The Measurement of Individual and Social Deprivation

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Abstract

The reduction of poverty is a widely-shared policy goal. Yet, there is no consensus on how to measure poverty. Moreover, current indices face serious drawbacks. First, they may identify an increase in poverty even if a policy makes all poor better-off. Second, most indices are not robust to measurement errors. Third, the ethical viewpoints expressed by these indices are often unclear. In this paper, I address these drawbacks. The axiomatic characterization builds on intuitive principles inspired by resource fairness. Social deprivation ought to be measured by the sum of specific indices of individual deprivation which: *(i)* respect the preferences of individuals; *(ii)* compare individuals based on the set of attainments they are deprived of; and *(iii)* are continuous and convex. I illustrate the criteria using Norwegian register data. Finally, I extend the results to differences in needs and categorical dimensions of deprivation.

Keywords: Poverty; multidimensional deprivation; fairness; differences in needs.

1 Introduction

Mark is income poor. He earns less than 60 percent of the median income. Nick is not. Should poverty reduction policies target individuals as Mark, rather than those as Nick? Not necessarily. Most economists and philosophers agree that income information is not sufficient to assess poverty. For instance, Mark and Nick might differ in other dimensions such as health, access to housing, education, liberties, etc. There is little consensus, however, on how to accommodate and combine these dimensions. Clearly, the necessity of reporting and summarizing the achievements of the fight against poverty led to a significant number of proposals (see the recent survey by Alkire et al. (2015)). At the same time, more and more authors express concerns for a number of drawbacks that have emerged (Alkire and Foster (2011b); Ravallion (2011); Thorbecke (2011); Aaberge and Brandolini (2015); Cowell (2015); Duclos and Tiberti (2016)).

Assume Mark is unemployed. He was offered a (reasonably attractive) job, fitting his education. It was a full-time employment paying enough for Mark to earn more than Nick. Yet, Mark rejected. By revealed choice, Mark prefers his current life style. If we were to force Mark (who is an income poor individual) to accept this job, he would be worse off. Surprisingly, most indices of poverty in the literature would associate such imposition with a reduction of poverty. Similarly, assume Nick has a full-time employment, which he supplements with a secondary job. With these two jobs he earns more than 60 percent of the median income, but needs to work more than 60 hours per week. Furthermore, Nick prefers a single full-time employment, even if this was to pay (slightly) less than 60 percent of the median income. Surprisingly, if Nick (who is not an income poor individual) decided to take such a job, most indices would measure an increase in poverty. The reason is that such indices do not respect the views of individuals about what constitutes a good life. In economic jargon, these indices violate the Pareto principle and, thus, do not respect preferences. Consequently, a policy that makes all (or some) poor better off may actually increase the level of measured poverty; conversely, a policy that decreases currently measured poverty may instead worsen the well-being of the poor. Duclos and Tiberti (2016) highlight that a related requirement, "monotonicity," is also generally violated. Monotonicity demands that poverty decreases when more goods are given to poor individuals. If individuals prefer more goods to less, respecting preferences is a stronger requirement than monotonicity. By imposing the Pareto principle, I here address these drawbacks.[1]

A different issue is that many indices are not robust to measurement errors. Small changes in the attainments of individuals may lead to large jumps in the level of measured poverty. This violation of "continuity" has several implications. Poverty indices are more volatile and, thus, require more data or longer monitoring to safely assess the effects of policies. Moreover, due to their discontinuities, these indices cannot accommodate social aversion to inequality. As a result, a transfer from a more deprived individual to a less deprived one may reduce the level of measured poverty. Here, I require the criterion to be continuous and to satisfy a multidimensional version of the Pigou-Dalton transfer principle.[2]

Finally, poverty indices are generally defined up to a range of parameters. These parameters constitute the *signature* of an index. The signature gives enough flexibility for a family of indices to accommodate different ethical views: for instance, by specifying how to identify the poor, whether to set a relative or absolute poverty line, how to aggregate across and within dimensions, or how to prioritize across individuals. Unfortunately, it is often difficult to link the signature of current indices to specific ethical views.[3]

[1]Of course, there are circumstances in which society might want to be paternalistic and disrespect preferences: for instance, when these suffer from schadenfreude, overconfindence, status-quo bias, inattention, myopia, or addiction. In these cases, society can correct individual preferences for these misbehaviors or even impose new ones. The results are unaffected by such change.

[2]The Pigou-Dalton transfer principle is the pillar of social aversion to inequality. It tells that a monetary transfer from a poorer to a richer individual increases inequality and, thus, reduces social welfare (see Aaberge and Brandolini, 2015).

[3]To address the lack of support for a single set of parameters, the recent literature has suggested adopting indices for a class of signatures (Davidson and Duclos (2000); Duclos et al. (2006)). While such robustness of poverty comparisons ensures more consensusual statements and alleviates the importance of each specific signature, it also reduces the capacity to compare policies and to assess individual and social deprivation.

The family of criteria characterized here offer an intuitive and transparent set of ethical choices, which I illustrate next.

Identification of the deprived individuals. The first ethical choice consists of defining the "no-deprivation set." The no-deprivation set is the set of commodities that ensure a (sufficiently) good quality of life to *any* individual. An individual is not deprived if she is assigned a bundle from the no-deprivation set or any bundle she finds equally desirable. The choice of this set is related to and generalizes the standard practice of setting a (multidimensional) poverty line.

Comparisons of individual deprivations. The next ethical choice requires to determine a family of "iso-deprivation contours." These sets can be thought of as production isoquants from standard microeconomic theory. Their role is to allow comparisons of deprivations across individuals. The idea goes as follows. Consider an iso-deprivation contour D. Assume individual i finds all attainments in D more desirable than her attainments vector. In contrast, individual j prefers her attainments vector to some of the alternatives in D. Then, individual i is more deprived than individual j. The form of the iso-deprivation contours express the substitutability/complementarity of the different dimensions of deprivation. For instance, it is natural to consider perfectly substitutable meals that provide the same nutritional input. In contrast, one might not want to set a linear trade-off between health and income: arguably, there might be sufficiently large reductions in health that no amount of money can compensate for and that necessarily make an individual more deprived.

Priority among deprived individuals. The last ethical choice pertains the sensitivity of the index of social deprivation to the concentration of deprivations. How much more importance should society place to the deprivation of the most deprived individuals? This choice is standard in the unidimensional poverty literature (see Foster et al. (1984)) and is here captured by an increasing and convex function, named "priority function."

These ethical choices constitute the signature of the family of criteria characterized here. Social deprivation is measured by the sum of specific indices of individual deprivation. Indices of individual deprivation are constructed as follows. First, if an individual is not deprived, her deprivation level is zero. If, instead, an individual is deprived, her index of deprivation reflects her preferences: attainments on a lower indifference curve determine larger levels of individual deprivation. Second, deprivation indices make interpersonal comparisons through the iso-deprivation contours: if an individual i is worse off than her least preferred alternative from an iso-deprivation contour D, while j is not, i's index of deprivation is larger than j's one. Third, deprivation indices accommodate social aversion to inequality: individuals' deprivation indices are convex, meaning that regressive transfers from individuals with higher deprivation to individuals with lower deprivations increase social deprivation.

The main result of the paper is to show that the above ethical choices (and the corresponding indices) are necessary and sufficient to satisfy a set of intuitive and compelling axioms. I already mentioned few of the requirements imposed here: *weak Pareto* requires society to respect individuals' preferences; *continuity* says that small changes of individuals' attainments lead to small changes in the level of social deprivation; and *equal-preference transfer* is a multidimensional version of the Pigou-Dalton principle, restricted to individuals with same preferences. An additional axiom is also standard: *separability* tells that the ranking of two alternatives is independent of the attainments of an individual who is unconcerned by the choice.

The central and novel axiom is *deprivation fairness*. First, society ought to take into account the deprivation of each individual. Second and more importantly, society cannot consider the deprivation of some individual more important than that of others. The intuition is simple. Assume individual i achieves a certain level of well-being and is the only one deprived in society. Assume that, no matter which attainments individual i has (ensuring that level of well-being), society always measures a larger social deprivation when some other individual j has larger attainments (and j being the only deprived). Then, the deprivation of i would count relatively less than that of j. *Deprivation fairness* prevents this kind of discrimination.

There are situations, however, where the lack of resources of some is socially more relevant than the deprivation of others. This is the case of differences in needs. Differences in needs emerge primarily due to household size and composition. For equal income, families with children are typically regarded as more income-deprived than families without children. For equal living space, families with two teenagers are typically regarded as more space-deprived than families with two toddlers. To account for such situations, I introduce the concept of basic needs. For each type of household, the basic needs is the set of large-enough attainment that ensure that the household is not deprived. Building on the idea behind *deprivation fairness*, I use basic needs to construct inter-household comparability of deprivation levels and extend the family of deprivation indices to differences in needs.

Finally, I also extend the results to categorical attainments, as it is often the case in the poverty measurement literature when accounting for health, access to education, access to infrastructure, etc.

In the next section, I illustrate the criteria and place the contribution within the literature; I also use the Norwegian register data to discuss and apply the results. Section 3 presents the framework, the axioms, and the main result. Section 4 extends the results to differences in needs and categorical dimensions of deprivation. Section 5 concludes. All proofs are contained in the appendix.

2 Related literature and empirical illustration

2.1 Poverty in income

The simplest setting for addressing the measurement of poverty is by assessing only one dimension, say after-tax income. An income distribution $y_N \equiv (y_1, ..., y_n) \in \mathbb{R}^n_+$ assigns a certain level of income $y_i \geq 0$ to each individual $i \in N \equiv \{1, ..., n\}$. How to measure poverty? Following the literature, the first step is to set a **poverty line** $z_y > 0$: individuals with income larger than z_y are non-deprived; individuals with income smaller or equal to z_y are income-deprived.

Example. *For the illustrations of the indices, I use the 2016 Norwegian register data. I focus on Norwegian single men that do not have children* *and are aged between 22 and 61.*[4] *My database consists of* 90462 *individuals. Let the poverty line be defined by 60 percent of the median after-tax income. In the data, the median after-tax income is about 333.000 NOK; expressing income in million NOK, the median income is (about)* $y_M = .333$ *and the poverty line is* $z_y = .6y_M = .2$.

Define each individuals' **income gap** as the difference (if any) between the poverty line and the assigned income; formally, $d(y_i) = z_y - y_i$ if $z_y \geq y_i$ and $d(y_i) = 0$ otherwise.

Then, a first intuitive way to aggregate income deprivation is the **mean income gap**:

$$D(y_N) = \frac{1}{n} \sum d(y_i). \tag{1}$$

Example (cont.). *In the data, the mean income gap is* $D(y_N) = .0011$. *If (about) 99,5 million NOK were appropriately distributed to the income poor, each individual would be above the poverty line.*

A drawback of (1) is its insensitivity to the concentration of income gap among individuals. Said differently, poverty is unchanged whether a certain income gap is held by few individuals or is shared among many. To avoid this drawback, the simple average can be replaced by the **generalized mean income gap**, defined by:

$$D_f(y_N) = \frac{1}{n} \sum f(d(y_i)), \tag{2}$$

where f is an increasing and convex real-valued function. When f is linear, society is unconcerned with the distribution of income gaps and (2) is ordinally equivalent to (1). As f becomes more and more convex, society places more and more weight on the individuals with largest income gaps. At the limit for f infinitely convex, poverty is defined by the income gap of the most deprived individual in society. Interestingly, in the one-dimensional setting, the criterion characterized in this paper is equivalent to (2).

[4] *The focus on a specific category of individuals significantly reduces the scope for accounting for differences in needs. I leave to future research the empirical assessment of deprivation in Norway using all types of households.*

A related family of poverty indices is that by Foster et al. (1984):

$$D_{FGT}(y_N;\theta) = \frac{1}{n}\sum\left(\frac{d(y_i)}{z_y}\right)^{\theta}. \qquad (3)$$

Two remarks are in order. First, the Foster-Greer-Thorbecke includes as a special case the head-count index (when $\theta = 0$), which measures social deprivation by the proportion of individuals that are income deprived. It accommodates strict aversion to inequality in deprivations only if $\theta>1$. The family Second, the distinguishing aspect of (3) is "scale invariance" (see Ebert and Moyes, 2002). Scale invariance requires the deprivation index to be unchanged when rescaling the income distribution (together with the poverty line). Despite its popularity, scale invariance is not compelling in a multidimensional setting and is not imposed here. The problem is that, with several commodities, individual i may prefer an attainments vector x_i to a different one x_i' and, at the same time, prefer the rescaled bundle $\alpha x_i'$ to αx_i for some $\alpha > 0$. This preference inversions make it impossible for any scale invariant criterion to respect individuals' preferences.

Example (cont.). *When $\theta = 0$, the head-count index is $D_{FGT}(y_N;0) = .0297$, i.e. 2692 of the 90463 individuals are income deprived. When $\theta = 1$, the index (3) is the relative mean income gap (equivalent to the mean income gap up to a multiplicative constant): $D_{FGT}(y_N;1) = z_y^{-1}D(y_N) = .0055$. When $\theta = 2$, poverty is larger when the income gap is concentrated among few individuals. More precisely, roughly the same level of poverty emerges whether (i) individual i is deprived with $y_i = 0$ while j is not ($y_j \geq .2$) or (ii) whether i and j are both deprived with $y_i = y_j = .06$ (that is 30 percent of z_y). In the data, $D_{FGT}(y_N;2) = .0012$.*

2.2 Multi-dimensional poverty in the literature

When two or more dimensions are considered, there is no consensus on how to identify, measure, and aggregate deprivations. Let the attainment space be $X \equiv \mathbb{R}_+^m$, with m finite.[5] A social state $x_N \in X^n$ specifies the attainment

[5] $\mathbb{R}_+^m$ denotes the non-negative orthant of $\mathbb{R}^m$.

of each individual i in each dimension m. I illustrate the multidimensional criteria by considering, together with the after-tax income, also the leisure time enjoyed by individuals.

An index of multidimensional poverty needs to aggregate the attainmnets of each individual (an $m \times n$ dimensioned social state) into a real value. It follows that a crucial ethical choice is the order of aggregation: one can aggregate over individuals first or over dimensions first.

The first aggregation method—over individuals first—closely builds on the one-dimensional criteria. Poverty is first assessed for each dimension separately: in order, setting a dimension specific poverty line, computing the dimension-specific attainment's gap of each individual, and aggregating these in a measure of dimension-specific poverty such as (1) or (3). In a second stage, these dimension-specific poverty levels are aggregated into a composite or "mashup" index.[6] A well-known example is the Human Development Index (Anand and Sen, 1994).

Example (cont.). *To illustrate, let an individual* i *be income poor if her income is less than* 60 *percent of the median income, i.e.* $y_i \leq z_y$. *The same individual is leisure poor if her leisure is less than* 75 *percent of the median leisure time. Fixing the weekly available time to 80 hours per week, the median leisure time is about* $l_M = 40$, *leading to a poverty line of* $z_l \equiv .75 l_M = 30$. *Then, the leisure poverty according to the Foster-Greer-Thorbecke index with* $\theta = 2$ *is* $D_{FGT}(l_N; 2) = 0,0018$. *A possible measure of multidimensional poverty is* $D(x_N) = D_{FGT}(y_N; 2) + D_{FGT}(l_N; 2) = .003$.[7]

Note, however, that Mashup indices disregard the joint distribution of deprivations across individuals, which means that they are unconcerned whether the dimension-specific deprivations are beared by few individuals only or spread out among many individuals. This also means that there is no assessment whether an individual is overall deprived or not. Consider the following

[6]Some authors argue that aggregation across dimensions is *ad hoc* and unnecessary. They instead suggest reporting the entire vector of dimension-specific deprivations (Hicks and Streeten, 1979). Unfortunately, this "dashboard" approach is often unable to provide a clearcut comparison of social states and, in particular when many dimensions are considered, it may prove difficult to interpret.

[7]Of course, one might consider weighting dimensions differently.

scenario. Individual i is both income and leisure deprived: even if working more than 50 hours per week, her income does not reach z_y (60 percent of the median income). Individual j, instead does not work and has an income in the top 1 percent of the income distribution. The level of measured poverty remains unchanged if i and j were to switch incomes. In this case, i would be leisure deprived, but earns a top 1 percent income; j is income deprived, but does not work. In the first case, individual i is clearly deprived, while j is clearly not. In the second case, it is not obvious that i or j are deprived. Even if they had the choice, they might both prefer their attainments to working (slightly less than) 40 hours per week at (slightly more than the) median income.

To avoid this drawback, Alkire and Foster (2011a) propose the adoption of a "dual cutoff method." The first cutoff is standard: a vector of dimension-specific poverty lines defines which dimensions (if any) an individual is deprived off and determines her dimension-specific attainment's gaps. This vector is here $z \equiv (z_y, z_l)$. The second cutoff establishes whether the dimension-specific deprivations are sufficient to indentify an individual as (overall) poor. For example, the "intersection approach" tells that an individual is poor if she is deprived in all dimensions. Then, multidimensional poverty is the sum of the (possibly weighted and transformed) dimension-specific attainments' gaps that each poor individual experiences.[8]

The second aggregation method—over dimensions first—is closely related to the "social welfare approach" to multidimensional poverty (Atkinson, 2003). Individuals' attainment gaps are first evaluated through a "utility-like" function and then aggregated across individuals.

A first alternative is to adopt individuals' cardinal and interpersonally comparable utility functions to evaluate their deprivation (Kingdon and Knight, 2006). The focus is on individuals' happiness or subjective well-being. The advantage of this approach is that it respects the preferences and choices of individuals. Yet, economists and philosophers have contended that poverty

[8]Clearly, the joint distribution of deprivations now matters. Yet, this is limited to the identification of the poor individuals. The distribution of dimension-specific attainments' gaps across poor individuals is still disregarded.

is about the lack of access to resources or attainments, rather than happiness (Sen, 1979).

A second alternative by Bourguignon and Chakravarty (2003) is to let society choose the utility-like function to evaluate the deprivations of each individual and then aggregate them additively across individuals (see also Duclos et al., 2006). As clarified by Atkinson (2003), this function reflects the preferences of society and incorporates three ethical choices, similar to the ones identified here. First, it requires setting a multidimensional poverty line, such as $z \equiv (z_y, z_l)$. Then, the dimension-specific deprivations are $d_y(y_i) \equiv \max[0, z_y - y_i]$ and $d_l(l_i) \equiv \max[0, z_l - l_i]$. Second, it requires setting "iso-poverty contours." Society chooses these level curves to specify which combinations of dimension-specific deprivation are associated the same level of individual deprivation. The form of such curves accommodates different degrees of substitutability and complementarity between dimensions. It can be described by a complete ordering of attainments. Fig.1 illustrates this construction. For example, individual i with attainments vector x_i is on a lower iso-poverty contour than individual j with attainments vector x_j. Third, society selects a common utility-like function to represent the iso-poverty contours. This choice fixes a specific "cardinal representation" and defines, through its convexity, how quickly the level of individual deprivation changes when climbing the contours.

Example (cont.). *Let the deprivation of each individual be measured by the following function:* $u(x_i) = [\beta d(y_i)^\gamma + (1-\beta) d(l_i)^\gamma]^{\frac{\theta}{\gamma}}$*, where* γ *measures the complementarity between deprivations in income and leisure,* β *is the relative weight attribute to the income dimension, and* θ *determines the priority attached to the most deprived individuals. For example,* $\gamma = 1$ *means that the iso-deprivation contours are linear;* $\beta = \frac{1}{2}$ *means that dimensions are equally important; and* $\theta = 2$ *specifies the aversion to inequality among the deprived. Then, the index of social deprivation is* $D_{BC} = .0014$.

A limitation of this approach is to disregard individuals' preferences. To illustrate, let j's attainments vector be above the poverty line, i.e. $x'_j = (1+\varepsilon) z$ with $\varepsilon > 0$. A policy offers individual j the possibility to switch to a different attainments vector x_j, which j prefers to x'_j. While j would exercise

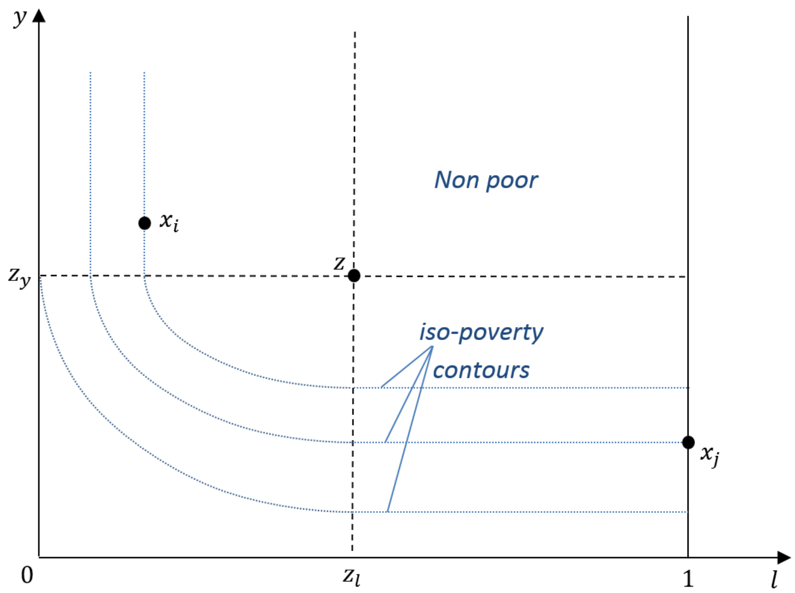

Figure 1: The multidimensional poverty line and the iso-poverty contours.

her opportunity to be better off, x_j might be below the poverty line (even if x'_j was not). In fact, all indices presented above—except the one based on happiness—would allow for poverty to increase when a non-poor individual switches to a better alternative. Does this mean that respecting preferences requires comparing individuals' deprivations by their happiness levels? As recently clarified by Fleurbaey and Maniquet (2011) and Piacquadio (2017) for the measurement of social welfare, the answer is negative. The trick is to measure deprivation with respect to utility-like functions that are different across individuals and represent individuals' preferences. The contribution of this paper is to characterize how society ought to choose these functions based on fairness principles.

A recent preference-sensitive proposal related to the present one is by Decancq et al. (2014) (see also Dimri and Maniquet (2017)). Two main differences emerge here. First, their poverty line is similar to that introduced by Bourguignon and Chakravarty (2003) and does not allow for complementarity/substitutability across dimensions. Second, society may promote regressive redistributions of attainments, i.e. from more deprived to less deprived individuals (even if they have the same preferences). Their criterion emerges as a special case of the family of indices characterized here when the preferences of individuals are homothetic and when society considers the attainments as perfect complements.

2.3 The present proposal

Three fundamental ethical choices emerge from the characterization result.

Identification of the deprived. An individual is deprived if any bundle from the **no-deprivation set** would make her better-off. The *no-deprivation set* is a subset of the attainments space. It includes all attainments that are sufficiently large to ensure that individuals with any such attainments are not deprived.

In one dimension, say income, this is equivalent to setting a poverty line. In the multidimensional setting, it includes as a special case the multidimensional poverty line z (as in Bourguignon and Chakravarty). Here, the *no-deprivation set* is more permissive: one can flexibly trade-off income and leisure and set any level of complementarity or substitutability between the different dimensions of deprivation. For instance, a sufficiently large income may be considered sufficient to compensate an individual for longer working hours and might ensure that an individual is not deprived.

Example (cont.). *For the sake of simplicity, let the no-deprivation set NDS be identified by the set of attainments vectors $(y, l) \in X$ such that:*

$$f(y, l) \equiv \left[\beta y^{\gamma} + (1-\beta) l^{\gamma}\right]^{\frac{1}{\gamma}} \geq \nu.$$

The parameters $\beta, \gamma, \nu \in \mathbb{R}$ are uniquely defined by the following three ethical choices. First, when working as much as the median labor supply, any income larger than (or equal to) 60 percent of median income ensures that an individual is not deprived. Second, when working 50 percent less than the median labor supply, than any income larger than (or equal to) 40 percent of median income ensures that an individual is not deprived. Finally, when working 33 percent more than the median labor supply, than any income larger than *(or equal to) the median income ensures that an individual is not deprived. These imply that $\beta = 0.177$, $\gamma = -3$, and $\nu = 0.33$.*

To clarify further, two differences with Bourguignon and Chakravarty (2003) emerge. The form of the deprivation set is here more general. Second, for an individual i to be considered deprived it is not sufficient that her attainments vector does not belong to NDS. To accout for i's preferences R_i, it is required that i would prefer any bundle $(y, l) \in NDS$ to her attainments (y_i, l_i).

Comparison of individual deprivations. Assume two individuals i and j are both deprived. Who is most deprived? The answer comes from the **iso-deprivation contours**. *Iso-deprivation contours* are level curves similar to the "iso-poverty contours" in Bourguignon and Chakravarty (2003). The iso-deprivation contour corresponding to the lowest level of deprivation corresponds to the *no-deprivation set* (without loss of generality, this level of deprivation is later normalized to 0). The smaller the attainments vector, the closer to the origin is the corresponding iso-poverty contour and the larger is the associated level of deprivation. The iso-deprivation contour corresponding to the highest level of deprivation is the origin (without loss of generality, this is later normalized to 1).

Importantly, *iso-deprivation contours* cannot be directly used to assess individuals' deprivations, as doing so would not respect individuals' preferences. Consider two individuals, i and j, with attainments $x_i, x_j \in X$. Individual i is considered more deprived than individual j if there exist an *iso-deprivation contour* such that i would be better off with any attainments vector from this contour, while j would not. Fig. 2 illustrates the construction of the *no-deprivation set* and the *iso-deprivation contours*. Individual i with preferences R_i is not income-poor. Yet, she is considered deprived since she would be better-off with any attainments vector from the no-deprivation set. Individual j with preferences R_j is income-poor. Yet, she is not considered deprived since she would not want to switch from x_j to x_M (or other attainments vectors from the *no-deprivation set*).

Example (cont.). *For simplicity, assume homothetic iso-deprivation contours. Then, the iso-deprivation contour of level $\lambda \in [0, 1]$, denoted*

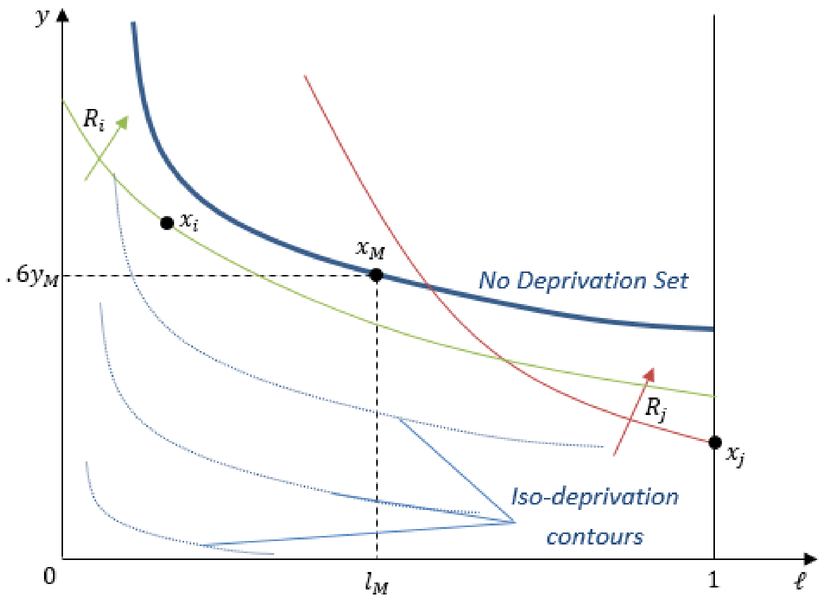

Figure 2: The no-deprivation set and the iso-deprivation contours.

$IDC(\lambda)$, consists of all the attainments vectors $(y, l) \in X$ such that:

$$\begin{cases} [\beta y^{\gamma} + (1-\beta) l^{\gamma}]^{\frac{1}{\gamma}} = (1-\lambda)\nu & if\ \lambda > 0, \\ [\beta y^{\gamma} + (1-\beta) l^{\gamma}]^{\frac{1}{\gamma}} \geq \nu & if\ \lambda = 0. \end{cases}$$

The first two ethical choices allow: *(i)* identifying which individuals are deprived; and *(ii)* ordinally compare any two deprived individuals in terms of their deprivation. The last step requires taking a stand on how to trade-off the deprivations of individuals. This requires setting a cardinal representation of individuals deprivations and defining how these are aggregated in an index of social deprivation.

Before this, I formalize numerically the above comparisons of deprivation across individuals. Loosely speaking, I define an index of deprivation for each individual such that whenever an individual has a larger index of deprivation, she is considered more deprived. Let the deprivation function of individual

i, i.e. $\lambda_i : X \to \mathbb{R}$, be defined as follows. At the attainments vector (y_i, l_i), i is deprived of level $\lambda_i = \lambda$ if λ is the largest scalar for which i finds (y_i, l_i) at least as desirable as any $(y, l) \in IDC(\lambda)$.[9]

Priority among deprived.

Priority among deprived individuals is introduced by requiring that society averses transfers of attainments from more deprived individuals to less deprived ones. Then, social deprivation is measured by the sum of opportunely transformed individual deprivations:

$$D(x_N) = \sum_{i \in N} f \circ \phi \circ \lambda_i (y_i, l_i), \tag{4}$$

where f is the **priority function** and the function ϕ ensures that, for each $i \in N$, $\phi \circ \lambda_i (y_i, l_i)$ is a convex representation of individuals' deprivation levels. The priority function plays a similar role as f in the generalized mean income gap (2): it establishes how much society is concerned with the most deprived individuals. The function ϕ provides cardinality to the indices of individual deprivation (it can be dropped when each λ_i is homogeneous of degree 1, more details in the next section and Fn.16).

Example (cont.). *To apply the above criterion, individuals' preferences need to be estimated. The methodology adopted here is to estimate the random utility model of Aaberge et al. (1999). The analysis generates more than 150 different types of individuals, depending on observables. Each type is associated a specific preference relation. For details, see Appendix A.*

The first result is that the number of individuals identified as deprived is larger than the number of individuals who earn less than 60 *percent of the median income, i.e.* 4.4 *percent of the population versus* 2.9 *percent. Two effects are in place. First, some individuals earn more than* 60 *percent of the median income, but since they work significantly more than median labor supply – while they would rather not – they are still considered as deprived. Second, some individuals earn less than* 60 *percent of the median income, but since they enjoy leisure "significantly" more than the median individual they*

[9] Note that such representations are purely ordinal, i.e. unique up to any common increasing transformation.

are not considered deprived. These are those individuals that, even if offered, would not accept a job that would move them out of income poverty. In total, 2.4 *percent of individuals are deprived according the preference-sensitive index adopted here, but are not according to income only. For* 0.9 *percent of individuals the converse holds. Setting the priority function to be the power function with exponent* 2*, social deprivation according to the present proposal (in per-capita terms) is* $D = \frac{1}{n}\sum_{i \in N}\left[\lambda_i\left(y_i, l_i\right)\right]^2 = 0,0005$.

This approach also highlights a new important aspect of poverty reduction policies. With standard poverty measurement, an income transfer targeting the income-deprived individuals must account for the behavioral responses of individuals: workers could adjust their labor supply downwards and thus benefit from the introduced transfer. A new wedge emerges here. Some of the income-deprived individuals should be held responsible for their status as they either deliberately chose to work little or would do so based on their preferences. Thus, a transfer based on income would accord them a non-deserved benefit. Moreover, some of the individuals who are not income deprived are so at the cost of large sacrifices in terms of leisure. If such individuals are better-off with the (full time) job of an income-deprived individual, they ought to be considered poor. Yet, the income transfer would levy on them an additional cost.

I leave to future research the analysis of the optimal poverty reduction policies, i.e. the design of the progressivity of the tax system, the analysis of the optimal tax credit system, or the introduction of a universal basic income. Two remarks are in order. First, the heterogeneity in individuals' preferences is a fundamental aspect of observed behavior that cannot be disregarded, in particular in economies where basic liberties, fundamental rights, and minimal education levels are well established and ensure that individuals can express their preferences through their behavior.[10] Second, even if data restrictions do not allow to identify the preferences of each individual, the

[10]The focus on preferences suggests that the approach is more appropriate when the no-deprivation set interprets level of attainments associated to a "socially satisfactory" life. When people struggle with survival, "necessity displaces desire," as recently suggested by Allen (2017) (see also Deaton (2016)). In such cases, differences in preferences, if any, are either insignificant or may be willingly disregarded by society.

joint distribution of preferences and attainments is sufficient to have a quite precise picture of the type of biases that disregarding individuals preferences would introduce. Of course, whether one wants to respect individuals' preferences when measuring poverty is an ethical choice that is left to the reader, philosophers, and policymakers. In the following, I discuss and characterize the preference-sensitive deprivation measures defined by (4).

3 Model, axioms, and characterization

3.1 The model

A society consists of a finite set of individuals $N = \{1, ..., n\}$ with $n \geq 3$. Each individual $i \in N$ is characterized by a preference relation R_i; R_i is a weak order on the m-dimensional Euclidean attainments space $X \equiv \mathbb{R}^m_+$, with m finite. The strict preference and indifference relations induced by R_i are denoted by P_i and I_i. Each preference relation R_i can be represented by a strictly increasing and concave numerical function $u_i : X \to \mathbb{R}$ such that $\lim_{|x_i| \to \infty} u_i(x_i) = \infty$. Moreover, for each individual i with preferences R_i, there is a different individual with the same preferences.[11]

A **social state** $x_N \equiv (x_1, ..., x_n)$ assigns an attainments vector x_i to each individual $i \in N$. The set of all possible social states is $X_N \equiv X^n$. A **(social) deprivation ranking**, denoted $\trianglerighteq$, is a weak ordering of social states. For each pair of social states $x_N, x'_N \in X_N$, $x_N \trianglerighteq x'_N$ means that x_N is characterized by at least as much deprivation as x'_N. The asymmetric and symmetric relations induced by $\trianglerighteq$ are denoted $\triangleright$ and $\backsimeq$. A **(social) deprivation index**, denoted $D : X_N \to \mathbb{R}$, is a numerical representation of the deprivation ranking $\trianglerighteq$, that is $x_N \trianglerighteq x'_N$ holds if and only if $D(x_N) \geq D(x'_N)$.

[11]This richness requirement can be replaced by a duplication invariance axiom: the criterion should rank consistently any two alternatives for the original society and the duplicated alternatives for the duplicated society.

3.2 The axioms

The first axiom says that if each individual finds her attainments vector at $x_N \in X_N$ at least as desirable as her attainments vector at $x'_N \in X_N$, then the social state x_N cannot have more social deprivation than the social state x'_N.

Weak Pareto: For each pair $x_N, x'_N \in X_N$, $x_i \, R_i \, x'_i$ for each $i \in N$ implies $x'_N \trianglerighteq x_N$.

Next, small changes in the social state do not cause large jumps in the level of social deprivation.

Continuity: For each $x_N \in X_N$, the set $\{x'_N \in X_N \, | x'_N \trianglerighteq x_N\}$ and the set $\{x'_N \in X_N \, | x_N \trianglerighteq x'_N\}$ are closed.

Next, the deprivation ranking of two social states is independent of the attainments vector of an individual who is unconcerned by these alternatives.[12]

Separability: For each pair $x_N, x'_N \in X_N$, if there is $i \in N$ such that $x_i = x'_i \equiv a_i$, then for each $b_i \in X$,

$$(a_i, x_{-i}) \trianglerighteq \left(a_i, x'_{-i}\right) \Longleftrightarrow (b_i, x_{-i}) \trianglerighteq \left(b_i, x'_{-i}\right).$$

The next axiom introduces the possibility of deprivation and prevents a certain form of discrimination: the deprivation of some individual cannot be considered more important than that of others. The first condition imposes the existence of a subset of (sufficiently large) attainments vectors C such that whether an individual is assigned such attainments or more does not affect social deprivation. The second condition concerns discrimination. Consider a social state where everyone is assigned such a sufficiently large attainments vector, i.e. $x^*_N \in C^n$. Let now an individual $i \in N$ be given a different attainment vector $\bar{x}_i \in X$ so that social deprivation is larger,

[12] I adopt the following notation: for each $x_N \in X_N$, each $i \in N$, and each $a_i \in X$, $(a_i, x_{-i}) \in X_N$ denotes the social state that assigns a_i to i and x_j to each $j \in N \backslash \{i\}$.

that is $(\bar{x}_i, x^*_{-i}) \rhd x^*_N$. Then, i is discriminated against if, for each attainments vector $x_i \in X$ that she finds equally desirable (i.e. such that $x_i \, I_i \, \bar{x}_i$), there is less social deprivation at the social state (x_i, x^*_{-i}) than at any social state $(x_j, x^*_{-j}) \in X_N$ which assigns a larger attainments vector $x_j > x_i$ to some other individual $j \in N \backslash \{i\}$.[13] In other words, even though individual i's deprivation is always at least as large as that of others, her deprivation would count less and would lead to a lower level of social deprivation. Such discrimination is prevented here.

Deprivation fairness: There exists a non-empty and closed set $C \subset X$ such that:

(i) for each $x^*_N \in C^n$, each $i \in N$, and each $x'_i \in X$ with $x'_i \geq x^*_i$, $(0, x^*_{-i}) \rhd x^*_N \unrhd (x'_i, x^*_{-i})$;

(ii) for each $x^*_N \in C^n$, each $i \in N$, and each $\bar{x}_i \in X$ with $(\bar{x}_i, x^*_{-i}) \rhd x^*_N$, there exists $x_i \in X$ with $x_i \, I_i \, \bar{x}_i$ such that for each $j \in N \backslash \{i\}$ and each $x_j \in X$, $x_i < x_j$ implies $(x_i, x^*_{-i}) \rhd (x_j, x^*_{-j})$.

The last axiom reinterprets the Pigou-Dalton transfer principle. Dalton (1920) suggested that a progressive transfer from a richer to a poorer individual (provided the richer/poorer relation is not reversed) leads to a more desirable distribution of income. Here, I impose that such transfer of attainments among equal-preference individuals weakly reduces social deprivation.

Equal-Preference Transfer: For each pair $j, k \in N$ such that $R_j = R_k \equiv R_0$, if there exist a pair $x_N, x'_N \in X_N$ and $\alpha \geq 0$ such that:

(i) $x_j - \alpha (x_j - x_k) = x'_j \; R_0 \; x'_k = x_k + \alpha (x_j - x_k)$;

(ii) for each $i \in N \backslash \{j, k\}$, $x_i = x'_i$;

then $x_N \unrhd x'_N$.

[13] Vector inequalities are denoted $\geq$, $>$, and $\gg$.

3.3 Individual and social deprivation

The **no-deprivation set** is a non-empty, closed, and convex subset $NDS \subset X$ such that, if $x \in NDS$ and $x' \geq x$, then $x' \in X$.

Let $\succsim$ be a weak order over X that is continuous, strictly antimonotonic over $X \backslash NDS$ (i.e. for each $x, x' \in X \backslash NDS$, $x \geq x'$ implies that $x' \succ x$), constant over NDS (i.e. for each $x, x' \in NDS$, $x \sim x'$), and has convex lower contours. The weak order $\succsim$ identifies a sequence of **iso-deprivation contours**, that is, sets of attainments equally ranked by $\succsim$. Let $\lambda : X \rightarrow [0,1]$ be a representation of $\succsim$ such that, without loss of generality, $\lambda(x) = 0$ for $x \in NDS$ and $\lambda(0) = 1$. Let $\lambda_i : X \rightarrow [0,1]$ be i's **individual deprivation function** for $\succsim$, defined by setting, for each $x_i \in X$, $\lambda_i(x_i) = \min\{\lambda(x) \,|x \, I_i \, x_i\}$.[14] Let the **normalized least joint convex function** ϕ be a real-valued, continuous, and increasing function such that: *(i)* $\phi \circ \lambda_i$ is convex for each $i \in N$; *(ii)* for ϕ is least convex among the functions satisfying *(i);* and, without loss of generality, *(iii)* $\phi(0) = 0$ and $\phi(1) = 1$.[15] The function ϕ provides cardinal meaning to the interpersonally comparable functions λ_i.[16] Let f be a real-valued and convex function, named **priority function**.

Each member of the family of deprivation indices characterized here can be written as follows:

$$D(x_N) = \sum_{i \in N} f \circ \phi \circ \lambda_i(x_i). \tag{5}$$

To repilogate, the ingredients of (5) are the following:

[14] The existence of a least desirable attainments vector for each iso-deprivation contour is ensured by continuity of $\succsim$ and the preferences assumption that, for some numerical representation u_i, $\lim_{|x_i| \rightarrow \infty} u_i(x_i) = \infty$.

[15] A function ϕ is least convex if any convex function ψ can be written by composing ϕ with a convex function f, i.e. $\psi = f \circ \phi$. It is unique up to an increasing affine transformation. The normalization of Condition *(iii)* ensures its uniqueness. See Debreu (1976), Kannai (1977), and Piacquadio (2017).

[16] The possibility of dropping ϕ from the formula (4) in the above example is due to the homotheticity of preferences and of the iso-deprivation contours. Then, the representation of preferences that is homogeneous of degree 1 is also a least concave function. Thus, $-\lambda_i$ is least convex. As ϕ would be an increasing linear transformation, it can be omitted.

1. the no-deprivation set NDS determines that $\lambda_i(x_i) = 0$ if i's attainments x_i are large enough ($x_i \in NDS$);

2. the iso-deprivation contours determine the extent of deprivation of each individual in a way that is interpersonally comparable: i is at least as deprived as j iff $\lambda_i(x_i) \geq \lambda_j(x_j)$;

3. the priority function f defines the priorities attributed to individuals based on their cardinalized deprivation indices $\phi \circ \lambda_i$.

The main result establishes the equivalence between deprivation rankings satisfying the introduced axioms and the social deprivation index in (5).

Theorem 1. *A deprivation ranking satisfies weak Pareto, continuity, separability, deprivation fairness, and equal-preference transfer if and only if it can be represented by a deprivation index as (5).*

4 Extensions

4.1 Differences in needs

As Atkinson writes (1987, p.753): "it should be noted that...families have been assumed to be identical in their needs and the poverty line has been taken as the same for all. In practice, the poverty line is different for families of different size and differing in other respects. There is therefore scope for disagreement not just about the *level* of the poverty line but also about its *structure*."

To takle differences in needs, I introduce the following changes. First, I weaken the axioms which derive their normative appeal from the implicit assumption of equal needs. Second, I introduce information about differences in needs and, correspondingly, axioms that allow incorporating such information in the deprivation ranking.

The first change is strightforward. Deprivation fairness and equal-preference transfer should be restricted to individuals with the same needs. By doing so, the axioms would lead to a need-specific no-deprivation set and to need-specific iso-deprivation contours. Thus, whenever two individuals are characterized by the same needs, their deprivations are compared throught the need-specific iso-deprivation contours. This leaves open the question of how to compare and aggregate deprivations across individuals with different needs.

The second change is more challenging. The standard answer is to base such comparisons on equivalence scales (see Lewbel (1989) and Ebert and Moyes (2003)). Equivalence scales, however, are constructed by assuming interpersonal comparability of utilities (or equivalently deprivation levels): needs are then implicitly defined by the different quantity of commodities needed to achieve the same level of utility. Here, no information is assumed about how to make interpersonal comparisons of utilities, which, instead, emerge endogenously from the axioms. Abiding by this approach, I suggest that the only additional information available is the set of attainments vectors that are considered sufficient to cover each individual's "basic needs," a multidimensional version of Atkinson's family-specific poverty lines.

Let Θ denote a finite set of types. The population is correspondingly partitioned, that is, there are a finite number of non-empty and disjoint set of individuals N^θ such that $N \equiv \bigcup_{\theta\in\Theta} N^\theta$. For each $\theta \in \Theta$, N^θ consists of n^θ individuals satisfying the assumptions of Section 3. For each $\theta \in \Theta$ and each $i \in N^\theta$, let the **basic needs** of i be a closed and non-empty set of attainments $B^\theta \subset X$. Let $B \equiv \left(B^\theta\right)_{\theta\in\Theta}$ define the basic needs of each type of individuals. For each $\theta \in \Theta$ and each $\alpha > 0$, let $\alpha B^\theta \equiv \left\{x \in X \,\middle|\, x = \alpha b \text{ with } b \in B^\theta\right\}$. With a slight abuse of notation, let $\alpha B^\theta\, P_i\, x_i$ mean that i considers all attainments in αB^θ more desirable than x_i and let $\alpha B^\theta \neg P_i\, x_i$ denote its negation, that is, there is at least an alternative in αB^θ that i finds at least as desirable as x_i.

I can now state formally a version of deprivation fairness that accounts for differences in needs. The first condition is unchanged. The second condition is only imposed on individuals belonging to the same type. The third condition is new and accounts for individuals with different needs. Let individual $i \in N^\theta$ find her attainments vector insufficient to cover her basic needs B^θ, that is $B^\theta\, P_i\, x_i$. In contrast, this is not true for individual $j \in N^{\theta'}$, that is $B^{\theta'} \neg P_j \, x_i$. Then, i should be considered at least as deprived as j. Importantly, this requirement is also imposed for proportional expansions and contractions of the set of basic needs, that is when for some $\alpha > 0$, $\alpha B^{\theta} \, P_i \, x_i$ and $\alpha B^{\theta'} \neg P_j \, x_j$, meaning that the relation between needs of individuals is preserved under rescaling of attainments.[17]

Needs-adjusted deprivation fairness: For each $\theta \in \Theta$, there exists a non-empty and closed set $C^{\theta} \subset X$ satisfying the following conditions. Let $C_N^{\Theta} \equiv \left\{ x_N \in X^n \,\middle|\, x_i \in C^{\theta} \text{ for each } \theta \in \Theta, i \in N^{\theta} \right\}$. Then:

(i) for each $x_N^* \in C_N^{\Theta}$, each $\theta \in \Theta$, each $i \in N^{\theta}$, and each $x_i' \in X$ with $x_i' \geq x_i^*$, $\left(0, x_{-i}^*\right) \rhd x_N^* \eqsim \left(x_i', x_{-i}^*\right)$;

(ii) for each $x_N^* \in C_N^{\Theta}$, each $\theta \in \Theta$, each $i \in N^{\theta}$, and each $\bar{x}_i \in X$ with $\left(\bar{x}_i, x_{-i}^*\right) \rhd x_N^*$, there exists $x_i \in X$ with $x_i \, I_i \, \bar{x}_i$ such that for each $j \in N_{\theta} \backslash \{i\}$ and each $x_j \in X$, $x_i < x_j$ implies $\left(x_i, x_{-i}^*\right) \rhd \left(x_j, x_{-j}^*\right)$;

(iii) for each $x_N^* \in C_N^{\Theta}$, each $\alpha > 0$, each pair $\theta, \theta' \in \Theta$, each $i \in N^{\theta}$, each $j \in N^{\theta'}$, and each pair $x_i, x_j \in X$ with $\alpha B^{\theta} \, P_i \, x_i$ and $\alpha B^{\theta'} \neg P_j \, x_j$, then $\left(x_i, x_{-i}^*\right) \trianglerighteq \left(x_j, x_{-j}^*\right)$.

Condition *(iii)* of *needs-adjusted deprivation fairness* introduces differences in needs for the measurement of individual and social deprivation. Combined with the other axioms, it forces the no-deprivation sets (one for each type) and the iso-deprivation contours to reflect the basic needs of each individual. In fact, for each type, the iso-deprivation contours are all homothetic and

[17] This rescaling condition is demanding. It is not obvious that proportional changes of the set of basic needs preserve interpersonal comparability in terms of deprivation of resources. Assume two different families have basic needs of, respectively, 1000\$/month and 800\$/month. Then, in a one-dimensional setting, this proportionality assumption also imposes that if these families earn, respectively, 500\$/month and 400\$/month, they are equally poor. In a multidimensional setting, preference diversity may complicate this relationship, but the idea remains controversial. Yet, whenever one can assess that proportionality is not desirable, more information is available on how deprivation changes with size. This information can then be used to avoid proportionality.

consist of the convex closure of the set of basic needs. What is left undefined is the size of the type-specific no-deprivation sets, which can be any common proportional expansion (for instance, when looking at relative poverty) or contraction (for instance, when looking at absolute poverty) of the basic needs.

Before stating this result, I present the multidimensional transfer principle. As anticipated, the only difference with *equal-preference transfer* is that transfers are now restricted to individuals with both same preferences and same needs.

Transfer among equals: For each $\theta \in \Theta$ and each pair $j, k \in B^{\theta}$ with $R_j = R_k \equiv R_0$, if there exist a pair $x_N, x'_N \in X_N$ and $\alpha \geq 0$ such that:

(i) $x_j - \alpha (x_j - x_k) = x'_j \; R_0 \; x'_k = x_k + \alpha (x_j - x_k)$;

(ii) for each $i \in N \backslash \{j, k\}$, $x_i = x'_i$;

then $x_N \trianglerighteq x'_N$.

Next, I define the deprivation indices that accommodate differences in needs. The type-specific no-deprivation set is defined by $\bar{\alpha} B^{\theta}$ for each $\theta \in \Theta$: it is proportional to the basic needs of each type of households. Let the index of deprivation for individual $i \in N$ of type $\theta \in \Theta$ be:

$$\lambda_i^{\theta} (x_i) = \max \left\{ 0, 1 - \frac{\alpha}{\bar{\alpha}} \left| x_i \, I_i \, x \text{ for some } x \in \alpha B^{\theta} \right. \right\}.$$

Let ϕ be the normalized least joint convex transformation associated to these deprivation indices. Formally, ϕ is a real-valued, continuous, and increasing function such that: *(i)* $\phi \circ \lambda_i^{\theta}$ is convex for each $\theta \in \Theta$ and each $i \in N^{\theta}$; *(ii)* ϕ is least convex among the functions satisfying *(i);* and, without loss of generality, *(iii)* $\phi (0) = 0$ and $\phi (1) = 1$. Finally, let f be the priority function, i.e. a continuous, increasing, and convex function. Then, the deprivation index $\bar{D}$ can be defined by setting for each $x_N \in X_N$:

$$\bar{D} (x_N) = \sum_{\theta \in \Theta} \sum_{i \in N^{\theta}} f \circ \phi \circ \lambda_i^{\theta} (x_i) \,. \tag{6}$$

The following result states that *weak Pareto*, *continuity*, *separability*, and the modiefied versions of *deprivation fairness* and *equal-preference transfer* characterize the deprivation index (6).

Theorem 2. *A deprivation ranking satisfies weak Pareto, continuity, separability, needs-adjusted deprivation fairness, and transfer among equals if and only if it can be represented by a deprivation index as (6).*

4.2 Categorical attainments

In the measurement of individual and social deprivation, it is often the case that attainments are categorical variables (see, among others, Alkire and Foster (2011a); Bossert et al. (2013); and Decancq et al. (2014)). I briefly address next how to extend the results when some dimensions are categorical.

Let S be the set of all possible combinations of categorical attainments, where the cardinality of S is finite. Define the extended attainments space as $X^+ \equiv X \times S$. Preferences R_i of each individual $i \in N$ are now defined on X^+ and are such that, for each categorical attainments $s \in S$, the weak order of attainments in X satisfies the assumptions of Section 3. Moreover, for each $x \in X$, each pair $s, s' \in S$, there exists $x' \in X$ such that $(x, s)\ I_i\ (x', s')$. A social state specifies an attainments vector $x_i^+ \equiv (x_i, s_i) \in X^+$ for each individual $i \in N$. A deprivation ranking $\trianglerighteq$ is a weak ordering of social states. A deprivation index represents such ranking by a function $D : X_N^+ \to \mathbb{R}$.

I suggest the axioms be changed as follows.

- **Weak Pareto**$^+$ and **separability**$^+$ are imposed on the extended attainments space X^+. These axioms are independent of the nature of the dimensions and do not pose any difficulties.[18]

- **Continuity**$^+$ and **equal-preference transfer**$^+$ are instead imposed on the space of continuous dimensions X (for each given vector of categorical attainments). Said differently, for each $s_N \in S^N$, the projection

[18] As an example, *weak Pareto*$^+$ now demands that for each pair $x_N^+, \bar{x}_N^+ \in X_N^+$, $x_i^+\ R_i\ \bar{x}_i^+$ for each $i \in N$ implies $\bar{x}_N^+ \trianglerighteq x_N^+$.

of the deprivation ranking on X^N satisfies the *continuity* and *equal-preference transfer* axioms introduced previously.[19]

- **Deprivation fairness**$^+$ is instead imposed for a reference vector of categorical attainments, equal across individuals. That is, there exists a vector of categorical attainments $s_N^* \in S_N$ with $s_i^* = s^*$ for each $i \in N$ that is kept fixed when imposing the previous version of deprivation fairness. As an example, if health is a categorical variable, s^* might be chosen as the attainment level corresponding to perfect health (as in Fleurbaey and Schokkaert (2009)).[20]

Then, the main result goes through. More precisely, the deprivation ranking satisfies the modified axioms if and only if it can be represented by a deprivation index

$$D^+ \left(x_N^+\right) = \sum_{i \in N} f \circ \phi \circ \tilde{\lambda}_i \left(x_i^+\right), \tag{7}$$

where:

- for a given $s^* \in S$, $\tilde{\lambda}_i \left(x_i^+\right)$ is defined by setting for each $x_i^+ \equiv (x_i, s_i) \in X^+$:
$$\tilde{\lambda}_i \left(x_i^+\right) = \min \left\{ \lambda(x) \left| (x, s^*) \ I_i \, x_i^+ \right. \right\}; \text{ and}$$

- ϕ is the normalized least joint convex function on X, that is a real-valued, continuous, and increasing function such that: *(i)* $\phi \circ \tilde{\lambda}_i$ is convex on X for each $i \in N$; *(ii)* ϕ is least convex among the functions satisfying *(i);* and, without loss of generality, *(iii)* $\phi(0) = 0$ and $\phi(1) = 1$.

[19] As an example, *continuity*$^+$ now demands that for each $x_N^+ \equiv (x_N, s_N) \in X_N^+$, the sets $\left\{ \bar{x}_N^+ \equiv (\bar{x}_N, \bar{s}_N) \in X_N \left| \bar{x}_N^+ \trianglerighteq x_N^+ \text{ with } s_N = \bar{s}_N \right. \right\}$ and $\left\{ \bar{x}_N^+ \equiv (\bar{x}_N, \bar{s}_N) \in X_N \left| x_N^+ \trianglerighteq \bar{x}_N^+ \text{ with } s_N = \bar{s}_N \right. \right\}$ are closed.

[20] Formally, deprivation fairness$^+$ now demands that there exists a non-empty and closed set $C \subset X$ and $s_N^* \in S_N$ with $s_i^* = s^*$ for each $i \in N$ such that: (i) for each $x_N^* \in C^n$, each $i \in N$, and each $x_i' \in X$ with $x_i' \geq x_i^*$, $\left((0, s^*), \left(x_{-i}^*, s_{-i}^*\right)\right) \triangleright (x_N^*, s_N^*) \simeq \left((x_i', s^*), \left(x_{-i}^*, s_{-i}^*\right)\right)$; and (ii) for each $x_N^* \in C^n$, each $i \in N$, and each $\bar{x}_i \in X$ with $\left((\bar{x}_i, s^*), \left(x_{-i}^*, s_{-i}^*\right)\right) \triangleright (x_N^*, s_N^*)$, there exists $x_i \in X$ with $(x_i, s_i^*) \ I_i \ (\bar{x}_i, s_i^*)$ such that for each $j \in N \backslash \{i\}$ and each $x_j \in X$, $x_i < x_j$ implies $\left((x_i, s^*), \left(x_{-i}^*, s_{-i}^*\right)\right) \triangleright \left((x_j, s^*), \left(x_{-j}^*, s_{-j}^*\right)\right)$.

- f is the priority function, that is a continuous, increasing, and convex function.

Theorem 3. *A deprivation ranking on X_N^+ satisfies weak Pareto$^+$, continuity$^+$, separability$^+$, deprivation fairness$^+$, and equal-preference transfer$^+$ if and only if it can be represented by a deprivation index as (7).*

To clarify, the no-deprivation set and the iso-deprivation contours continue to be defined on the subspace of continuous dimensions X. The reason is that only X has the properties required to define interpersonal comparability and cardinality of deprivations based on individuals' attainments vectors. Yet, this doesn't prevent the family of deprivation indices to respect preferences. In fact, the deprivation indices allow comparisons over the entire space of categorical attainments by "Pareto indifference," which demands that whenever all individuals are indifferent between two social states, social deprivation is unchanged.

5 Conclusions

The indices of individual and social deprivation characterized in this paper introduce a novel way to assess poverty. These indices: *(i)* are multidimensional; *(ii)* compare individuals by the set of attainments they are deprived of; *(iii)* are robust to measurement errors; *(iv)* express aversion to inequality among the poor; and *(v)* respect individuals' preferences. Moreover, such indices can account for differences in needs and extend to categorical dimensions of deprivation. The following table summarizes the main characteristics of the current proposal in relation to well-known alternatives.

	Foster et al. (1984)	Mashup indices (i.e. HDI)	Bourguignon & Chakravarty (2003)	Alkire and Foster (2011)	Current proposal
Multidimensional	NO	YES	YES	YES	YES
Attainment(s) based	YES	YES	YES	YES	YES
Robust to measurement errors	if $\theta \geq 1$	NO	YES	NO	YES
Averse to inequality among equal-pref. poor	if $\theta \geq 1$	NO	NO	NO	YES
Respect preferences	NO	NO	NO	NO	YES

While overcoming several drawbacks of currently adopted indices, the results of this paper also emphasize that fundamental ethical choices are unavoidable and remain open to debate. These choices include the dimensions to take into account, the weights to attribute to each dimension, and the priority to place on the most deprived individuals. Importantly, such ethical choices necessarily depend on each specific application (i.e. absolute or relative poverty, local or global poverty), on the data available for the poverty assessment, and on the ethical views that society wants to embrace.

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A Empirical exercise: details

A.1 Estimation of preferences

The database is the 2016 Norwegian register data, restricted to single men without children with age between 22 and 61. It consists of 90462 individuals. For each individual I use information about their after-tax income, labor supply, age, education, wage rate.

Each individual $i \in N$ is assumed to have preferences defined over after-tax income y_i, leisure l_i, and unobservable factors q (i.e. non-pecuniary benefits related to the specific job). Income is measured in million NOK. Let h_i denote the number of hours worked per year, the leisure is normalized as follows $l_i = 1 - \frac{h}{4160}$ ($l_i = 1$ corresponds to 80 hours of work per week). Preferences of individual i can be represented by:

$$U_i(y, l, z) = v_i(y, l)\, \varepsilon(q),$$

which consists of an individual-specific "deterministic" component $v_i(y, l)$ and a common "stochastic" component $\varepsilon(q)$. Following Aaberge et al. (1999), assume that $\varepsilon(q)$ is extreme-value distributed of type III. Let the deterministic component of preferences be given by:

$$\ln v_i(y, l) = \left[\beta_i y^{\zeta} + (1 - \beta_i)\, l^{\zeta}\right]^{\frac{1}{\zeta}}.$$

Each individual is assigned a random opportunity set, consisting of triplets such as (y, l, q). Each individual selects the alternative that maximizes her preferences U_i. Then, the parameters β_i and ζ are identified to maximize the likelihood between the observed distribution of after-tax income and leisure pairs and the estimated ones. The random opportunity sets and the preference parameters β_i are type specific and depend on the observable characteristics of individuals. Preferences depend on age, age squared, and 4 different levels of education: this leads to 156 different types of individuals.[21]

Formally, $\beta_i \equiv \frac{\beta_0}{\beta_0 + \delta_0 + \delta X_i}$ where $\beta_0 = 0.15$ and $\delta_0 = 0.75$ are constants and $\delta \in \mathbb{R}^5$ is a vector collecting the estimates for, in order, the multipliers for age (divided by 100) $\delta_1 = -1.21$, age squared (divided by 100^2) $\delta_2 = 1.88$, secondary school (1 if highest degree) $\delta_3 = -0,09$, college or university lower degree education $\delta_4 = -0,06$, and university higher degree education $\delta_5 = -0,18$. For simplicity, the elasticity of substitution between

[21] In the empirical application $\varepsilon(q)$ is set to unity and, thus, disregarded. I leave to future research the attempt to use the information on the unobservable component estimated for each individual to improve the measurement of individual and social deprivation.

consumption and leisure is assumed to be equal across individuals and estimated as $\zeta = -2.55$. All coefficients are highly significant.[22] The interpretation is as follows, individuals' willingness to work (or the relative importance given to consumption) first increases and then decreases with age, achieving a peak at age 32. As for education, the highest willingness to work is associated to the individuals with university higher degree education; the lowest is of those with no education; the individuals with highschool education are slightly more inclined to work than those with college or university lower degree education.

A.2 Individual and social deprivation

I first recall some definitions. The *no-deprivation set* is:

$$NDS \equiv \left\{ (y,l) \in X \left| \left[\beta y^{\gamma} + (1-\beta) l^{\gamma} \right]^{\frac{1}{\gamma}} \geq \nu \right. \right\},$$

where β, γ, ν are ethical parameters. The parameters are uniquely identified by the ethical choices discussed in Subsection 2.3. The *iso-deprivation contour* of level $\lambda \in [0,1]$ is the set:

$$IDC(\lambda) \equiv \left\{ (y,l) \in X \left| \left[\beta y^{\gamma} + (1-\beta) l^{\gamma} \right]^{\frac{1}{\gamma}} = (1-\lambda)\nu \right. \right\}.$$

Clearly, $IDC(0) = Fr\{NDS\}$ and $IDC(1) = \{0\}$.[23] Consider individual $i \in N$ with preferences R_i. Her individual deprivation function $\lambda_i : X \to \mathbb{R}$ is defined by setting for each $(y,l) \in X$,

$$\lambda_i(y,l) \equiv \min \left\{ \lambda \in [0,1] \left| (y,l) \ I_i \ (y',l') \text{ for some } (y',l') \in IDC(\lambda) \right. \right\}.$$

By homotheticity of preferences and of the *iso-deprivation contours*, the above individual's deprivation function is homogeneous of degree 1 and, thus, a least convex function. This allows disregarding the cardinalizing function

[22] I report the t-values in parenthesis: β_0 (22.0); ζ (−91.4); δ_0 (16.9); δ_1 (−6.0); δ_2 (7.8); δ_3 (−12.0); δ_4 (−6.9); and δ_5 (−16.0).

[23] By monotonicity of preferences, it is irrelevant whether the *iso-deprivation contour* of level 0 is the entire *no-deprivation set* or only its frontier.

"ϕ:" said differently, for any ordinally equivalent representation of individual i's deprivation, the endogenous cardinalizing function ϕ would have transformed these in an affine transformation of λ_i. Let $V_i \equiv \min_{(y,l)\in NDS} v_i(y,l)$ be the minimum level of well-being of i when she is not deprived. By simple manipulation, the individual deprivation function can be rewritten as:

$$\lambda_i(y,l) = \max\left\{0, \frac{V_i - v_i(y,l)}{V_i}\right\}.$$

Social deprivation is measured by the sum of a convex transformation of the individuals deprivations. Using a power transformation with exponent 2, gives the following per capita index of social deprivation:

$$D(x_N) \equiv \frac{1}{n}\sum_i \left[\lambda_i(y_i,l_i)\right]^2.$$

B Proofs

B.1 Proof of Theorem 1

The proof that a deprivation ranking with a representation (5) satisfies the axioms is quite strightforward.

For each $i \in N$, $-\lambda_i$ is a non-decreasing transformation of a representation of preferences R_i: it is a representation of preferences whenever i is deprived (for each x_i such that $NDS\, R_i\, x_i$); it is constant whenever i is not deprived (for each x_i such that $C\, \neg P_i\, x_i$). Thus, *weak Pareto* follows. Since the functions f, ϕ, and, for each $i \in N$, λ_i are continuous, the deprivation ranking $\trianglerighteq$ is *continuous.* Since the representation of $\trianglerighteq$ is additive over individuals, *separability* holds. By convexity of f and, for each $i \in N$, $\phi \circ \lambda_i$, *equal-preference transfer* follows.

Since each function λ_i is largest at the 0 attainment vector, first strictly decreasing and then constant, there exists a set C^n (for example, $C = NDS$) such that Condition (i) of *no-deprivation fairness* holds. Finally, assume that Condition (ii) of *no-deprivation fairness* is violated. Then, there exists a $x^*_N \in C^n$, $i \in N$, and $\bar{x}_i \in X$ with $\left(\bar{x}_i, x^*_{-i}\right) \triangleright x^*_N$, such that for each $x_i \in X$

with $x_i \, I_i \, \bar{x}_i$, each each $j \in N \backslash \{i\}$ and each $x_j \in X$ with $x_i < x_j$, it holds that $\left(x_j, x^*_{-j}\right) \trianglerighteq \left(x_i, x^*_{-i}\right)$. By construction of λ_i, the index of deprivation of i at $\bar{x}_i$ is the index λ associated to the smallest lower contour set of $\succsim$ such that i finds each element of this set at least as desirable as $\bar{x}_i$. Let x'_i be (one of) i's least preferred bundles in this set (the existence is ensured by continuity of preferences and the assumption that $\lim_{|x_i| \to \infty} u_i(x_i) = \infty$). By continuity of preferences, $x'_i \, I_i \, \bar{x}_i$. Moreover, each $x_j > x'_i$ belongs to a smaller lower contour set of $\succsim$. By construction, the index of deprivation associated to x_j is such that $\lambda_j(x_j) < \lambda_i(\bar{x}_i)$. This contradicts $\left(x_j, x^*_{-j}\right) \trianglerighteq \left(x_i, x^*_{-i}\right)$ and proves that Condition (ii) holds.

I next show the reverse implication. The proof is divided in three steps.

Step 1. *If a deprivation ranking $\trianglerighteq$ satisfies weak Pareto, continuity, and separability, then there exists a continuous function $D : X_N \to \mathbb{R}$, and, for each $i \in N$, a non-increasing real-valued function g_i and a representation $U_i : X \to \mathbb{R}$ of her preferences R_i, such that for each pair $x_N, x'_N \in X_N$, $x_N \trianglerighteq x'_N$ if and only if*

$$D(x_N) \equiv \sum_{i \in N} g_i \circ U_i(x_i) \geq \sum_{i \in N} g_i \circ U_i(x'_i) \equiv D(x'_N).$$

Proof. By *continuity*, there exists a continuous function $\bar{D} : X_N \to \mathbb{R}$ that represents $\trianglerighteq$. By *weak Pareto*, there exists a continuous function $\tilde{D} : \mathbb{R}^n \to \mathbb{R}$ non increasing in each argument and, for each $i \in N$, a continuous function $U_i : X \to \mathbb{R}$ representing preferences R_i such that for each $x_N \in X_N$, $\bar{D}(x_N) = \tilde{D}(U_1(x_1), ..., U_n(x_n))$. By *separability* (and the assumption that $|N| \geq 3$), there exists a increasing function $H : \mathbb{R} \to \mathbb{R}$ and, for each $i \in N$, a continuous and non increasing function $g_i : \mathbb{R} \to \mathbb{R}$ such that $\bar{D}(x) = H\left(\sum_{i \in N} w_i \circ U_i(x_i)\right)$. Let $D : X_N \to \mathbb{R}$ be such that for each $x_N \in X_N$, $D(x_N) \equiv \sum_{i \in N} g_i \circ U_i(x_i)$. Since H is increasing, D also represents the deprivation ranking $\trianglerighteq$, proving the result. □

Step 2. *If the deprivation ranking $\trianglerighteq$ also satisfies deprivation fairness,* *then there exists a no-deprivation set $NDS \subset X$, a weak order $\succsim$ identifying the iso-deprivation contours, and a strictly increasing and continuous real-valued function g such that, for each individual $i \in N$, $g_i \circ U_i = g \circ \lambda_i$, where λ_i is defined in Section 3. That is, the deprivation ranking $\trianglerighteq$ can be represented by $D = \sum_{i \in N} g \circ \lambda_i$.*

Proof. Deprivation fairness directly postulates (Condition i) the existence of a non-empty and closed set $C \subset X$ such that for each $i \in N$, each $x_N \in X_N$ with $x_i \in C$, and each $a_i \in X$ with $a_i \geq x_i$, $(0, x_{-i}) \triangleright (x_i, x_{-i}) \simeq (a_i, x_{-i})$. Thus, $0 \notin C$. Let $\underline{x}_i$ be (one of) the attainments vector(s) that i finds least desirable in C; it's existence follows from each i's preferences admitting a strictly increasing and concave representation $u_i : X \to \mathbb{R}$ such that $\lim_{|x_i| \to \infty} u_i(x_i) = \infty$ and since C is non-empty and closed. Then, using the representation of Step 1, $g_i \circ U_i(0) > g_i \circ U_i(x_i) = g_i \circ U_i(a_i)$ for each $a_i \geq x_i$ with $x_i \in C$. By monotonicity of preferences, $g_i \circ U_i(0) > g_i \circ U_i(\underline{x}_i) = g_i \circ U_i(x_i)$ for each $x_i \, R_i \, \underline{x}_i$.

Let $x^*_N \in C$. I show next that $\left(0, x^*_{-i}\right) \simeq \left(0, x^*_{-j}\right)$ for each $i, j \in N$. Assume not: then, without loss of generality, $\left(0, x^*_{-j}\right) \triangleright \left(0, x^*_{-i}\right)$. By continuity of individual preferences and of the deprivation ranking, there exists an attainment vector $\varepsilon_j \gg 0$ such that $\left(\varepsilon_j, x^*_{-j}\right) \triangleright \left(0, x^*_{-i}\right)$. This directly violates *deprivation fairness* (Condition ii), where $\bar{x}_i = 0$. This result also implies that $g_i \circ U_i(0) - g_i \circ U_i(\underline{x}_i) = g_j \circ U_j(0) - g_j \circ U_j\left(\underline{x}_j\right)$ for each $i, j \in N$.

Let $K \equiv [0, g_i \circ U_i(0) - g_i \circ U_i(\underline{x}_i)]$. By *continuity*, for each $k \in K$ and each $i \in N$, there exists an attainments vector $x_i(k)$ such that $g_i \circ U_i(x_i(k)) - g_i \circ U_i(\underline{x}_i) = k$. It follows that $\left(x_i(k), \underline{x}_{-i}\right) \simeq \left(x_j(k), \underline{x}_{-j}\right)$ for each $i, j \in N$. For each $i \in N$, let the upper-contour sets at $x_i(k)$ be denoted $UCS_i(k) \equiv \{x_i \in X \,| x_i \, R_i \, x_i(k)\}$. The intersection of these upper-contour sets is $UCS(k) \equiv \bigcap_{i \in N} UCS_i(k)$. Clearly, by definition of upper contour set, it cannot be that $x_i(k) \; P_i \; x$ for some $x \in UCS(k)$. Assume instead that, for each $x \in UCS(k)$, $x \, P_i \, x_i(k)$. By continuity of U_i, there exists an attainments vector $x_i^+ \in X$ such that, for each $x \in UCS(k)$, $x \, P_i \, x_i^+ \, P_i \, x_i(k)$ and $\left(x_j(k), \underline{x}_{-j}\right) \triangleright \left(x_i^+, \underline{x}_{-i}\right)$. Since $x_i^+ \notin UCS(k)$, for each $x_i \, I_i \, x_i^+$ there exists $j \neq i$ and $x_j > x_i$ with $x_j \, I_j \, x_j(k)$ and, by transitivity of the deprivation ranking, $\left(x_j, \underline{x}_{-j}\right) \rhd \left(x_i, \underline{x}_{-i}\right)$. This is a violation of *deprivation fairness* (Condition *ii*). Thus, for each $i \in I$ and each $k \in K$, $x_i(k)\ I_i\, w_i$ with w_i one of the least desirable attainment vectors for i in $UCS(k)$. Moreover, for each $i, j \in N$ and each $k, k' \in K$, $g_i \circ U_i(x_i(k)) \geq g_j \circ U_j(x_j(k'))$ if and only if $k \geq k'$.

I next construct the weak order $\succsim$ and the corresponding iso-deprivation contours. First, let NDS be the convex hull of $UCS(0)$. This is non-empty, closed, and, by definition, convex; it is thus a no-deprivation set. For each $k \in K$, let $C(k)$ be the convex hull of $UCS(k)$. Note that by concavity of preferences, for each $i \in I$, i's least desirable attainment vectors in $UCS(k)$ are the same as in $C(k)$. Define $\succsim$ by setting, for each $x, x' \in X$, $x \succsim x'$ if and only if $\max[k \in K\, |x \in C(k)] \geq \max[k \in K\, |x' \in C(k)]$. This weak order is continuous, strictly antimonotonic over $X \backslash C$ (i.e. for each $x, x' \in X\backslash C$, $x \geq x'$ implies that $x' \succ x$), constant over C (i.e. for each $x, x' \in C$, $x \sim x'$), and, by convexity of individuals' preferences, has convex lower contours.

Now, let $\lambda : X \to [0,1]$ be a representation of $\succsim$ such that, without loss of generality, $\lambda(x) = 0$ for $x \in NDS$ and $\lambda(0) = 1$.[24] Given λ, individual i's deprivation function λ_i is defined by setting for each $x_i \in X$, $\lambda_i(x_i) = \min\{\lambda(x)\, |x\, I_i\, x_i\}$.

Finally, for each $i, j \in N$ and each $k, k' \in K$, $\lambda_i(x_i(k)) \geq \lambda_j(x_j(k'))$ if and only if $k \geq k'$, which holds if and only if $g_i \circ U_i(x_i(k)) \geq g_j \circ U_j(x_j(k'))$. Thus, there exists a real-valued and increasing function g such that $g_i \circ U_i = g \circ \lambda_i$. □

Step 3. *If the deprivation ranking $\trianglerighteq$ also satisfies equal-preference transfer, then there exists a priority function f and a least joint convex function ϕ such that $g = f \circ \phi$. That is, the deprivation ranking $\trianglerighteq$ can be represented by $D = \sum_{i \in N} f \circ \phi \circ \lambda_i$.*

Proof. Let $j, k \in N$ be such that $R_j = R_k \equiv R_0$. By Step 2, $\lambda_j = \lambda_k \equiv \lambda_0$. Let $x_N, x'_N \in X^n$ be such that: *(i)* $x'_j = x'_k = \frac{x_j + x_k}{2}$; *(ii)* for each

[24] As an example, let $\bar{\alpha} \equiv \min[\alpha \in \mathbb{R}_+ |\alpha \mathbf{1}_\ell \in NDS]$. Clearly, $\bar{\alpha} > 0$. Then, for each $x \in X$, let $\lambda(x) = \frac{\bar{\alpha} - \alpha}{\bar{\alpha}}$ if either $x = \alpha \mathbf{1}_\ell$ or if $x \sim \alpha \mathbf{1}_\ell$.

$i \in N \setminus \{j,k\}$, $x_i = x'_i$. By *equal-preference transfer*, $x \trianglerighteq x'$. By the previous steps, this is equivalent to

$$g \circ \lambda_0 \left(\frac{x_j + x_k}{2} \right) \leq \frac{g \circ \lambda_0 (x_j) + g \circ \lambda_0 (x_k)}{2}.$$

Since this holds for each pair of attainments vectors of i and j, $g \circ \lambda_0$ is convex. As the argument holds for each pair of individuals with the same preferences, for each $i \in N$, $g \circ \lambda_i$ is convex. Let ϕ be a least joint convex transformation. It's existence follows from Lemma 1 of Piacquadio (2017) by noting that $-\phi$ is least joint concave. Then, there exists a priority function f (continuous, increasing, and convex) such that $g = f \circ \phi$ and $\trianglerighteq$ can be represented by $D = \sum_{i \in N} f \circ \phi \circ \lambda_i$. □

B.2 Proof of Theorem 2

I only show that the axioms imply the deprivation index (6). I follow the same steps of the proof of Theorem 1.

Step 1 directly extends. If a deprivation ranking $\trianglerighteq$ satisfies *weak Pareto*, *continuity*, and *separability*, then there exists a continuous function $D : X_N \to \mathbb{R}$, and, for each $\theta \in \Theta$ and each $i \in N_\theta$, a non-increasing real-valued function g_i^θ and a representation $U_i^\theta : X \to \mathbb{R}$ of her preferences R_i, such that for each pair $x_N, x'_N \in X_N$, $x_N \trianglerighteq x'_N$ if and only if

$$D(x_N) \equiv \sum_{\theta \in \Theta} \sum_{i \in N_\theta} g_i^\theta \circ U_i^\theta (x_i) \geq \sum_{\theta \in \Theta} \sum_{i \in N_\theta} g_i^\theta \circ U_i^\theta (x'_i) \equiv D(x'_N).$$

With the next step, I also impose *needs-adjusted deprivation fairness* on the deprivation ranking $\trianglerighteq$.

Step 2. *If the deprivation ranking $\trianglerighteq$ also satisfies* needs-adjusted deprivation fairness, *then there exists a constant $\bar{\alpha} > 0$ (identifying the deprivation functions λ_i^θ in Section 4) and a strictly increasing and continuous real-valued function g such that, for each $\theta \in \Theta$ and each $i \in N^\theta$, $g_i^\theta \circ U_i^\theta = g \circ \lambda_i^\theta$. That is, the deprivation ranking $\trianglerighteq$ can be represented by $D = \sum_{\theta \in \Theta} \sum_{i \in N^\theta} g \circ \lambda_i^\theta$.*

Proof. The first two conditions of *needs-adjusted deprivation fairness* are equivalent to those of *deprivation fairness*, except that the second holds only for equal-type individuals. Thus, repeating Step 2 of the proof of Theorem 1, for each $\theta \in \Theta$, there exists a weak order $\succsim^{\theta}$ on X (identifying the iso-deprivation contours) and a no-deprivation set NDS^{θ} such that $\succsim^{\theta}$ is continuous, strictly antimonotonic over $X \backslash NDS^{\theta}$ (i.e. for each $x, x' \in X \backslash NDS^{\theta}$, $x \geq x'$ implies that $x' \succ x$), constant over NDS^{θ} (i.e. for each $x, x' \in NDS^{\theta}$, $x \sim x'$), and has convex lower contours. For each $\theta \in \Theta$, let $\phi^{\theta} : X \rightarrow [0,1]$ be a representation of $\succsim$ such that, without loss of generality, $\phi^{\theta}(x) = 0$ for $x \in NDS^{\theta}$ and $\phi^{\theta}(0) = 1$. For each $\theta \in \Theta$ and each $i \in N^{\theta}$, let ϕ_i^{θ} be such that $\phi_i^{\theta}(x_i) = \min\left\{\phi^{\theta}(x) \,|x \, I_i \, x_i\right\}$ and define:

$$\alpha_i^{\theta} \equiv \min\left\{\alpha \in \mathbb{R}_+ \,\middle|\alpha B^{\theta} \, R_i \, x_i \text{ for some } x_i \in X \text{ with } \phi_i^{\theta}(x_i) = 0\right\}.$$

I prove next that there exists $\bar{\alpha}$ such that $\alpha_i^{\theta} = \bar{\alpha}$ for each $\theta \in \Theta$ and each $i \in N^{\theta}$. Assume not, then there exist $\theta, \theta' \in \Theta$, $i \in N^{\theta}$, and $j \in N^{\theta'}$ such that $\alpha_i^{\theta} < \alpha_j^{\theta'}$. For each $\alpha > 0$, let $x_i(\alpha), x_j(\alpha) \in X$ be such that $\alpha B^{\theta} \, I_i \, x_i(\alpha)$ and $\alpha B^{\theta'} \, I_j \, x_j(\alpha)$. Let $\alpha \in \left(\alpha_i^{\theta}, \alpha_j^{\theta'}\right)$ and $x_N^* \in NDS_N^{\Theta}$. Since $\alpha > \alpha_i^{\theta}$, $\alpha B^{\theta} \, P_i \, x_i\left(\alpha_i^{\theta}\right)$ and $x_i(\alpha) \; P_i \, x_i\left(\alpha_i^{\theta}\right)$. By definition of ϕ_i^{θ}, $\phi_i^{\theta}(x_i(\alpha)) = \phi_i^{\theta}\left(x_i\left(\alpha_i^{\theta}\right)\right) = 0$ and, consequently, $\left(x_i(\alpha), x_{-i}^*\right) \sim x_N^*$. Conversely, since $\alpha < \alpha_j^{\theta'}$, $\alpha B^{\theta'} \neg P_j \, x_j\left(\alpha_j^{\theta'}\right)$, $\phi_j^{\theta'}\left(x_j\left(\alpha_j^{\theta'}\right)\right) > \phi_j^{\theta'}(x_j(\alpha)) = 0$, and $\left(x_j(\alpha), x_{-i}^*\right) \rhd x_N^*$. Let $\alpha^- \in \left(\alpha_i^{\theta}, \alpha\right)$ and $\alpha^+ \in \left(\alpha, \alpha_j^{\theta'}\right)$. By monotonicity of preferences, $\alpha B^{\theta} \, P_i \, x_i(\alpha^-)$ and $\alpha B^{\theta'} \neg P_j \, x_j(\alpha^+)$. By *efficiency*, $\left(x_j(\alpha^+), x_{-j}^*\right) \rhd \left(x_i(\alpha^-), x_{-i}^*\right)$. This is a contradiction of Condition (iii), proving the existence of $\bar{\alpha}$.

I next show that for each $\alpha, \alpha' \in [0, \bar{\alpha}]$ with $\alpha \leq \alpha'$, each pair $\theta, \theta' \in \Theta$, each $i \in N^{\theta}$, each $j \in N^{\theta'}$, and each $x_N^* \in NDS_N^{\Theta}$, $\left(x_i(\alpha), x_{-i}^*\right) \unrhd \left(x_j(\alpha'), x_{-j}^*\right)$. The same argument as above shows that the converse relation , i.e. $\left(x_j(\alpha'), x_{-j}^*\right) \rhd \left(x_i(\alpha), x_{-i}^*\right)$, leads to a contradiction of Condition (iii). For each $\theta \in \Theta$ and $i \in N^{\theta}$, define:

$$\lambda_i^{\theta}(x_i) = \max\left\{0, 1 - \frac{\alpha}{\bar{\alpha}} \,\middle| x_i \, I_i \, x \text{ for some } x \in \alpha B^{\theta}\right\}.$$

Finally, for each pair $\theta, \theta' \in \Theta$, each $i \in N^{\theta}$, each $j \in N^{\theta'}$, and each $\alpha, \alpha' \in [0, \bar{\alpha}]$, $\lambda_i^{\theta}(x_i(\alpha)) \geq \lambda_j^{\theta'}(x_j(\alpha'))$ if and only if $\alpha \leq \alpha'$. By Step 1, $\lambda_i^{\theta}(x_i(\alpha)) \geq \lambda_i^{\theta}(x_j(\alpha'))$ if and only if $g_i^{\theta} \circ U_i^{\theta}(x_i(\alpha)) \geq g_j^{\theta} \circ U_i^{\theta}(x_i(\alpha'))$. Thus, there exists a real-valued and increasing function g such that $g_i^{\theta} \circ U_i^{\theta} = g \circ \lambda_i^{\theta}$ and $\trianglerighteq$ can be represented by $D = \sum_{\theta \in \Theta} \sum_{i \in N^{\theta}} g \circ \lambda_i^{\theta}$. □

Step 3. *If the deprivation ranking $\trianglerighteq$ also satisfies equal-preference transfer, then there exists a priority function f and a least joint convex function ϕ such that $g = f \circ \phi$. That is, the deprivation ranking $\trianglerighteq$ can be represented by $D = \sum_{\theta \in \Theta} \sum_{i \in N^{\theta}} f \circ \phi \circ \lambda_i^{\theta}$.*

Proof. See the proof of the corresponding step of Th.1. □

B.3 Proof of Theorem 3

Again, the proof that the ranking satisfies the axioms is omitted.

I prove the converse implication. By *deprivation fairness*$^{+}$, there exists a vector of categorical attainments $s^* \in S$ such that, if each i's attainment is $s_i = s^*$, the same conditions in *deprivation fairness* are imposed. Thus, I first focus on X_N^{+} such that $s_N = s_N^*$. All the axioms in Theorem 1 hold on this space. Thus, the deprivation ranking $\trianglerighteq$ (on X_N^{+} with $s_N = s_N^*$) can be represented by $D = \sum_{i \in N} f \circ \phi \circ \tilde{\lambda}_i$, where $\tilde{\lambda}_i$ is defined in Section 4.

I next extend the representation to X_N^{+}. By assumption, for each $x_N^{+} \in X_N^{+}$ there exists a social state $\bar{x}_N^{+} \in X_N^{+}$ with $s_N = s_N^*$ such that, for each $i \in N$, $(x_i, s_i)\ I_i\ (\bar{x}_i, s^*)$. By *weak Pareto*$^{+}$ and *continuity*$^{+}$, $x_N^{+} \simeq \bar{x}_N^{+}$. Thus, the result follows.

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