z}^C, \mathbf{a}^C)$ is a not a LNE of the election.

In contrast to Gallego and Schofield (2016) where there are no SIGs valences, in the weak local Nash equilibrium (LNE) of the election of this model, candidates do *not* adopt the electoral mean as their campaign strategy. Candidates' policy and advertising campaigns are determined by two components. One given by candidates' weighted electoral mean, where the endogenous weight is the weight given by candidates to voters in their campaigns. Since candidates weight voters differently, giving more weight to undecided than to committed voters, policies and advertising campaigns are more reflective of the ideal policies and campaign tolerance levels of uncommitted voters. The second one corresponds to the marginal effect that the SIGs valences have voters' uitility weighted by the relative importance voters give to these valences. Since these two components are respectively associated with the electoral and SIG pulls, in equilibrium candidates campaigns balance the electoral and SIG pulls.

References

[1] Ansolabehere S, de Figueiredo JM, Snyder J. 2003. Why is there so little Money in U.S.Politics? J*ournal of Economic Perspectives*, 17: 105-130.

[2] Ansolabehere S, Snyder J. 2000 Valence Politics and Equilibrium in Spatial Election Models. *Public Choice* 103: 327-336.

[3] Ashworth S, Bueno de Mesquita E. 2009. Elections with Platform and Valence Competition. *Games and Economic Behavior* 67: 191–216.

[4] Bartels L. 2008. *Unequal Democracy*. Princeton: Princeton University Press.

[5] Bonica A. 2014. Mapping the Ideological Marketplace. *American Journal of Political Science*, 58(2): 367–386.

[6] Brams S, Davis M. 1973. Resource allocation models in presidential campaigning: implications for democratic representation. *Annals of the New York Academy of Sciences* 219(1): 105-123.

[7] Coate S. 2004. Political Competition with Campaign Contributions and Informative Advertising. *Journal of the European Economic Association* 2: 772-804.

[8] Colantoni C, Levesque T, Ordeshook P. 1975. Campaign Resource Allocations Under the Electoral College. *American Political Science Review*, 69(1), 141-154.

[9] Fernando G. (2016, November 30). How social media helped Trump take over the world. Retrieved August 1, 2017, from http://www.news.com.au/finance/work/leaders/how-donald-trump-used-social-media-to-take-over-the-world/news-story/97567c6a71fb6aa8940fb2d4f133dd68

[10] Gallego M, Schofield N. 2016. Modelling the effect of campaign advertising on US presidential elections. In *The Political Economy of Social Choices*, Gallego M, Schofield N [Eds]. Springer.

[11] Grossman G, Helpman E. 1994. Protection for Sale. *American Economic Review* 84: 833-50.

[12] Grossman GM, Helpman E. 1996. Electoral Competition and Special Interest Politics. *The Review of Economic Studies* 63: 265-286.

[13] Grossman, GM, Helpman E. 2001. *Special Interest Groups.*Cambridge, MA.: MIT Press.

[14] Herrera H, Levine D, Martinelli C. 2008. Policy Platforms, Campaign Spending and Voter Participation. *Journal of Public Economics* 92: 501-513.

[15] Huber GA, Arceneaux K. 2007. Identifying the Persuasive Effects of Presidential Advertising. *American Journal of Political Science*, 51(4), 957-977.

[16] Kenski K, Hardy BW, Jamieson KH. 2010. *The Obama Victory.* Oxford: Oxford University Press.

[17] Lefebvre P, Martimort D. 2017. "When Olson meets Dahl": From innefficient group formation to ineffcient policy-making. Unpublished manuscript. Plenary lecture presented at the Association for Public Economic Theory in Paris, July.

[18] Lindbeck A, Weibull J. 1987. Balanced-Budget Redistribution as the Outcome of Political Competition. *Public Choice*, 52(3) 273-97.

[19] Lizzeri A, Persico N. 2001. The Provision of Public Goods under Alternative Electoral Incentives. *American Economic Review* 91(1) 225-39.

[20] Meirowitz A. 2008. Electoral Contests, Incumbency Advantages and Campaign Finance. *Journal of Politics*, 70: 681-699.

[21] Persson T, Tabellini G. 1999. The Size and Scope of Government: Comparative Politics with Rational Politicians. *European Economic Review* 43(4-6): 699-735.

[22] Rivero C. 2016. How marketing helped Donald Trump win the 2016 election. The Washington Post, Posted November 17, 2016 at https://www.washingtonpost.com/graphics/politics/2016-election/trump-campaign-marketing/.

[23] Schofield N. 2006. Equilibria in the Spatial Stochastic Model of Voting with Party Activists, *Review of Economic Design* 10 (3): 183-203.

[24] Schofield N. 2007. The mean voter theorem: Necessary and sufficient conditions for convergent equilibrium, *Review of Economic Studies* 74, 965-980.

[25] Schofield N, Schnidman E. 2011. Support for Political Leaders. *Homo Oeconomicus*, 28: 7-47.

[26] Snyder JM. 1989. Election Goals and the Allocation of Campaign Resources. *Econometrica*, 57(3): 637–660.

[27] Snyder J, Ting M. 2008. Interest groups and the electoral control of politicians. Journal of Public Economics. 92(3–4): 482–500.

[28] TRAIN K. 2003. *Discrete Choice Methods for Simulation*. Cambridge, U.K., Cambridge University Press.

10 Appendix

10.1 Second order conditions (SOC)

Section 5 derived candidates' best response campaign functions. We now determine the conditions under which candidates' vote share functions, V_j in (4) for all $j \in \mathcal{C}$, are at a local maximum at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e., for $(\mathbf{z}^C, \mathbf{a}^C)$ to be a LNE of the election.

Candidate j's critical choices, (z_j^C, a_j^C), are implicitly defined by (13). We use the *Hessian matrix of second order partial derivatives* of candidate j's vote share function, V_j in (4), to determine whether V_j is at a maximum at (z_j^C, a_j^C) holding the electoral campaigns of all other candidates at $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$. To find this Hessian we need the second order partial derivatives of the probability that i votes for j, $\mathbf{D}^2 p_{ij}$ evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$, $\mathbf{D}^2 p_{ij}^C$, using p_{ij} in (3). Then using $\mathbf{D}^2 p_{ij}^C$ we find the Hessian of j's vote share, $\mathbf{D}^2 V_j$ at $(\mathbf{z}^C, \mathbf{a}^C)$.

$\mathbf{D}^2 p_{ij}^C$ is the partial derivative of (7) wrt z_j and a_j evaluated at (z_j^C, a_j^C), i.e.,

$$\begin{aligned}
\mathbf{D}^2 p_{ij}^C &\equiv \mathbf{D}^2 p_{ij}(\mathbf{z}^C, \mathbf{a}^C) \equiv \begin{pmatrix} \frac{\partial^2 p_{ij}}{\partial (z_j)^2} & \frac{\partial^2 p_{ij}}{\partial a_j \partial z_j} \\ \frac{\partial^2 p_{ij}}{\partial z_j \partial a_j} & \frac{\partial^2 p_{ij}}{\partial (a_j)^2} \end{pmatrix} \\
&= p_{ij}^C (1 - p_{ij}^C) \left[(1 - 2p_{ij}^C) \left[\boldsymbol{\iota} \mathbf{d}_{(\mathbf{z}^C, \mathbf{a}^C)}^{ij} + \mathbf{D} v_j^C \right]^{\mathbf{T}} \left[\boldsymbol{\iota} \mathbf{d}_{(\mathbf{z}^C, \mathbf{a}^C)}^{ij} + \mathbf{D} v_j^C \right] - (\boldsymbol{\iota} - \mathbf{D}^2 v_j^C) \right] \\
&= p_{ij}^C (1 - p_{ij}^C) \left[(1 - 2p_{ij}^C) \boldsymbol{\Delta}_{ij}^C - (\boldsymbol{\iota} - \mathbf{D}^2 v_j^C) \right] \qquad (32)
\end{aligned}$$

$$\text{where} \quad p_{ij}^C \equiv p_{ij}(\mathbf{z}^C, \mathbf{a}^C) \qquad (33)$$

$$\boldsymbol{\iota} \equiv \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}, \quad \mathbf{d}_{(\mathbf{z}^C, \mathbf{a}^C)}^{ij} \equiv \left[\begin{array}{cc} (x_i - z_j^C) & (t_i - a_j^C) \end{array} \right], \qquad (34)$$

$$\mathbf{D} v_j^C \equiv \mathbf{D} v_j(z_j^C, a_j^C) \equiv \begin{pmatrix} \frac{\partial \sigma_j}{\partial z_j}(z_j^C) & \frac{\partial \alpha_j}{\partial a_j}(a_j^C) \end{pmatrix}, \qquad (35)$$

$$\mathbf{D}^2 v_j^C \equiv \mathbf{D} v_j(z_j^C, a_j^C) \equiv \begin{pmatrix} \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C) & \frac{\partial^2 \sigma_j}{\partial z_j \partial a_j} \equiv 0 \\ \frac{\partial^2 \alpha_j}{\partial a_j \partial z_j} \equiv 0 & \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C) \end{pmatrix} \qquad (36)$$

and where $\mathbf{T}$ denotes the transpose of a vector. The probability that i votes for j at $(\mathbf{z}^C, \mathbf{a}^C)$,

p_{ij}^C in (33) is given by p_{ij} in (3) when evaluated at $(\mathbf{z}^C, \mathbf{a}^C)$. The diagonal of matrix $\boldsymbol{\iota}$ in (34), (b, e), gives the *importance* voters assign to the *direct* effect that differences with j's policy and ad campaign have on their utility function in (1). The vector $\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ij}$ in (32) and (34) measures the *distance* between i's ideals (x_i, t_i) and j's critical choices (z_j^C, a_j^C).

The vector $\mathbf{D}v_j^C$ in (32) and (35) gives the marginal effect that the SIG valences $(\sigma_j(z_j), \alpha_j(a_j))$ have on i's utility from j in (1) when evaluated at the critical values (z_j^C, a_j^C) which is positive for both SIG valences since $\frac{\partial \sigma_j}{\partial z_j}(z_j) > 0$ for all $z_j \in \mathcal{Z}$ and $\frac{\partial \alpha_j}{\partial a_j}(a_j) > 0$ for all $a_j \in \mathcal{A}$.[20] The matrix $\mathbf{D}^2 v_j^C$ in (36) gives the rate at which the SIG valence functions impact i's utility from j at (z_j^C, a_j^C) which is negative for both SIG valences since $\frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j) < 0$ for all $z_j \in \mathcal{Z}$ and $\frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j) < 0$ for all $a_j \in \mathcal{A}$. So that the effect of the SIG valences on voters' utilities increases in the corresponding variable at a decreasing rate. Moreover, since the SIG valences exert no cross partial effect on voters' utility in (1), meaning that $\frac{\partial \sigma_j}{\partial a_j} = 0$ for all $z_j \in \mathcal{Z}$ and $\frac{\partial \alpha_j}{\partial z_j} = 0$ for all $a_j \in \mathcal{A}$, then $\frac{\partial^2 \sigma_j}{\partial z_j \partial a_j} \equiv 0$ and $\frac{\partial^2 \alpha_j}{\partial a_j \partial z_j} \equiv 0$ as shown in (36).

The rate at which voter i's *marginal* utility changes in z_j and a_j is given by the matrix $\left(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C\right)$ in (32) defined by

$$\boldsymbol{\iota} - \mathbf{D}^2 v_j^C \equiv \begin{pmatrix} b - \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C) & 0 \\ 0 & e - \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C) \end{pmatrix}. \tag{37}$$

Since $\frac{\partial^2 \sigma_j}{\partial (z_j)^2}$ and $\frac{\partial^2 \alpha_j}{\partial (a_j)^2}$ are by assumption both negative, then $b - \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C) > 0$ and $e - \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C) > 0$, so that the trace of $(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)$, used later on, is positive and given by

$$Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C) \equiv \left[b - \frac{\partial^2 \sigma_j}{\partial (z_j)^2}(z_j^C)\right] + \left[e - \frac{\partial^2 \alpha_j}{\partial (a_j)^2}(a_j^C)\right] > 0. \tag{38}$$

Note that $Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)$ is the same for all voters.

The transpose of vector $\left[\boldsymbol{\iota}\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ij} + \mathbf{D}v_j^C\right]$ in (32) is given by

$$\left[\boldsymbol{\iota}\mathbf{d}_{(\mathbf{z}^C,\mathbf{a}^C)}^{ij} + \mathbf{D}v_j^C\right]^{\mathbf{T}} \equiv \begin{pmatrix} b(x_i - z_j^C) + \frac{\partial \sigma_j}{\partial z_j}(z_j^C) \\ e(t_i - a_j^C) + \frac{\partial \alpha_j}{\partial a_j}(a_j^C) \end{pmatrix} = \begin{pmatrix} b(x_i - \sum_{i \in \mathcal{N}} w_{ij}^C x_i) \\ e(t_i - \sum_{i \in \mathcal{N}} w_{ij}^C t_i) \end{pmatrix} \tag{39}$$

where the last term follows after substituting in j's critical choices, (z_j^C, a_j^C), in (13). Let A,

[20] To save spsace in the on-line equations we sometimes write $\frac{\partial \sigma_j}{\partial z_j}(z_j^C)$ as $\frac{\partial \sigma_j (z_j^C)}{\partial z_j}$ and $\frac{\partial \alpha_j}{\partial a_j}(a_j^C)$ and $\frac{\partial \alpha_j (a_j^C)}{\partial a_j}$.

B and C stand for the short hand version of the following terms

$$\begin{aligned} A &= \left[b(x_i - z_j^C) + \frac{\partial \sigma_j(z_j^C)}{\partial z_j} \right]^2, \quad B = \left[e(t_i - a_j^C) + \frac{\partial \alpha_j(a_j^C)}{\partial a_j} \right]^2 \\ \text{and} \quad C &= \left[b(x_i - z_j^C) + \frac{\partial \sigma_j(z_j^C)}{\partial z_j} \right] \left[e(t_i - a_j^C) + \frac{\partial \alpha_j(a_j^C)}{\partial a_j} \right] \end{aligned} \tag{40}$$

The matrix $\mathbf{\Delta}_{ij}^C$ in (32) is given by

$$\mathbf{\Delta}_{ij}^C \equiv \left[\boldsymbol{\iota} \mathbf{d}_{(\mathbf{z}^C, \mathbf{a}^C)}^{ij} + \mathbf{D} v_j^C \right]^{\mathbf{T}} \left[\boldsymbol{\iota} \mathbf{d}_{(\mathbf{z}^C, \mathbf{a}^C)}^{ij} + \mathbf{D} v_j^C \right] \tag{41}$$

$$\equiv \begin{pmatrix} b(x_i - z_j^C) + \frac{\partial \sigma_j}{\partial z_j}(z_j^C) \\ e(t_i - a_j^C) + \frac{\partial \alpha_j}{\partial a_j}(a_j^C) \end{pmatrix} \begin{pmatrix} \left[b(x_i - z_j^C) + \frac{\partial \sigma_j}{\partial z_j}(z_j^C) \right] & \left[e(t_i - a_j^C) + \frac{\partial \alpha_j}{\partial a_j}(a_j^C) \right] \end{pmatrix}$$

$$= \begin{pmatrix} A & C \\ C & B \end{pmatrix} \tag{42}$$

where in the third line the terms A, B and C are given by (40). Subtituting in the term in the last line of (39), the matrix $\mathbf{\Delta}_{ij}^C$ can be re-written as

$$\begin{aligned} \mathbf{\Delta}_{ij}^C &= \begin{pmatrix} b(x_i - \sum_{i \in \mathcal{N}} w_{ij}^C x_i)^2 b & b(x_i - \sum_{i \in \mathcal{N}} w_{ij}^C x_i)(t_i - \sum_{i \in \mathcal{N}} w_{ij}^C t_i) e \\ b(x_i - \sum_{i \in \mathcal{N}} w_{ij}^C x_i)(t_i - \sum_{i \in \mathcal{N}} w_{ij}^C t_i) e & e(t_i - \sum_{i \in \mathcal{N}} w_{ij}^C t_i)^2 e \end{pmatrix} \\ &= \begin{pmatrix} b(x_i - z_j^w)^2 b & b(x_i - z_j^w)(t_i - a_j^w) e \\ b(x_i - z_j^w)(t_i - a_j^w) e & e(t_i - a_j^w)^2 e \end{pmatrix}. \end{aligned} \tag{43}$$

where in the last line we subsitute j's weighted electoral mean, (z_j^w, a_j^w), given in (8). Thus, $\mathbf{\Delta}_{ij}^C$ is the $\boldsymbol{\iota}-$*weighted variance-covariance matrix* of i's ideals around j's weighted electoral mean (z_j^w, a_j^w) where $\boldsymbol{\iota}$ is given by (34). The SIG valences, $\sigma_j(z_j^C)$ and $\alpha_j(a_j^C)$, affect (43) through (z_j^w, a_j^w) as they depend on the weight j gives i, w_{ij}^C in (9) which depends on p_{ij}^C in (10) and thus depends on voter i's utility in (1) that depend on the SIG valences.

Using (42) and (43), the trace of $\mathbf{\Delta}_{ij}^C$ is always positive and given by

$$\begin{aligned} Tr(\mathbf{\Delta}_{ij}^C) &= \left[b(x_i - z_j^C) + \frac{\partial \sigma_j(z_j^C)}{\partial z_j} \right]^2 + \left[e(t_i - a_j^C) + \frac{\partial \alpha_j(a_j^C)}{\partial a_j} \right]^2 \\ &= b(x_i - z_j^w)^2 b + e(t_i - a_j^w)^2 e > 0. \end{aligned} \tag{44}$$

To find if j is maximizing its vote share function at $(\mathbf{z}^C, \mathbf{a}^C)$, we need the *Hessian* of second order partial derivatives of j's vote share, $\mathbf{D}^2 V_j(\mathbf{z},\mathbf{a})$ evaluated at $(\mathbf{z}^C,\mathbf{a}^C)$, given by

$$\begin{aligned}\mathbf{D}^2 V_j^C &\equiv \mathbf{D}^2 V_j(\mathbf{z}^C,\mathbf{a}^C) = \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \mathbf{D}^2 p_{ij}^C \\ &= \sum\nolimits_{i\in\mathcal{N}} p_{ij}^C(1-p_{ij}^C)\left[(1-2p_{ij}^C)\boldsymbol{\Delta}_{ij}^C - (\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)\right] \qquad (45)\end{aligned}$$

where V_j^C is given by (4) so that after substituting $\mathbf{D}^2 p_{ij}$ in (32) the last line follows with $\boldsymbol{\Delta}_{ij}^C$ given by (41), $(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)$ by (37) and V_j^C by (4).

To show that $(\mathbf{z}^C,\mathbf{a}^C)$ is a *weak* local Nash equilibrium (LNE), i.e., that candidate j's vote share V_j is at a local maximum at $(\mathbf{z}^C,\mathbf{a}^C)$, the Hessian of all candidates, $\mathbf{D}^2 V_j^C$ for all $j\in\mathcal{C}$, must be negative semi-definite which happens when the trace is negative as in this case the eigenvalues of $\mathbf{D}^2 V_j^C$ are both non-positive.

We derive the *necessary* second order condition (SOC) for j's expected vote share function to be at a maximum at $(\mathbf{z}^C,\mathbf{a}^C)$, i.e., the conditions under which the trace of $\mathbf{D}^2 V_j^C$ is negative then show those under $(\mathbf{z}^C,\mathbf{a}^C)$ is a LNE of this stochastic electoral game.

10.2 Necessary SOC for j to adopt (z_j^C, a_j^C) is its campaign

Before examining the necessary condition for candidates to adopt $(\mathbf{z}^C,\mathbf{a}^C)$ as their campaign, we give the following definitions that help simply notation later on.

Definition 6 *Let $\mathbf{C}_{ij}^C$ be voter i's characteristic matrix for candidate j at $(\mathbf{z}^C,\mathbf{a}^C)$, i.e.,*

$$\begin{aligned}\mathbf{C}_{ij}^C &\equiv \mathbf{C}_{ij}(\mathbf{z}^C,\mathbf{a}^C) \equiv f_{ij}^C\boldsymbol{\Delta}_{ij}^C - (\boldsymbol{\iota}-\mathbf{D}^2 v_j^C) \qquad (46)\\ &= \begin{pmatrix} f_{ij}^C b(x_i-z_j^w)^2 b & f_{ij}^C b(x_i-z_j^w)(t_i-a_j^w)e \\ f_{ij}^C b(x_i-z_j^w)(t_i-a_j^w)e & f_{ij}^C e(t_i-a_j^w)^2 e\end{pmatrix} \\ &\quad - \begin{pmatrix} b-\frac{\partial^2\sigma_j(z_j^C)}{\partial(z_j)^2} & 0 \\ 0 & e-\frac{\partial^2\alpha_j(a_j^C)}{\partial(a_j)^2}\end{pmatrix} \\ \textit{where}\quad f_{ij}^C &\equiv (1-2p_{ij}^C) \qquad (47)\end{aligned}$$

is the characteristic factor of $\mathbf{C}_{ij}^C$ and where $\boldsymbol{\Delta}_{ij}^C$ is given in (43) and $\left(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C\right)$ in (37) and where the second line follows after substituting in $\boldsymbol{\Delta}_{ij}^C$ in (43).

The trace of $\mathbf{C}_{ij}^C$ is given by

$$\begin{aligned}Tr(\mathbf{C}_{ij}^C) &\equiv f_{ij}^C Tr(\boldsymbol{\Delta}_{ij}^C) - Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C) \qquad (48)\\ &= f_{ij}^C\left[b(x_i-z_j^w)^2 b + e(t_i-a_j^w)^2 e\right] - \left[b-\frac{\partial^2\sigma_j}{\partial(z_j)^2}(z_j^C) + e - \frac{\partial^2\alpha_j}{\partial(a_j)^2}(a_j^C)\right]\end{aligned}$$

where $Tr(\mathbf{\Delta}_{ij}^C)$ is given by (44) and $Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)$ by (38). Note that $Tr(\mathbf{\Delta}_{ij}^C)$ and $Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)$ are also given by (26) and (25) in the text.

Using $q_{ij}^C \equiv p_{ij}^C(1-p_{ij}^C)$ in (11), $\mathbf{D}^2 V_j^C$ in (45) can be re-written as a weighted function of voters' characteristic matrices, $\mathbf{C}_{ij}^C$ in (46) for $i \in \mathcal{N}$, i.e.,

$$\mathbf{D}^2 V_j^C = \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} q_{ij}^C \mathbf{C}_{ij}^C. \tag{49}$$

Using $Tr(\mathbf{C}_{ij}^C)$ in (48), the trace of $\mathbf{D}^2 V_j^C$ is then given by

$$\begin{aligned} Tr\left(\mathbf{D}^2 V_j^C\right) &= \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} q_{ij}^C Tr(\mathbf{C}_{ij}^C) \\ &= \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} q_{ij}^C \left[f_{ij}^C Tr(\mathbf{\Delta}_{ij}^C) - Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)\right]. \end{aligned} \tag{50}$$

The first line in (50) shows that the trace of $Tr\left(\mathbf{D}^2 V_j^C\right)$ depends on whether the q_{ij}^C weighted sum of $Tr(\mathbf{C}_{ij}^C)$ across voters is positive or negative. Clearly, if $Tr(\mathbf{C}_{ij}^C) < 0$ for all $i \in \mathcal{N}$ then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$. These strong conditions on voters are not necessary as we know show that if for *enough* voters, $Tr(\mathbf{C}_{ij}^C)$ in (48) is negative, then $Tr\left(\mathbf{D}^2 V_j^C\right)$ in (50) is negative.

We now determine the conditions under which $Tr(\mathbf{C}_{ij}^C)$ in (48) is negative to then derive those under which $Tr\left(\mathbf{D}^2 V_j^C\right)$ is negative.

From (48), $Tr(\mathbf{C}_{ij}^C) < 0$ when

$$f_{ij}^C < \frac{Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)}{Tr(\mathbf{\Delta}_{ij}^C)}, \tag{51}$$

so that after substituting in f_{ij}^C from (47) into (51) and some manipulation, we obtain

$$\frac{1}{2} - \frac{1}{4}\frac{Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)}{Tr\left(\mathbf{\Delta}_{ij}^C\right)} < p_{ij}^C \tag{52}$$

where p_{ij}^C is given by (33). Define the left hand side (LHS) of (52) as voter i's *necessary pivotal probability* for candidate j at $(\mathbf{z}^C, \mathbf{a}^C)$, i.e.,

$$\varphi_{ij}^C \equiv \varphi_{ij}(\mathbf{z}^C, \mathbf{a}^C) \equiv \frac{1}{2} - \frac{1}{4}\frac{Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)}{Tr\left(\mathbf{\Delta}_{ij}^C\right)} = \frac{1}{2} - \frac{1}{4}\frac{\left[b - \frac{\partial^2\sigma_j}{\partial(z_j)^2}(z_j^C) + e - \frac{\partial^2\alpha_j}{\partial(a_j)^2}(a_j^C)\right]}{b(x_i - z_j^w)^2 b + e(t_i - a_j^w)^2 e} \tag{53}$$

where the last term follows after substituting in $Tr(\boldsymbol{\iota}-\mathbf{D}^2 v_j^C)$ in (38) and $Tr\left(\mathbf{\Delta}_{ij}^C\right)$ in (44). Since the rate at which the SIGs valences affect i's marginal utility, $\mathbf{D}^2 v_j^C$, depends on j's campaign (z_j^C, a_j^C), and since candidates adopt different campaigns, voter i's necessary pivotal probability varies across candidates.

Given the campaigns of all other candidates, $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$, using (53), the *necessary condition on voter i for j to adopt (z_j^C, a_j^C) as j's electoral campaign* given in (52) becomes

$$\varphi_{ij}^C < p_{ij}^C. \tag{54}$$

This condition says that if the probability that i votes for j at $(\mathbf{z}^C, \mathbf{a}^C)$, p_{ij}^C, is *higher* than i's necessary pivotal probability for j at $(\mathbf{z}^C, \mathbf{a}^C)$, φ_{ij}^C, then $Tr(\mathbf{C}_{ij}^C) < 0$. For *some* voters, the probability of voting for j is *too low* in the sense that $\varphi_{ij}^C > p_{ij}^C$, for these voters $Tr(\mathbf{C}_{ij}^C) > 0$.

Having determined the condition under which $Tr(\mathbf{C}_{ij}^C) < 0$ for $i \in \mathcal{N}$, we now study those under which $Tr\left(\mathbf{D}^2 V_j^C\right)$ in (50) is negative. Average the LHS of (54) over all voters to obtain j's *necessary pivotal vote share*, Φ_j^C at $(\mathbf{z}^C, \mathbf{a}^C)$,

$$\Phi_j^C \equiv \Phi_j(\mathbf{z}^C, \mathbf{a}^C) \equiv \frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \varphi_{ij}^C \equiv \frac{1}{2} - \frac{1}{4}\frac{1}{n}\sum\nolimits_{i\in\mathcal{N}} \frac{Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)}{Tr\left(\boldsymbol{\Delta}_{ij}^C\right)} \tag{55}$$

where the last term follows after substituting φ_{ij}^C in (53) and where $Tr(\boldsymbol{\iota} - \mathbf{D}^2 v_j^C)$ is given by (38) and $Tr\left(\boldsymbol{\Delta}_{ij}^C\right)$ by (44). Since $Tr\left(\boldsymbol{\Delta}_{ij}^C\right)$ depends on (z_j^C, a_j^C), so does j's necessary pivotal vote share, implying that pivotal probabilities, Φ_j^C for all $j \in \mathcal{C}$, vary across candidates. Similarly, averaging the RHS of (54) across voters gives j's *expected vote share*, V_j^C in (4).

If $Tr(\mathbf{C}_{ij}^C)$ in (48) is negative for all $i \in \mathcal{N}$ then j's vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C in (4), is greater than j's pivotal vote share, Φ_j^C in (55), i.e., $\Phi_j^C < V_j^C$ and the trace of the Hessian of j's expected vote share, $Tr\left(\mathbf{D}^2 V_j^C\right)$ in (50), negative.

We do not need that all voters vote for j with high enough probability, i.e., that $\varphi_{ij}^C < p_{ij}^C$ for all $i \in \mathcal{N}$, for $Tr\left(\mathbf{D}^2 V_j^C\right)$ to be negative. To show this partition the set of *voters*, $\mathcal{N}$, into three subsets according to whether $Tr(\mathbf{C}_{ij}^C) \gtreqless 0$. Let $\mathcal{N}^{T-}$ be the set of voters for whom $Tr(\mathbf{C}_{ij}^C) < 0$, i.e., for whom $\varphi_{ij}^C < p_{ij}^C$; $\mathcal{N}^{T+}$ those for whom $Tr(\mathbf{C}_{ij}^C) > 0$, i.e., for whom $\varphi_{ij}^C > p_{ij}^C$; and $\mathcal{N}^{T0}$ those for whom $Tr(\mathbf{C}_{ij}^C) = 0$, i.e., for whom $\varphi_{ij}^C = p_{ij}^C$.

Since $q_{ij}^C \equiv p_{ij}^C(1 - p_{ij}^C)$ in (11) is positive for all $i \in \mathcal{N}$, the sign of $q_{ij}^C Tr(\mathbf{C}_{ij}^C)$ depends on the sign of $Tr(\mathbf{C}_{ij}^C)$ in (48). When aggregating $q_{ij}^C Tr(\mathbf{C}_{ij}^C)$ over $\mathcal{N}^{T-}$, $\mathcal{N}^{T+}$ and $\mathcal{N}^{T0}$, we get

$$\sum\nolimits_{i\in\mathcal{N}^{T-}} q_{ij}^C Tr(\mathbf{C}_{ij}^C) < 0, \quad \sum\nolimits_{i\in\mathcal{N}^{T+}} q_{ij}^C Tr(\mathbf{C}_{ij}^C) > 0 \quad \text{and} \quad \sum\nolimits_{i\in\mathcal{N}^{T0}} q_{ij}^C Tr(\mathbf{C}_{ij}^C) = 0.$$

The trace of $\mathbf{D}^2 V_j^C$ in (50) can be decomposed into

$$Tr\left(\mathbf{D}^2 V_j^C\right) = \sum\nolimits_{i\in\mathcal{N}} q_{ij}^C Tr(\mathbf{C}_{ij}^C) = \sum\nolimits_{i\in\mathcal{N}^{T-}} q_{ij}^C Tr(\mathbf{C}_{ij}^C) + \sum\nolimits_{i\in\mathcal{N}^{T+}} q_{ij}^C Tr(\mathbf{C}_{ij}^C)$$

Thus, $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$ iff $\left|\sum_{i\in\mathcal{N}^{T-}} q_{ij}^C Tr(\mathbf{C}_{ij}^C)\right| > \left|\sum_{i\in\mathcal{N}^{T+}} q_{ij}^C Tr(\mathbf{C}_{ij}^C)\right|$. Therefore, if $Tr(\mathbf{C}_{ij}^C) < 0$ for *enough* voters, i.e., if $\varphi_{ij}^C < p_{ij}^C$ in (54) is satisfied for *enough* voters then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$. This says that when there are enough voters for whom the probability of voting for j is high enough, i.e., if $\mathcal{N}^{T-}$ is large enough, then $Tr\left(\mathbf{D}^2 V_j^C\right) < 0$.

Thus, the *necessary condition* for j to adopt (z_j^C, a_j^C) as its electoral campaign, holding the campaigns of all other candidates constant at $(\mathbf{z}_{-j}^C, \mathbf{a}_{-j}^C)$, is that, j's expected vote share at $(\mathbf{z}^C, \mathbf{a}^C)$, V_j^C in (4), be greater than j's necessary pivotal vote share, Φ_j^C in (55), i.e., that

$$\Phi_j^C < V_j^C. \tag{56}$$

When candidate j's vote share at $(\mathbf{z}^C, \mathbf{a}^C)$ is *too low*, in the sense that $\Phi_j^C > V_j^C$, j does *not* adopt (z_j^C, a_j^C) as its electoral campaign as j's expected vote share is at a *minimum or at a saddle point* at (z_j^C, a_j^C). By changing its policy and/or ad campaign j increases its expected vote share. When j changes its electoral campaign, other candidates may also find it in their interest to also change their electoral campaign. In this case, candidates' critical campaigns $(\mathbf{z}^C, \mathbf{a}^C)$ are *not* a LNE of the election.