Identification of Structural Vector Autoregressions Through Higher Unconditional Moments[*]

Alain Guay[†]   Michel Normandin[‡]

October 2018

Abstract

This paper pursues two objectives. First, we determine the local identification conditions of SVAR processes through the third and fourth unconditional moments of the reduced-form innovations. Our findings provide novel insights when the entire system is not identified, as they highlight which subset of structural parameters is identified and which is not. Second, we elaborate a tractable testing procedure to verify whether the identification conditions hold, prior to the estimation of the structural parameters of the SVAR process. To do so, we design a new bootstrap procedure that improves the small-sample properties of rank tests for the symmetry and kurtosis of the structural shocks.

JEL classification: C12, C32, C51.

Keywords: Bootstrap procedure, excess kurtosis, identification condition, rank test, skewness, structural vector autoregression.

1. Introduction

Econometric methods for simultaneous equation models highlight the importance of verifying the identification conditions before proceeding to the estimation exercise of the structural parameters. Namely, it is only if the identification conditions hold that it becomes feasible to estimate all the structural parameters. In this vein, this paper pursues two objectives. First, we derive the conditions for local identification of Structural Vector Autoregressive (SVAR) processes through higher unconditional moments. Second, we develop a tractable method to verify whether a SVAR process is identified, prior to the estimation of the structural parameters.

SVAR processes represent systems of simultaneous, dynamic, linear equations, in which the structural parameters reflect the contemporaneous interactions across the current variables of interest and the dynamic feedbacks between these current variables and their lagged values. Such processes are frequently used in macroeconomics to assess the dynamic responses of the variables of interest to various structural shocks.

A first strand of the SVAR literature relies on the standard assumption that the structural shocks are orthogonal and extracts the information contained in the unconditional covariances of the reduced-form innovations to identify the structural parameters. As is well known, this information is insufficient to identify all the parameters, so that short-run restrictions (e.g. Sims, 1980), long-run restrictions (e.g. Blanchard and Quah, 1989), and/or sign restrictions (e.g. Uhlig, 2005) need to be placed. If the restrictions are economically motivated, then the imposition of enough restrictions gives rise to economic identification in the sense that the dynamic responses become interpretable given that the structural shocks are economically meaningful. However, it is not possible to verify jointly the validity of all the restrictions by applying formal statistical tests.

A second strand of the literature exploits the information related to certain statistical properties of the data, in addition to the unconditional covariances of the reduced-form innovations (see Kilian and Lütkepohl, 2017, Chapter 14). If this information is rich enough then this strategy yields local identification, without resorting to any identifying restrictions, and, hence, the dynamic response matrices are unique up to changes in sign and permutations of columns.[1] It also produces

---

[1]Changing the signs of columns means that negative instead of positive structural shocks (and vice versa) are

statistical (rather than economic) identification as nothing guarantees that the dynamic responses and structural shocks have an economic interpretation. In this framework, it is possible to verify the validity of certain classes of restrictions (e.g. short- and long-run restrictions) that would have been required if only the unconditional covariances of the reduced-form innovations were taken into account. This is convenient, for example, to formally select among alternative sets of restrictions reflecting competing economic theories.

One method relying on the statistical properties of the data specifies the time-varying variances of the structural shocks, while preserving the standard assumption that theses shocks are orthogonal. In this context, all the structural parameters involved in the SVAR are identified, without placing any restrictions, when at least all, but one, structural shocks display distinct time-varying variances. Note, however, that the method requires to take a stand about whether the time-varying variances are determined by fixing a priori the dates of the structural breaks, are specified via GARCH processes, or are modeled by regime switching processes with Markov chains or smooth transitions (e.g. Rigobon, 2003; Normandin and Phaneuf, 2004; Lanne, Lütkepohl, and Maciejowska, 2010; Lütkepohl and Netšunajev, 2014; Lütkepohl and Schlaak, 2018). Another approach is based on unconditional non-normal distributions of the structural shocks, but assumes that these shocks are independent. In this environment, all the structural parameters are identified, when at least all, but one, structural shocks are non-normally distributed (see Comon, 1994; Eriksson and Koivunen, 2004; Herwartz, 2015; Gouriéroux, Monfort, and Renne, 2017).[2] Observe, however, that the assumption of independent structural shocks is more restrictive than the standard one stating that the shocks are orthogonal; that is, it is not always possible to recover independent structural shocks from non-normal reduced-form innovations through linear transformations.

A key goal of this paper is to determine the local, statistical identification conditions of SVAR processes through the third and fourth unconditional moments of the reduced-form innovations. For this purpose, we assume that the structural shocks display zero cross-sectional covariances, coskewnesses, and excess cokurtoses. Note that this can be viewed as a natural extension to the

---

considered. Permuting the columns implies alternative orderings of the structural shocks.

[2] Alternatively, Gouriéroux, Monfort, and Renne (2018) show that all structural parameters are identified under the assumption that the structural shocks are mutually independent and some conditions on the cumulants of order three or higher for each structural shock.

third and fourth unconditional comoments of the standard assumption that the structural shocks are orthogonal. Moreover, our assumption admits the possibility that the structural shocks exhibit time-varying conditional variances (although we do need to specify the process governing these variances) and is milder than the assumption stating that the shocks are independent. In our context, not only the covariances of the reduced-form innovations, but also the coskewnesses and excess cokurtoses of these innovations can be exploited to identify extra structural parameters, and, hence, to relax some of the identifying restrictions required when the information contained in the third and fourth moments is ignored. Formally, we derive the order (necessary) and rank (sufficient) conditions for local, statistical identification, where the latter are obtained by extending the developments of Lütkepohl (2007, Chapter 9). We further express these conditions in terms of simple formulas, which exclusively involve the numbers of structural shocks displaying skewness and excess kurtosis. Given this information, it is most easy for empirical researchers to determine whether or not the structural system is identified.

Our results regarding the identification of the entire structural system parallel the existing results. That is, all the structural parameters are identified when at least all, but one, structural shocks exhibit skewness and/or excess kurtosis. Our findings further provide novel insights when the entire SVAR process is not identified, as they highlight which subset of structural parameters is identified and which is not. This leads to three important implications. First, one can establish which structural subsystem is identified. Note that this subsystem documents the dynamic responses of all the variables included in the SVAR process to the structural shocks which are asymmetric and/or non-mesokurtic.[3] Hence, these responses can be traced without imposing any restrictions on the structural parameters. Second, one can determine the structural parameters for which some restrictions must be placed on in order to achieve the identification of the entire system. Such restrictions are required to evaluate the dynamic responses to the structural shocks which are symmetric and mesokurtic. Third, one can test the validity of economic and statistical restrictions (by treating these as overidentifying restrictions) that are commonly placed on the

---

[3] For briefness, throughout the text symmetric (asymmetric) and mesokurtic (non-mesokurtic) variables refer to variables with symmetric (asymmetric) and mesokurtic (non-mesokurtic) distributions. Also, a symmetric (asymmetric) distribution implies a zero (non-zero) skewness, whereas a mesokurtic (non-mesokurtic) distribution implies a zero (non-zero) excess kurtosis.

structural subsytem that is identified through higher unconditional moments.

Another prime aim of this paper is to elaborate a tractable testing procedure to verify whether the identification conditions hold, prior to the estimation of the structural parameters involved in the SVAR process. As stated above, our identification conditions require the knowledge of the numbers of asymmetric and non-mesokurtic structural shocks. At first glance, this may seem problematic for practitioners, as the structural shocks become measurable only once the structural system is estimated.[4] However, we demonstrate that the numbers of structural shocks displaying skewness and excess kurtosis correspond to the ranks of the coskewness and excess cokurtosis matrices of the reduced-form innovations, where these matrices are easily constructed from sample estimates of the moments of the reduced-form residuals — without having to proceed to the estimation of the structural system.

In this paper, we design a new bootstrap procedure to approximate the finite-sample distributions in order to test the ranks of the coskewness and excess cokurtosis matrices of the reduced-form innovations. We show that this procedure allows to overcome size distortions. Specifically, for symmetry both the Wald and likelihood-ratio versions of the rank test with bootstrap critical values feature empirical sizes that are almost identical to the nominal sizes, regardless of the number of observations in the sample. In comparison, the Wald test with asymptotic distributions has empirical sizes that slightly deviate from the nominal ones, and the likelihood-ratio test with limiting distributions has empirical sizes that are substantially smaller than the nominal counterparts. For kurtosis, the bootstrap versions of the Wald and likelihood-ratio test statistics are essentially free of size distortions for all sample sizes. In sharp contrast, the Wald and likelihood-ratio tests with asymptotic distributions imply empirical sizes that are systematically close to zero, even for large samples.

Finally, we illustrate our developments by identifying the effects of fiscal policies on economic activity; a topic that has received renewed interest in light of the recent Great Recession. For this purpose, we perform the analysis on a trivariate SVAR process which includes taxes, public

---

[4]As a result, existing studies do not verify whether the structural shocks are asymmetric or non-mesokurtic before proceeding to the estimation of the structural system; see for example Moneta, Entner, Hoyer, and Coad (2013), Lanne, Meitz, and Saikkonen (2017), and Gouriéroux, Monfort, and Renne (2017).

spending, and output for the U.S. The empirical results for the Wald and likelihood-ratio bootstrap versions for the rank tests indicate that all the structural shocks are symmetric and only the tax shock is non-mesokurtic. Based on this information, the identification conditions reveal that the subsystem relating all the variables to the tax shock is identified, so that the effects of this shock can be assessed without imposing any restrictions on the structural parameters. In contrast, the subsytem relating the variables to the public spending shock is under-identified, and, hence, the effects of this shock can only be evaluated under certain restrictions. Also, we show that the restrictions invoked in the seminal study of Blanchard and Perotti (2002) imply that the subsystem relating the variables to the spending shock becomes over-identified. We further document that the effects of the spending shock highly depend on the nature of the identifying restrictions used.

This paper is organized as follows. Section 2 motivates, from a simple example, the identification through the third and fourth unconditional moments. Section 3 derives the order and rank conditions for the identification of the structural parameters involved in SVAR processes. Section 4 develops a tractable procedure to test whether the identification conditions hold, before the estimation of the structural parameters. Section 5 presents an application related to the identification of the structural parameters determining the dynamic responses of output to fiscal shocks. Section 6 concludes.

2. Motivation

This section motivates the strategy of identifying SVAR processes through higher unconditional moments. To do so, we provide a simple example to gain some intuition about how the information related to asymmetric and non-mesokurtic distributions can be exploited to achieve identification. Specifically, we consider the following bivariate SVAR process (in innovation form):

$$\nu_{y,t} = -\alpha_d \nu_{p,t} + \omega_d \epsilon_{d,t}, \tag{1}$$

$$\nu_{p,t} = \alpha_s \nu_{y,t} + \omega_s \epsilon_{s,t}. \tag{2}$$

This system expresses a downward-sloping demand curve (1) and an upward-sloping (inverse) supply curve (2) of a good. The terms $\nu_{y,t}$ and $\nu_{p,t}$ represent the reduced-form innovations associated with the quantity and price of the good, while $\epsilon_{d,t}$ and $\epsilon_{s,t}$ are structural shocks capturing the demand and supply shocks with the following unconditional scedastic structure: $\sigma_{\epsilon,dd} = E[\epsilon_{d,t}^2] = 1$, $\sigma_{\epsilon,ss} = E[\epsilon_{s,t}^2] = 1$, and $\sigma_{\epsilon,ds} = E[\epsilon_{d,t}\epsilon_{s,t}] = 0$. The positive parameters α_d and α_s are related to the slopes of the demand and supply curves, whereas the positive parameters ω_d and ω_s are related to the shifts of the curves following demand and supply shocks.

System (1)–(2) involves four parameters that have to be identified: α_d, α_s, ω_d, and ω_s. As usual, three of these parameters, say for illustration purposes, α_d, ω_d, and ω_s, can potentially be identified through the distinct elements of the unconditional covariance matrix of the reduced-form innovations: $\sigma_{\nu,yy} = E[\nu_{y,t}^2]$, $\sigma_{\nu,pp} = E[\nu_{p,t}^2]$, and $\sigma_{\nu,yp} = E[\nu_{y,t}\nu_{p,t}]$. Importantly, the remaining parameter, α_s, can potentially be identified through higher unconditional moments, reflecting, for example, asymmetric and non-mesokurtic distributions.

As a starting point, Figures 1 and 2 depict the densities and the scatter plot of simulated shocks for the case where $\epsilon_{d,t}$ and $\epsilon_{s,t}$ are normally distributed. The simulations are generated for the following parametrization of equations (1)–(2): $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $\epsilon_{d,t} \sim N(0,1)$, and $\epsilon_{s,t} \sim N(0,1)$. As expected, $\nu_{y,t}$ and $\nu_{p,t}$ are also normally distributed and the realizations of these innovations form a spherical cloud in the $(\nu_{y,t}, \nu_{p,t})$ plan. In this context, shifts in the demand and supply curves are as likely to generate the realizations of $\nu_{y,t}$ and $\nu_{p,t}$. Consequently, these realizations are not informative about the slope of either of the two curves, so that α_s cannot be identified. In this context, possible identification strategies are to impose one short-run or long-run restriction to identify α_s.

Figures 1 and 2 also show the case where $\epsilon_{d,t}$ follows a mixture of normal distributions: $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326, 1)$ with probability 0.2113, whereas $\epsilon_{s,t} \sim N(0,1)$. The resulting density of $\epsilon_{d,t}$ is mesokurtic as it is characterized by a zero excess kurtosis, as for the normal distribution, but it displays an asymmetry given that it is left skewed.[5] This third moment of $\epsilon_{d,t}$ implies a negative skewness that is more pronounced for $\nu_{y,t}$

---

[5] The unconditional moments of the demand shock are as follows: the expectation is $E[\epsilon_{d,t}] = 0$, the variance is

than for $\nu_{p,t}$. As a result, the scatter plot of $\nu_{y,t}$ and $\nu_{p,t}$ exhibits an elliptical shape along the supply curve. This occurs because large negative values are more often observed for $\epsilon_{d,t}$ (than for $\epsilon_{s,t}$), and this induces substantial leftward shifts of the demand curve (relative to those associated with the supply curve). These shifts of the demand curve imply movements along the supply curve, so that it becomes possible to identify the slope of the supply curve, α_s.

Finally, Figures 3 and 4 display the case where $\epsilon_{d,t}$ follows a Student's t-distribution: $1.291 \times \epsilon_{d,t} \sim t(5)$ and $\epsilon_{s,t} \sim N(0,1)$. For this parametrization, the density of $\epsilon_{d,t}$ is symmetric, similarly to the normal distribution, but it is leptokurtic given that it has fat tails.[6] The fourth moment of $\epsilon_{d,t}$ translates into a large positive excess kurtosis for $\nu_{y,t}$ and a small excess kurtosis for $\nu_{p,t}$. This leads to an elliptical shape along the supply curve, where the extreme realizations of $\epsilon_{d,t}$ (compared to those of $\epsilon_{s,t}$) generate pronounced leftward and rightward shifts of the demand curve (relative to those associated with the supply curve). Again, these shifts of the demand curve imply movements along the supply curve, so the slope parameter α_s is identified.

The examples presented so far highlight that the parameter α_s is identified when the unconditional distribution of the demand shock $\epsilon_{d,t}$ is either asymmetric or non-mesokurtic. Note that α_s is also identified when $\epsilon_{d,t}$ exhibits a time-varying conditional variance. This is because conditional heteroskedasticity typically implies unconditional leptokurtic (even for the case of conditional mesokurtic distributions), and, as discussed above, it is precisely the presence of unconditional non-mesokurtic demand shock that leads to the identification of α_s.

Taken altogether, these examples suggest that exploiting the information of the structural shocks related to higher unconditional moments, such as the third and fourth moments, help to identify additional parameters of a SVAR process (relative to the case where only the second unconditional moments are considered).

$E[\epsilon_{d,t}^2] = 1$, the skewness is $s_{\epsilon,d,dd} = E[\epsilon_{d,t}^3] = -0.9907$, and the excess kurtosis is $\kappa_{\epsilon,dd,dd}^e = E[\epsilon_{d,t}^4] - 3 = 0$. Also, the demand and supply shocks display zero covariance, $E[\epsilon_{d,t}\epsilon_{s,t}] = 0$, coskewnesses, $E[\epsilon_{d,t}^2\epsilon_{s,t}] = E[\epsilon_{d,t}\epsilon_{s,t}^2] = 0$, and excess cokurtoses, $E[\epsilon_{d,t}^3\epsilon_{s,t}] = E[\epsilon_{d,t}\epsilon_{s,t}^3] = 0$ and $E[\epsilon_{d,t}^2\epsilon_{s,t}^2] - 1 = 0$.

[6]Specifically, the unconditional moments of the demand shock are the following: the expectation is $E[\epsilon_{d,t}] = 0$, the variance is $E[\epsilon_{d,t}^2] = 1$, the skewness is $s_{\epsilon,d,dd} = E[\epsilon_{d,t}^3] = 0$, and the excess kurtosis is $\kappa_{\epsilon,dd,dd}^e = E[\epsilon_{d,t}^4] - 3 = 6$. Also, the demand and supply shocks display zero covariance, $E[\epsilon_{d,t}\epsilon_{s,t}] = 0$, coskewnesses, $E[\epsilon_{d,t}^2\epsilon_{s,t}] = E[\epsilon_{d,t}\epsilon_{s,t}^2] = 0$, and excess cokurtoses, $E[\epsilon_{d,t}^3\epsilon_{s,t}] = E[\epsilon_{d,t}\epsilon_{s,t}^3] = 0$ and $E[\epsilon_{d,t}^2\epsilon_{s,t}^2] - 1 = 0$.

3. Identification

In this section, we first present the SVAR specification. We then derive the order and rank conditions of local identification through higher unconditional moments.

3.1 Specification

We consider a structural system that takes the form of the following p-order SVAR process:

$$\Phi x_t = \Phi_0 + \sum_{\tau=1}^{p} \Phi_\tau x_{t-\tau} + \epsilon_t. \tag{3}$$

The $(n \times 1)$ vector x_t includes the variables of interest. The $(n \times 1)$ vector ϵ_t contains the structural shocks. These shocks are assumed to display zero cross-sectional unconditional covariances, coskewnesses, and excess cokurtoses. The $(n \times 1)$ vector Φ_0 incorporates n unrestricted intercepts. The non-singular $(n \times n)$ matrix Φ captures n^2 unrestricted contemporaneous interactions among the variables.[7] The $(n \times n)$ matrix Φ_τ contains n^2 unrestricted dynamic feedbacks between the variables.

The first four unconditional moments of the structural shocks of system (3) are obtained from the following expressions:

$$M_\epsilon = E[\epsilon_t], \tag{4}$$

$$\Sigma_\epsilon = E[\epsilon_t \epsilon_t'], \tag{5}$$

$$S_\epsilon = E[\epsilon_t \epsilon_t' \otimes \epsilon_t'], \tag{6}$$

$$K_\epsilon^e = K_\epsilon - K_{\tilde{\epsilon}} = E[\epsilon_t \epsilon_t' \otimes \epsilon_t' \otimes \epsilon_t'] - E[\tilde{\epsilon}_t \tilde{\epsilon}_t' \otimes \tilde{\epsilon}_t' \otimes \tilde{\epsilon}_t'], \tag{7}$$

where E is the unconditional expectation operator and $\otimes$ denotes the Kronecker product. As is common practice, the $(n \times 1)$ vector of expectations is fixed to $M_\epsilon = [\mu_{\epsilon,i}] = 0$ and the $(n \times n)$ covariance matrix is set to $\Sigma_\epsilon = [\sigma_{\epsilon,ij}] = I$ (for $i, j = 1, \ldots, n$), where the latter expression implies that all covariances are assumed to be null, $\sigma_{\epsilon,ij} = 0$ (for $i \neq j$). Also, the $(n \times n^2)$ coskewness matrix concatenates n symmetric $(n \times n)$ submatrices: $S_\epsilon = [S_{\epsilon,1}, \ldots, S_{\epsilon,n}]$, where $S_{\epsilon,k} = [s_{\epsilon,k,ij}] = E[\epsilon_{k,t} \epsilon_{i,t} \epsilon_{j,t}]$. The n unconstrained skewnesses of the structural shocks may be non-zero, $s_{\epsilon,k,kk} \neq 0$,

---

[7]The assumption of non singularity ensures that there is no redundant variables included in the SVAR process.

whereas all coskewnesses are assumed to be null, $s_{\epsilon,k,ii} = s_{\epsilon,k,ij} = 0$ (for $i, j \neq k$). Finally, the $(n \times n^3)$ excess cokurtosis matrix, K_ϵ^e, is the difference between the cokurtosis matrix, K_ϵ, of the true structural shocks, ϵ_t, and the cokurtosis matrix, $K_{\tilde{\epsilon}}$, associated with hypothetical mesokurtic structural shocks, $\tilde{\epsilon}_t$. The excess cokurtosis matrix stacks n^2 symmetric $(n \times n)$ submatrices: $K_\epsilon^e = [K_{\epsilon,11}^e, \ldots, K_{\epsilon,1n}^e, \ldots, K_{\epsilon,n1}^e, \ldots, K_{\epsilon,nn}^e]$, where $K_{\epsilon,k\ell}^e = [\kappa_{\epsilon,k\ell,ij}^e] = E[\epsilon_{k,t}\epsilon_{\ell,t}\epsilon_{i,t}\epsilon_{j,t}] - E[\tilde{\epsilon}_{k,t}\tilde{\epsilon}_{\ell,t}\tilde{\epsilon}_{i,t}\tilde{\epsilon}_{j,t}]$. The n unconstrained excess kurtoses may be non-zero, $\kappa_{\epsilon,kk,kk}^e \neq 0$, whereas the excess cokurtoses are assumed to be null, $\kappa_{\epsilon,kk,ii}^e = \kappa_{\epsilon,kk,ki}^e = \kappa_{\epsilon,kk,ij}^e = \kappa_{\epsilon,k\ell,ij}^e = 0$.[8]

Next, the reduced form associated with system (3) corresponds to the following p-order VAR process:

$$x_t = \Gamma_0 + \sum_{\tau=1}^{p} \Gamma_\tau x_{t-\tau} + \nu_t, \tag{8}$$

where $\Gamma_0 = \Theta\Phi_0$, $\Gamma_\tau = \Theta\Phi_\tau$, and the non-singular matrix $\Theta = \Phi^{-1}$ captures the impact responses of the variables of interest to the various structural shocks, whereas ν_t includes the reduced-form innovations. These innovations are related to the structural shocks as follows:

$$\nu_t = \Theta\epsilon_t. \tag{9}$$

Also, the first four unconditional moments of the reduced-form innovations are:

$$M_\nu = E[\nu_t], = \Theta M_\epsilon, \tag{10}$$

$$\Sigma_\nu = E[\nu_t\nu_t'] = \Theta\Sigma_\epsilon\Theta', \tag{11}$$

$$S_\nu = E[\nu_t\nu_t' \otimes \nu_t'] = \Theta S_\epsilon(\Theta' \otimes \Theta'), \tag{12}$$

$$K_\nu^e = K_\nu - K_{\tilde{\nu}} = E[\nu_t\nu_t' \otimes \nu_t' \otimes \nu_t'] - E[\tilde{\nu}_t\tilde{\nu}_t' \otimes \tilde{\nu}_t' \otimes \tilde{\nu}_t'] = \Theta K_\epsilon^e(\Theta' \otimes \Theta' \otimes \Theta'). \tag{13}$$

Here, $M_\nu = [\mu_{\nu,i}] = 0$ given that $M_\epsilon = 0$ and $\Sigma_\nu = [\sigma_{\nu,ij}] = \Theta\Theta'$ since $\Sigma_\epsilon = I$. Moreover, $S_\nu = [S_{\nu,1}, \ldots, S_{\nu,n}]$ with $S_{\nu,k} = [s_{\nu,k,ij}]$ and $K_\nu = [K_{v,11}, \ldots, K_{v,1n}, \ldots, K_{v,n1}, \ldots, K_{v,nn}]$ with $K_{v,k\ell} = [\kappa_{v,k\ell,ij}]$ for $v_t = \nu_t, \tilde{\nu}_t$, where ν_t captures the true reduced-form innovations and $\tilde{\nu}_t$ contains hypothetical mesokurtic reduced-form innovations. As is well known, the symmetric matrix Σ_ν

---

[8]As an example, the bivariate system (1)–(2) implies that all the elements of S_ϵ are null, except potentially the (1,1) and (2,4) elements which correspond to the skewnesses of the demand shock, $s_{\epsilon,d,dd}$, and supply shock, $s_{\epsilon,s,ss}$. Also, all the elements of K_ϵ^e are null, with the possible exceptions of the (1,1) and (2,8) elements which capture the excess kurtoses of the demand shock, $\kappa_{\epsilon,dd,dd}^e = (\kappa_{\epsilon,dd,dd} - 3)$, and supply shock, $\kappa_{\epsilon,ss,ss}^e = (\kappa_{\epsilon,ss,ss} - 3)$.

contains $\frac{n(n+1)}{2}$ distinct elements. Furthermore, the matrices S_ν and K_ν^e include $\frac{n(n+1)(n+2)}{6}$ and $\frac{n(n+1)(n+2)(n+3)}{24}$ distinct elements.[9]

3.2 Identification Conditions

We now determine the order and rank conditions for local identification, which establish the conditions for identifying the parameters associated with the structural form (3) from the distinct elements and the rank associated with the reduced form (8). For expositional purposes, these conditions are mainly derived from two cases; the first case exploits the skewness of the structural shocks, whereas the second case focuses on the excess kurtosis of the structural shocks. This presentation accords with our simple illustration highlighting that identification can be achieved when the unconditional distributions of the structural shocks are either asymmetric or non-mesokurtic (see Section 2). For each case, we elaborate the conditions required to identify the impact responses involved in Θ and the skewnesses or excess kurtoses of the structural shocks included in S_ϵ or K_ϵ^e from the unconditional moments of the reduced-form innovations contained in Σ_ν and S_ν or K_ν^e.[10] For completeness, note that, once these parameters are identified, it is trivial to identify the other structural parameters included in Φ_0 and Φ_τ (where $\tau = 1, \ldots, p$) through the relations $\Phi_0 = \Theta^{-1}\Gamma_0$ and $\Phi_\tau = \Theta^{-1}\Gamma_\tau$.

3.2.1 Order Conditions

We denote by η and ρ the number of parameters involved in the structural form and the number of distinct elements in the reduced form. The order conditions are given by $\rho = \eta$ and $\rho > \eta$, which represent necessary conditions for the exact- and over-identification of the entire structural system.

We begin by examining our first case, which exploits the skewness of the structural shocks. On the one hand, the number of parameters in the structural form is $\eta = n^2 + m_s$, given that there are n^2 and m_s parameters to identify in the impact response and skewness matrices, Θ and S_ϵ — where m_s is the number of asymmetric structural shocks. Note that Θ contains $n^2 - (n - n_s)m_s$ non-

---

[9] For example, the bivariate system (1)–(2) implies that Σ_ν incorporates 3 distinct elements, namely $\sigma_{\nu,yy}$, $\sigma_{\nu,yp}$, and $\sigma_{\nu,pp}$. Also, S_ν includes 4 distinct elements: $s_{\nu,y,yy}$, $s_{\nu,y,yp}$, $s_{\nu,y,pp}$, and $s_{\nu,p,pp}$. Finally, K_ν^e involves 5 distinct elements: $\kappa_{\nu,yy,yy}^e$, $\kappa_{\nu,yy,yp}^e$, $\kappa_{\nu,yy,pp}^e$, $\kappa_{\nu,yp,pp}^e$, and $\kappa_{\nu,pp,pp}^e$.

[10] Note that under local identification the matrix Θ is unique up to changes in sign and permutations of columns.

zero parameters and $(n - n_s)m_s$ zero elements — where n_s is the number of skewed reduced-form innovations. To clarify this, we partition relation (9) as:

$$\begin{pmatrix} \nu_{s,t} \\ \nu_{ns,t} \end{pmatrix} = \begin{pmatrix} \Theta_{s,s} & \Theta_{s,ns} \\ \Theta_{ns,s} & \Theta_{ns,ns} \end{pmatrix} \begin{pmatrix} \epsilon_{s,t} \\ \epsilon_{ns,t} \end{pmatrix}, \tag{14}$$

where $\nu_{s,t}$ and $\nu_{ns,t}$ are subvectors that collect, respectively, the n_s and $(n - n_s)$ skewed and non-skewed reduced-form innovations, $\epsilon_{s,t}$ and $\epsilon_{ns,t}$ contain the m_s and $(n - m_s)$ asymmetric and symmetric structural shocks, and $\Theta_{i,j}$ are conformable submatrices of impact responses (for $i, j = s, ns$).[11] We then consider the configuration where $n > n_s \geq m_s$.[12] In this environment, the $(n - n_s)$ reduced-form innovations contained in $\nu_{ns,t}$ are symmetric, as long as they are not related to the m_s skewed structural shocks included in $\epsilon_{s,t}$. This absence of relation occurs when $\Theta_{ns,s} = 0$, which implies that Θ involves $(n - n_s)m_s$ zero elements. For instance, when $n = 2$ and $n_s = m_s = 1$ (where, say, $\epsilon_{s,t} = \epsilon_{1,t}$ is the only asymmetric structural shock so that $s_{\epsilon,1,11} \neq 0$), then the coskewnesses $s_{\nu,2,ij} = (\theta_{21}\theta_{i1}\theta_{j1})s_{\epsilon,1,11}$ (for $i, j = 1, 2$) obtained form (12) are null only if $\Theta_{ns,s} = \theta_{21} = 0$.

On the other hand, the number of distinct elements from the reduced from is $\rho = \left[\frac{n(n+1)}{2}\right] + \left[\frac{n(n+1)(n+2)}{6}\right]$. As already mentioned, there are $\frac{n(n+1)}{2}$ and $\frac{n(n+1)(n+2)}{6}$ distinct elements in Σ_ν and S_ν. Intuitively, the information contained in S_ν helps to identify the parameters in $\Theta_{s,s}$, $\Theta_{ns,s}$, and S_ϵ, whereas Σ_ν helps to identify the parameters in $\Theta_{s,ns}$ and $\Theta_{ns,ns}$. To see this, let's come back to the configuration where $n = 2$ and $n_s = m_s = 1$ (where $\epsilon_{s,t} = \epsilon_{1,t}$ is skewed). In this case, the four distinct elements involved in S_ν — which corresponds to $s_{\nu,1,11} = \theta_{11}^3 s_{\epsilon,1,11} \neq 0$, $s_{\nu,1,12} = \theta_{11}^2 \theta_{21} s_{\epsilon,1,11} = 0$, $s_{\nu,1,22} = \theta_{11}\theta_{21}^2 s_{\epsilon,1,11} = 0$, and $s_{\nu,2,22} = \theta_{21}^3 s_{\epsilon,1,11} = 0$ — lead to the over-identification of the three structural parameters incorporated in $\Theta_{s,s} = \theta_{11} \neq 0$, $\Theta_{ns,s} = \theta_{21} = 0$, and S_ϵ. Also, the three distinct elements in Σ_ν — which are $\sigma_{\nu,11} = \theta_{11}^2 + \theta_{12}^2$, $\sigma_{\nu,12} = \theta_{11}\theta_{21} + \theta_{12}\theta_{22}$, and $\sigma_{\nu,22} = \theta_{21}^2 + \theta_{22}^2$ — yield the over-identification of the remaining two parameters $\Theta_{s,ns} = \theta_{12}$ and $\Theta_{ns,ns} = \theta_{22}$.

We now turn to the second case which focuses on the excess kurtosis of the structural shocks.

---

[11]Note that all the elements included in $\nu_{s,t}$ and $\epsilon_{s,t}$ are asymmetric, but some of these elements may also be non-mesokurtic. Moreover, the elements in $\nu_{ns,t}$ and $\epsilon_{ns,t}$ are symmetric, but possibly non-mesokurtic.

[12]This configuration satisfies the conditions $n \geq n_s \geq m_s$, which ensure that the impact response matrix Θ is non-singular.

In this context, the partitioned representation of relation (9) becomes:

$$\begin{pmatrix} \nu_{\kappa,t} \\ \nu_{n\kappa,t} \end{pmatrix} = \begin{pmatrix} \Theta_{\kappa,\kappa} & \Theta_{\kappa,n\kappa} \\ \Theta_{n\kappa,\kappa} & \Theta_{n\kappa,n\kappa} \end{pmatrix} \begin{pmatrix} \epsilon_{\kappa,t} \\ \epsilon_{n\kappa,t} \end{pmatrix}, \tag{15}$$

where $\nu_{\kappa,t}$ and $\nu_{n\kappa,t}$ are subvectors that collect, respectively, the n_κ and $(n - n_\kappa)$ non-mesokurtic and mesokurtic reduced-form innovations, while $\epsilon_{\kappa,t}$ and $\epsilon_{n\kappa,t}$ contain the m_κ and $(n - m_\kappa)$ non-mesokurtic and mesokurtic structural shocks.[13]

Invoking analogous arguments as those elaborated above implies that $\eta = n^2 + m_\kappa$ — there are n^2 parameters in the impact response matrix Θ, among which $\Theta_{n\kappa,\kappa}$ includes $(n - n_\kappa)m_\kappa$ zero parameters, and m_κ non-zero excess kurtosis in K_ϵ^e. Also, $\rho = \left[\frac{n(n+1)}{2} \right] + \left[\frac{n(n+1)(n+2)(n+3)}{24} \right]$, because there are $\frac{n(n+1)}{2}$ and $\frac{n(n+1)(n+2)(n+3)}{24}$ distinct elements in Σ_ν and K_ν^e. Here, the information contains in K_ν^e helps to identify the parameters in $\Theta_{\kappa,\kappa}$, $\Theta_{n\kappa,\kappa}$ and K_ϵ^e, whereas Σ_ν helps to identify the parameters in $\Theta_{\kappa,n\kappa}$ and $\Theta_{n\kappa,n\kappa}$.

Finally, we present the general case which takes into account both the skewness and excess kurtosis of the structural shocks. To do so, the relation (9) is partitioned according to the characteristics of the reduced-form innovations and structural shocks as following:

$$\begin{pmatrix} \nu_{ss,t} \\ \nu_{\kappa\kappa,t} \\ \nu_{s\kappa,t} \\ \nu_{ns\kappa,t} \end{pmatrix} = \begin{pmatrix} \Theta_{ss,ss} & \Theta_{ss,\kappa\kappa} & \Theta_{ss,s\kappa} & \Theta_{ss,ns\kappa} \\ \Theta_{\kappa\kappa,ss} & \Theta_{\kappa\kappa,\kappa\kappa} & \Theta_{\kappa\kappa,s\kappa} & \Theta_{\kappa\kappa,ns\kappa} \\ \Theta_{s\kappa,ss} & \Theta_{s\kappa,\kappa\kappa} & \Theta_{s\kappa,s\kappa} & \Theta_{s\kappa,ns\kappa} \\ \Theta_{ns\kappa,ss} & \Theta_{ns\kappa,\kappa\kappa} & \Theta_{ns\kappa,s\kappa} & \Theta_{ns\kappa,ns\kappa} \end{pmatrix} \begin{pmatrix} \epsilon_{ss,t} \\ \epsilon_{\kappa\kappa,t} \\ \epsilon_{s\kappa,t} \\ \epsilon_{ns\kappa,t} \end{pmatrix}, \tag{16}$$

or more compactly

$$\nu_t = \begin{pmatrix} \Theta_{ss} & \Theta_{\kappa\kappa} & \Theta_{s\kappa} & \Theta_{ns\kappa} \end{pmatrix} \epsilon_t. \tag{17}$$

Here, $\nu_{ss,t}$, $\nu_{\kappa\kappa,t}$, $\nu_{s\kappa,t}$, and $\nu_{ns\kappa,t}$ are subvectors that collect, respectively, the n_{ss}, $n_{\kappa\kappa}$, $n_{s\kappa}$, and $(n - n_{ss} - n_{\kappa\kappa} - n_{s\kappa})$ reduced-form innovations that are exclusively skewed, only non-mesokurtic, both asymmetric and non-mesokurtic, and both symmetric and mesokurtic. The numbers of skewed and non-mesokurtic reduced-form innovations correspond to $n_s = n_{ss} + n_{s\kappa}$ and $n_\kappa = n_{\kappa\kappa} + n_{s\kappa}$. Likewise, the subvectors $\epsilon_{ss,t}$, $\epsilon_{\kappa\kappa,t}$, $\epsilon_{s\kappa,t}$, and $\epsilon_{ns\kappa,t}$ contain, respectively, the m_{ss}, $m_{\kappa\kappa}$, $m_{s\kappa}$, and $(m - m_{ss} - m_{\kappa\kappa} - m_{s\kappa})$ structural shocks that are exclusively skewed, only non-mesokurtic,

---

[13] All the terms incorporated in $\nu_{\kappa,t}$ and $\epsilon_{\kappa,t}$ are non-mesokurtic, but some of these terms may also be asymmetric. Furthermeore, the terms in $\nu_{n\kappa,t}$ and $\epsilon_{n\kappa,t}$ are mesokurtic, but possibly asymmetric.

both asymmetric and non-mesokurtic, and both symmetric and mesokurtic. The numbers of skewed and non-mesokurtic structural shocks are $m_s = m_{ss} + m_{s\kappa}$ and $m_\kappa = m_{\kappa\kappa} + m_{s\kappa}$. We also define $\Theta_{ss} = \begin{pmatrix} \Theta'_{ss,ss} & \Theta'_{\kappa\kappa,ss} & \Theta'_{s\kappa,ss} & \Theta'_{ns\kappa,ss} \end{pmatrix}'$, $\Theta_{\kappa\kappa} = \begin{pmatrix} \Theta'_{ss,\kappa\kappa} & \Theta'_{\kappa\kappa,\kappa\kappa} & \Theta'_{s\kappa,\kappa\kappa} & \Theta'_{ns\kappa,\kappa\kappa} \end{pmatrix}'$, $\Theta_{s\kappa} = \begin{pmatrix} \Theta'_{ss,s\kappa} & \Theta'_{\kappa\kappa,s\kappa} & \Theta'_{s\kappa,s\kappa} & \Theta'_{ns\kappa,s\kappa} \end{pmatrix}'$, and $\Theta_{ns\kappa} = \begin{pmatrix} \Theta'_{ss,ns\kappa} & \Theta'_{\kappa\kappa,ns\kappa} & \Theta'_{s\kappa,ns\kappa} & \Theta'_{ns\kappa,ns\kappa} \end{pmatrix}'$ — where the elements of the matrices $\Theta_{\kappa\kappa,ss}$, $\Theta_{ns\kappa,ss}$, $\Theta_{ss,\kappa\kappa}$, $\Theta_{ns\kappa,\kappa\kappa}$, $\Theta_{ss,s\kappa}$, $\Theta_{snk,s\kappa}$, $\Theta_{\kappa\kappa,s\kappa}$ and $\Theta_{ns\kappa,s\kappa}$ are equal to zero.[14] In this environment, $\eta = n^2 + [m_s + m_\kappa]$ and $\rho = \left[\frac{n(n+1)}{2} \right] + \left[\frac{n(n+1)(n+2)}{6} \right] + \left[\frac{n(n+1)(n+2)(n+3)}{24} \right]$. Intuitively, S_ν and K_ν^e help to identify the parameters in Θ_{ss}, $\Theta_{\kappa\kappa}$, $\Theta_{s\kappa}$, S_ϵ, and K_ϵ^e, whereas Σ_ν helps to identify the parameters in Θ_{snk}.

3.2.2 Rank Condition

In this section, we formally derive the rank condition and simple formulas which allow practitioners to evaluate easily this rank condition. The rank condition $r = \eta$ represents the sufficient condition for the local identification of the entire structural system, where r corresponds to the rank associated with the unconditional moment matrices of the reduced-form innovations. Extending the developments of Lütkepohl (2007, Chapter 9), we derive this condition from the ranks of the Jacobian matrices associated with the structural parameters to identify.

If it turns out that the entire structural system is not identified, then our approach further allows to establish which structural parameters are identified and which are not. This gives rise to two important implications. First, it permits to assess which structural subsystem is identified. This subsystem documents the effects induced by the asymmetric and/or non-mesokurtic structural shocks. Second, it enables to determine the structural parameters for which some restrictions must be placed on in order to achieve the identification of the entire system. This is required to document the effects of the symmetric and mesokurtic structural shocks. As far as we know, these key implications have never been examined in previous studies.

Again, we first consider the case which exploits the asymmetry of the structural shocks. As

---

[14]System (16) is related to (14) as $\nu_{s,t} = \begin{pmatrix} \nu_{ss,t} & \nu_{s\kappa,t} \end{pmatrix}'$, $\nu_{ns,t} = \begin{pmatrix} \nu_{\kappa\kappa,t} & \nu_{ns\kappa,t} \end{pmatrix}'$, $\epsilon_{s,t} = \begin{pmatrix} \epsilon_{ss,t} & \epsilon_{s\kappa,t} \end{pmatrix}'$, and $\epsilon_{ns,t} = \begin{pmatrix} \epsilon_{\kappa\kappa,t} & \epsilon_{ns\kappa,t} \end{pmatrix}'$. Likewise, system (16) is related to (15) as $\nu_{\kappa,t} = \begin{pmatrix} \nu_{\kappa\kappa,t} & \nu_{s\kappa,t} \end{pmatrix}'$, $\nu_{n\kappa,t} = \begin{pmatrix} \nu_{ss,t} & \nu_{ns\kappa,t} \end{pmatrix}'$, $\epsilon_{\kappa,t} = \begin{pmatrix} \epsilon_{\kappa\kappa,t} & \epsilon_{s\kappa,t} \end{pmatrix}'$, and $\epsilon_{n\kappa,t} = \begin{pmatrix} \epsilon_{ss,t} & \epsilon_{ns\kappa,t} \end{pmatrix}'$. For system (16), note further that the reduced-form innovations involved in $\nu_{ss,t}$ are mesokurtic. It can be shown that this occurs because $\nu_{ss,t}$ is a linear combination of the structural shocks contained in $\epsilon_{ss,t}$ and $\epsilon_{ns\kappa,t}$, where these shocks display zero excess kurtoses and zero excess cokurtoses. Similarly, the statistical innovations collected in $\nu_{\kappa\kappa,t}$ are symmetric.

explained above, the number of parameters involved in the structural form is $\eta = n^2 + m_s$. Also, the rank associated with the reduced form is equal to the rank of the following Jacobian matrix:

$$J = \begin{bmatrix} J_{\theta_s} & J_{\theta_{ns}} & J_{s_\epsilon} \end{bmatrix} = \begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & J_{\sigma_\nu,s_\epsilon} \\ J_{s_\nu,\theta_s} & J_{s_\nu,\theta_{ns}} & J_{s_\nu,s_\epsilon} \end{bmatrix}. \tag{18}$$

Here, $J_{\theta_s} = \begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} \end{bmatrix}'$, $J_{\theta_{ns}} = \begin{bmatrix} J'_{\sigma_\nu,\theta_{ns}} & J'_{s_\nu,\theta_{ns}} \end{bmatrix}'$, $J_{s_\epsilon} = \begin{bmatrix} J'_{\sigma_\nu,s_\epsilon} & J'_{s_\nu,s_\epsilon} \end{bmatrix}'$, and $J_{y,x} = \frac{\partial y}{\partial x'}$. Moreover, the vector σ_ν vectorizes the lower triangular part of the symmetric covariance matrix Σ_ν, and the vector s_ν collects the distinct elements of the coskewness matrix S_ν. Finally, the vector θ_s stacks the columns of the matrix $\Theta_s = \begin{pmatrix} \Theta'_{s,s} & \Theta'_{ns,s} \end{pmatrix}'$ in system (14), the vector θ_{ns} contains the elements of the matrix $\Theta_{ns} = \begin{pmatrix} \Theta'_{s,ns} & \Theta'_{ns,ns} \end{pmatrix}'$, and the vector s_ϵ includes the non-zero elements of the skewness matrix S_ϵ. The analytical derivatives involved in (18) are detailed in the Appendix.

The rank of the Jacobian matrix (18), $r = rk[J]$, can be evaluated from the analytical derivatives. From these derivatives, we deduce simple formulas to evaluate the rank r, which can be easily assessed from the number of variables involved in the system, n, and the number of skewed structural shocks, m_s. Specifically, the rank corresponds to the sum of three components: $r = r_s + r_{ns} + r_{s_\epsilon}$, with $r_s = rk[J_{\theta_s}] = n \times m_s$, $r_{ns} = rk[J_{\theta_{ns}}] = \sum_{i=0}^{n-m_s}(n-i) - m_s$, and $r_{s_\epsilon} = rk[J_{s_\epsilon}] = m_s$.

The components $r_s = n \times m_s$ and $r_{s_\epsilon} = m_s$ reveal that the information contained in the second and third moments of the reduced-form innovations, Σ_ν and S_ν, allows to identify all the $n \times m_s$ elements of the matrix Θ_s relating the reduced-form innovations to the skewed structural shocks, as well as all the m_s non-zero elements of the skewness matrix S_ϵ. The intuition for this result can be gained from the two following features. First, $rk[J_{s_\nu,\theta_s}] = n \times m_s$ and $rk[J_{s_\nu,s_\epsilon}] = m_s$, but $rk\begin{bmatrix} J_{s_\nu,\theta_s} & J_{s_\nu,s_\epsilon} \end{bmatrix} = n \times m_s$. This implies that the coskewness matrix S_ν identifies the elements of Θ_s and S_ϵ jointly, but not separately. To illustrate this, consider the configuration where $n = 2$ and $n_s = m_s = 1$ (where $\epsilon_{s,t} = \epsilon_{1,t}$ is skewed), so that $\Theta_s = \begin{pmatrix} \Theta'_{s,s} & \Theta'_{ns,s} \end{pmatrix}'$ with $\Theta_{s,s} = \theta_{11}$ and $\Theta_{ns,s} = \theta_{21} = 0$. In this context, the four distinct elements involved in S_ν — which corresponds to $s_{\nu,1,11} = \theta_{11}^3 s_{\epsilon,1,11} \neq 0$, $s_{\nu,1,12} = \theta_{11}^2 \theta_{21} s_{\epsilon,1,11} = 0$, $s_{\nu,1,22} = \theta_{11}\theta_{21}^2 s_{\epsilon,1,11} = 0$, and $s_{\nu,2,22} = \theta_{21}^3 s_{\epsilon,1,11} = 0$ — identify the parameters θ_{11}, θ_{21}, and $s_{\epsilon,1,11}$ jointly, but not individually. Second, $J_{\sigma_\nu,\theta_s} \neq 0$ whereas $J_{\sigma_\nu,s_\epsilon} = 0$. This implies that the covariance matrix Σ_ν disentangles the parameters involved in Θ_s from those contained in S_ϵ, so that it becomes possible to identify

14

individually each parameter in Θ_s and S_ϵ. Coming back to the previous example, the three distinct elements in Σ_ν — which are $\sigma_{\nu,11} = \theta_{11}^2 + \theta_{12}^2$, $\sigma_{\nu,12} = \theta_{11}\theta_{21} + \theta_{12}\theta_{22}$, and $\sigma_{\nu,22} = \theta_{21}^2 + \theta_{22}^2$ — disentangle the parameters θ_{11} and θ_{21} from $s_{\epsilon,1,11}$, given that the variances and covariance are related to θ_{11} and θ_{21} but not to $s_{\epsilon,1,11}$.

The component $r_{ns} = \sum_{i=0}^{n-m_s}(n-i) - m_s$ indicates whether the remaining information contained in the second moments of the reduced-form innovations, Σ_ν, allows to identify all the $n \times (n - m_s)$ elements of the matrix Θ_{ns} relating the reduced-form innovations to the symmetric structural shocks. The intuition for this result is obtained from the following features: $J_{s_\nu,\theta_{ns}} = 0$ and $J_{\sigma_\nu,\theta_{ns}} \neq 0$. This implies that only the information captured in Σ_ν, independent of that already used to identify Θ_s, can be exploited to identify the parameters included in Θ_{ns}.

Our findings parallel the existing results. These results highlight that, under the more restrictive assumption of independent structural shocks, all the structural parameters are identified when at least all, but one, structural shocks are non-normally distributed (see Comon, 1994; Eriksson and Koivunen, 2004; Herwartz, 2015; Gouriéroux, Monfort, and Renne, 2017). Our findings state that the entire structural system is identified when at least all, but one, structural shocks are skewed. Specifically, when all structural shocks are asymmetric, $m_s = n$, then all the structural parameters are identified as $\eta = r = n^2 + n$ — where $\eta = n^2 + m_s = n^2 + n$ and $r = r_s + r_{ns} + r_{s_\epsilon}$, with $r_s = n^2$, $r_{ns} = 0$, and $r_{s_\epsilon} = n$. When all, but one, structural shocks are skewed, $m_s = n - 1$, then all the structural parameters are identified as $\eta = r = n^2 + n - 1$, where $r_s = n(n - 1)$, $r_{ns} = n$, and $r_{s_\epsilon} = n - 1$.

Importantly, our approach further provides insights when the entire structural system is not identified. In particular, as already explained above, the moments Σ_ν and S_ν allow to identify the $n \times m_s$ structural parameters included in Θ_s and the m_s distinct elements involved in S_ϵ. Hence, the subsystem relating all the reduced-form innovations to the skewed structural shocks is always identified. This subsystem traces the effects generated by the structural shocks displaying skewness. For example, the impact responses of the variables associated with the skewed reduced-form innovations are given by $\Theta_{s,s}$, whereas those of the variables related to the symmetric reduced-form innovations are equal to $\Theta_{ns,s} = 0$. Hence, if the structural shocks of interest are skewed, then

their effects can be assessed without imposing any restrictions on the structural parameters.

Moreover, the under-identification of the entire structural system occurs when the moments Σ_ν do not permit to identify all the $n \times (n - m_s)$ elements contained in Θ_{ns}. As a result, certain restrictions on these structural parameters must be imposed. For illustration purposes, consider the following (linear) short-run restrictions $R\theta_{ns} = q$. In this context, the rank condition holds when:

$$rk[J^+] = rk \begin{bmatrix} J_{\theta_s}^+ & J_{\theta_{ns}}^+ & J_{s_\epsilon}^+ \end{bmatrix} = rk \begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & J_{\sigma_\nu,s_\epsilon} \\ J_{s_\nu,\theta_s} & J_{s_\nu,\theta_{ns}} & J_{s_\nu,s_\epsilon} \\ 0 & R & 0 \end{bmatrix} = \eta, \qquad (19)$$

where J^+ is the augmented Jacobian matrix, $J_{\theta_s}^+ = \begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} & 0' \end{bmatrix}'$, $J_{\theta_{ns}}^+ = \begin{bmatrix} J'_{\sigma_\nu,\theta_{ns}} & J'_{s_\nu,\theta_{ns}} & R' \end{bmatrix}'$, and $J_{s_\epsilon}^+ = \begin{bmatrix} J'_{\sigma_\nu,s_\epsilon} & J'_{s_\nu,s_\epsilon} & 0' \end{bmatrix}'$. The rank condition (19) states that $(\eta - r)$ linearly independent restrictions on θ_{ns} are needed to identify the entire structural system. Hence, if the structural shocks of interest are symmetric, then their effects can only be gauged when $(\eta - r)$ restrictions are placed on θ_{ns}. In expression (19), the short-run restrictions imply $(\eta - r)$ constraints on the impact responses of the variables to the symmetric structural shocks. It is straightforward to show that relevant long-run restrictions imply $(\eta - r)$ constraints on the dynamic responses (evaluated over an infinite horizon) of the variables to the symmetric shocks.

We next analyze the case which focuses on the excess kurtosis of the structural shocks. Under the short-run restrictions $R\theta_{n\kappa} = q$, the rank condition is verified if:

$$rk[J^+] = rk \begin{bmatrix} J_{\theta_\kappa}^+ & J_{\theta_{n\kappa}}^+ & J_{\kappa_\epsilon^e}^+ \end{bmatrix} = rk \begin{bmatrix} J_{\sigma_\nu,\theta_\kappa} & J_{\sigma_\nu,\theta_{n\kappa}} & J_{\sigma_\nu,\kappa_\epsilon^e} \\ J_{\kappa_\nu^e,\theta_\kappa} & J_{\kappa_\nu^e,\theta_{n\kappa}} & J_{\kappa_\nu^e,\kappa_\epsilon^e} \\ 0 & R & 0 \end{bmatrix} = \eta. \qquad (20)$$

Here, the vectors κ_ν^e and κ_ϵ^e incorporate the distinct elements of the matrices K_ν^e and K_ϵ^e. Also, the vector θ_κ collects the parameters of the matrix $\Theta_\kappa = \begin{pmatrix} \Theta'_{\kappa,\kappa} & \Theta'_{n\kappa,\kappa} \end{pmatrix}'$ in system (15), while the vector $\theta_{n\kappa}$ includes the elements of the matrix $\Theta_{n\kappa} = \begin{pmatrix} \Theta'_{\kappa,n\kappa} & \Theta'_{n\kappa,n\kappa} \end{pmatrix}'$. Again, the analytical derivatives involved in (20) are relegated in the Appendix.

Recall that the number of structural parameters to identify in this case is $\eta = n^2 + m_\kappa$. We first consider the eventuality that no restrictions are placed on the structural parameters, that is $R = 0$. Then, the rank of J^+ corresponds to $r = r_\kappa + r_{n\kappa} + r_{\kappa_\epsilon^e}$ with $r_\kappa = rk[J_{\theta_\kappa}^+] = n \times m_\kappa$, $r_{n\kappa} = rk[J_{\theta_{n\kappa}}^+] = \sum_{i=0}^{n-m_\kappa}(n-i) - m_\kappa$, and $r_{\kappa_\epsilon^e} = rk\left[J_{\kappa_\epsilon^e}^+\right] = m_\kappa$. Consequently, the entire structural system is identified, that is $\eta = r$, when at least all, but one, structural shocks display excess kurtosis.

16

Also, invoking analogous arguments as those developed above reveals that, whether or not $\eta = r$, the subsystem relating all the reduced-form innovations to the non-mesokurtic structural shocks is identified, as the information contained in Σ_ν and K_ν^e always allows to recover the structural parameters involved in Θ_κ and K_ϵ^e. Hence, if the structural shocks of interest display excess kurtosis, then their effects can be documented without imposing any restrictions on the structural parameters.

We now contemplate the eventuality that some restrictions are imposed on the structural parameters ($R \neq 0$). These restrictions are required when the remaining information captured in Σ_ν does not allow to identify all the structural parameters in $\Theta_{n\kappa}$. In this context, the entire structural system is identified when $(\eta - r)$ linearly independent restrictions are imposed on these structural parameters, where the restrictions can take the form of the short-run restrictions $R\theta_{n\kappa} = q$. Thus, if the structural shocks of interest do not exhibit excess kurtosis, then their effects can only be determined when $(\eta - r)$ restrictions are placed on $\Theta_{n\kappa}$.

Finally, we establish a proposition providing the rank condition for the identification of the structural parameters, under short-run restrictions, for the general case where the structural shocks display skewness and/or excess kurtosis.

Proposition 1 *Given the unconditional moments of the reduced-form innovations, Σ_ν, S_ν, and K_ν^e, the system of equations (11)–(13) has a locally unique solution if and only if*

$$
\begin{aligned}
rk[J^+] &= rk\begin{bmatrix} J_{\theta_{ss}}^+ & J_{\theta_{\kappa\kappa}}^+ & J_{\theta_{s\kappa}}^+ & J_{\theta_{ns\kappa}}^+ & J_{s_\epsilon}^+ & J_{\kappa_\epsilon^e}^+ \end{bmatrix} \\
&= rk\begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} & J_{\sigma_\nu,\theta_{ns\kappa}} & J_{\sigma_\nu,s_\epsilon} & J_{\sigma_\nu,\kappa_\epsilon^e} \\ J_{s_\nu,\theta_{ss}} & J_{s_\nu,\theta_{\kappa\kappa}} & J_{s_\nu,\theta_{s\kappa}} & J_{s_\nu,\theta_{ns\kappa}} & J_{s_\nu,s_\epsilon} & J_{s_\nu,\kappa_\epsilon^e} \\ J_{\kappa_\nu^e,\theta_{ss}} & J_{\kappa_\nu^e,\theta_{\kappa\kappa}} & J_{\kappa_\nu^e,\theta_{s\kappa}} & J_{\kappa_\nu^e,\theta_{ns\kappa}} & J_{\kappa_\nu^e,s_\epsilon} & J_{\kappa_\nu^e,\kappa_\epsilon^e} \\ 0 & 0 & 0 & R & 0 & 0 \end{bmatrix} = \eta,
\end{aligned}
\tag{21}
$$

where the vector θ_{ss} stacks by columns the $n \times m_{ss}$ parameters involved in the matrix Θ_{ss} relating the reduced-form innovations to the structural shocks displaying only skewness in system (17), the vector $\theta_{\kappa\kappa}$ contains the $n \times m_{\kappa\kappa}$ parameters of the matrix $\Theta_{\kappa\kappa}$ linking the reduced-form innovations to the structural shocks exhibiting exclusively excess kurtosis, the vector $\theta_{s\kappa}$ includes the $n \times m_{s\kappa}$ parameters of the matrix $\Theta_{s\kappa}$ associating the reduced-form innovations to the structural shocks featuring both skewness and excess kurtosis, $\theta_{ns\kappa}$ incorporates the $n \times [n - (m_{ss} + m_{\kappa\kappa} + m_{s\kappa})]$

parameters of the matrix $\Theta_{ns\kappa}$ relating reduced-form innovations to the structural shocks which are both symmetric and mesokurtic, and the matrix R forms the short-run restrictions $R\theta_{ns\kappa} = q$.

The analytical derivatives involved in (21) are reported in the Appendix. As explained above, the number of structural parameters to identify is $\eta = n^2 + m_s + m_\kappa$. When no restrictions are imposed on the structural parameters ($R = 0$), then $rk\left[J^+\right] = r$ with $r = r_{ss} + r_{\kappa\kappa} + r_{s\kappa} + r_{ns\kappa} + r_{s_\epsilon} + r_{\kappa_\epsilon^e}$, $r_{ss} = rk[J_{\theta_{ss}}^+] = n \times m_{ss}$, $r_{\kappa\kappa} = rk[J_{\theta_{\kappa\kappa}}^+] = n \times m_{\kappa\kappa}$, $r_{s\kappa} = rk[J_{\theta_{\kappa\kappa}}^+] = n \times m_{s\kappa}$, $r_{ns\kappa} = rk[J_{\theta_{ns\kappa}}^+] = \sum_{i=0}^{n-(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})}(n-i) - (m_{ss} + m_{\kappa\kappa} + m_{s\kappa})$, $r_{s_\epsilon} = rk\left[J_{s_\epsilon}^+\right] = m_s$, and $r_{\kappa_\epsilon^e} = rk\left[J_{\kappa_\epsilon^e}^+\right] = m_\kappa$. In this context, **Proposition 1** has two implications. First, the entire structural system is identified, that is $\eta = r$, when at least all, but one, structural shocks exhibit skewness and/or excess kurtosis. Second, whether or not $\eta = r$, the subsystem relating all the reduced-form innovations to the asymmetric and/or non-mesokurtic structural shocks is identified, given that the information contained in Σ_ν, S_ν, and K_ν^e always allows to recover the structural parameters involved in Θ_{ss}, $\Theta_{\kappa\kappa}$, $\Theta_{s\kappa}$, S_ϵ, and K_ϵ^e — that is $[r_{ss} + r_{\kappa\kappa} + r_{s\kappa}] + [r_{s_\epsilon} + r_{\kappa_\epsilon^e}] = [n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa}] + [m_s + m_\kappa]$. When some restrictions are placed on the structural parameters ($R \neq 0$), these restrictions are required if the remaining information captured in Σ_ν does not allow to identify all the structural parameters contained in $\Theta_{ns\kappa}$ — that is $r_{ns\kappa} < n \times [n - (m_{ss} + m_{\kappa\kappa} + m_{s\kappa})]$. In this environment, **Proposition 1** states that the entire structural system becomes identified only if $(\eta - r)$ linearly independent restrictions are imposed on $\Theta_{ns\kappa}$. Overall, these results reveal that the rank condition can be readily evaluated from the number of variables involved in the system, n, and the numbers of asymmetric and/or non-mesokurtic structural shocks, m_{ss}, $m_{\kappa\kappa}$, and $m_{s\kappa}$.

4. Testing Procedure

In this section, we elaborate, for the first time in the literature, a testing procedure to verify the symmetry and excess kurtosis of the structural shocks, prior to the estimation of the SVAR process. Specifically, we develop a tractable procedure to verify whether the order and rank conditions hold by assessing the numbers of asymmetric and/or non-mesokurtic structural shocks. We then outline a bootstrap procedure to improve the small-sample properties of rank tests designed to verify the numbers of structural shocks displaying skewness and/or excess kurtosis.

4.1 Verification of the Order and Rank Conditions

The order and rank conditions for identification are useful as long as they are verified before proceeding to the estimation of the SVAR (3); it is only if they hold that it becomes feasible to estimate all the structural parameters involved in the system. As explained above, the order conditions, $\rho \geq \eta$, and the rank condition, $r = \eta$, can be verified from the numbers of asymmetric and/or non-mesokurtic structural shocks.

However, the structural shocks become measurable only once the SVAR is estimated.[15] To circumvent this problem, we develop a method to test the number of asymmetric and/or non-mesokurtic structural shocks, which relies exclusively on the reduced-form innovations — where the latter can be evaluated from the reduced form (8) before the estimation of the structural form (3). Specifically, the number of skewed structural shocks, m_s, corresponds to the rank of the coskewness matrix of the reduced-form innovations, S_ν. To see this, note that expression (12) implies that $rk[S_\nu] = rk[S_\epsilon]$ given that Θ is a non-singular matrix and, as a result, $(\Theta' \otimes \Theta')$ is a full-rank matrix. Also, $rk[S_\epsilon] = m_s$ because the assumption of zero cross-sectional coskewnesses of the structural shocks implies that the quadratic form of the corresponding skewness matrix is $S_\epsilon S_\epsilon' = diag\left(s_{\epsilon,1,11}^2 \quad \cdots \quad s_{\epsilon,n,nn}^2\right)$, and $s_{\epsilon,i,ii}^2 \neq 0$ only for $i = 1, \ldots, m_s$ when m_s structural shocks are skewed.

Analogously, the number of non-mesokurtic structural shocks, m_κ, is given by the rank of the excess cokurtosis matrix of the reduced-form innovations, K_ν^e. That is, equation (13) implies that $rk[K_\nu^e] = rk[K_\epsilon^e]$ given that Θ is a non singular. Also, $rk[K_\epsilon^e] = m_\kappa$ since the assumption of zero cross-sectional excess cokurtoses of the structural shocks leads to $K_\epsilon^e K_\epsilon^{e\prime} = diag\left(\kappa_{\epsilon,11,11}^{e2} \quad \cdots \quad \kappa_{\epsilon,nn,nn}^{e2}\right)$, and $\kappa_{\epsilon,ii,ii}^{e2} \neq 0$ only for $i = 1, \ldots, m_\kappa$.

---

[15]Empirically, asymmetric (either postive or negative skewness) and leptokurtic behaviors have been extensively documented for stock and bond returns as well as for exchange rates and commodity prices (see for example, Clark, 1973; Boothe and Glassman, 1987; Bekaert and Harvey, 1997; Fujiwara, Körber, and Nagakura, 2013). Likewise, positive excess kurtosis have been detected for several macroeconomic series, including indicators related to the economic activity — e.g. real GDP, the components of the real aggregate expenditure, industrial production, and unemployment — as well as a variety of indices of the cost of living — e.g. GDP deflator and CPI (see for example, Blanchard and Watson, 1986; Kilian, 1998; Bai and Ng, 2005; Lanne, Meitz, and Saikkonen, 2017; Gouriéroux, Montford, and Renne, 2017). Note that the studies just reported highlight the existence of skewness and/or excess kurtosis for the variables of interest or for the reduced-form innovations related to these variables, but never for the structural shocks.

Based on the arguments developed above, we present a proposition to determine the number of structural shocks displaying either skewness, excess kurtosis, or both.

Proposition 2 *Given the unconditional third and fourth moments of the reduced-form innovations, S_ν and K_ν^e, the full rank of the impact matrix Θ and the assumption of zero cross-sectional coskewnesses and excess cokurtoses of the structural shocks imply that the number of asymmetric and/or non-mesokurtic structural shocks, $m_{ss} + m_{kk} + m_{sk}$, is equal to the rank of the matrix $\Psi_\nu = \begin{pmatrix} S_\nu & K_\nu^e \end{pmatrix}$.*

Proposition 2 is obtained as follows. First, equations (12) and (13) are used to highlight that $rk[\Psi_\nu] = rk[\Psi_\epsilon]$ with $\Psi_\nu = \begin{pmatrix} \Theta S_\epsilon(\Theta' \otimes \Theta') & \Theta K_\epsilon^e(\Theta' \otimes \Theta' \otimes \Theta') \end{pmatrix}$ and $\Psi_\epsilon = \begin{pmatrix} S_\epsilon & K_\epsilon^e \end{pmatrix}$, given that Θ is a non-singular matrix. Then, $rk[\Psi_\epsilon] = m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$ because the assumption of zero cross-sectional coskewnesses and excess cokurtoses of the structural shocks leads to $\Psi_\epsilon \Psi_\epsilon' = diag\left(s_{\epsilon,1,11}^2 + \kappa_{\epsilon,11,11}^{e2} \quad \cdots \quad s_{\epsilon,n,nn}^2 + \kappa_{\epsilon,nn,nn}^{e2} \right)$, and $s_{\epsilon,i,ii}^2 + \kappa_{\epsilon,ii,ii}^{e2} \neq 0$ for the m_{ss} structural shocks displaying exclusively skewness, the $m_{\kappa\kappa}$ shocks exhibiting only excess kurtosis, and the $m_{s\kappa}$ shocks featuring both shewness and excess kurtosis.

In summary, the ranks of S_ν, K_ν^e, and Ψ_ν allow to determine m_s, m_κ, and $m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$ before the estimation of the structural form (3). Then, the numbers of structural shocks displaying exclusively skewness, m_{ss}, excess kurtosis, $m_{\kappa\kappa}$, and both, $m_{s\kappa}$, are readily deduced — given that $m_s = m_{ss} + m_{s\kappa}$ and $m_\kappa = m_{\kappa\kappa} + m_{s\kappa}$.[16]

4.2   Bootstrap Procedure

In the rank test to determine m_s, m_κ, or $m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$, we use the following likelihood-ratio (LR) and Wald (W) statistics:[17]

---

[16]Specifically, $m_{s\kappa}$ is determined from $m_{ss} + m_{\kappa\kappa} + m_{s\kappa} = (m_s - m_{s\kappa}) + (m_\kappa - m_{s\kappa}) + m_{s\kappa}$, where $m_{ss} + m_{\kappa\kappa} + m_{s\kappa} = rk[\Psi_\nu]$, $m_s = rk[S_\nu]$, and $m_\kappa = rk[K_\nu^e]$. Then, m_{ss} and $m_{\kappa\kappa}$ are determined from $m_{ss} = m_s - m_{s\kappa}$ and $m_{\kappa\kappa} = m_\kappa - m_{s\kappa}$.

[17]See Anderson (1951) and Robin and Smith (2000).

$$\widehat{CRT}_{r^*}^{LR} \quad = \quad (T-p) \sum_{i=r^*+1}^{n} \ln(1 + \hat{\lambda}_i), \tag{22}$$

$$\widehat{CRT}_{r^*}^{W} \quad = \quad (T-p) \sum_{i=r^*+1}^{n} \hat{\lambda}_i, \tag{23}$$

where $\hat{\lambda}_i$ are the estimates of the eigenvalues of the quadratic form of the matrix S_u, K_u^e, or Ψ_u (with $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_n \geq 0$) and r^* is the rank of this matrix under the null hypothesis. The matrices S_u, K_u^e, and Ψ_u are constructed from the sample estimates of the coskweness $\hat{s}_{u,k,ij} = \frac{1}{T-p} \sum_{t=p+1}^{T} \hat{u}_{k,t} \hat{u}_{i,t} \hat{u}_{j,t}$ and cokurtosis $\hat{\kappa}_{u,k\ell,ij} = \frac{1}{T-p} \sum_{t=p+1}^{T} \hat{u}_{k,t} \hat{u}_{\ell,t} \hat{u}_{i,t} \hat{u}_{j,t}$ of the estimated normalized reduced-form innovations, as well as the cokurtoses $\kappa_{\tilde{u},kk,kk} = 3$, $\kappa_{\tilde{u},kk,ii} = \sigma_{\tilde{u},kk} \sigma_{\tilde{u},ii} = 1$ (for $i \neq k$), and $\kappa_{\tilde{u},kk,ki} = \kappa_{\tilde{u},kk,ij} = \kappa_{\tilde{u},k\ell,ij} = 0$ (for $\ell, i, j \neq k$) of hypothetical normal reduced-form innovations. Moreover, the estimate of the normalized reduced-form innovations corresponds to $\hat{u}_t = \hat{\Omega}^{-1} \hat{\nu}_t$, where $\hat{\nu}_t$ represents the OLS residuals of the reduced form (8) and $\hat{\Omega}$ is a lower triangular matrix obtained from the Cholesky decompostion of the estimated covariance matrix of the OLS residuals; i.e. $\hat{\Sigma}_\nu = \hat{\Omega}\hat{\Omega}'$.[18] Robin and Smith (2000) show that, under some regularity conditions, the statistics (22) and (23) have limiting distributions that are weighted sums of independent chi-squared variables, despite that the estimators of $vec(S_u)$, $vec(K_u^e)$, and $vec(\Psi_u)$ have not full rank asymptotic covariance matrices.[19] These distributions are used to find the asymptotic critical values for the statistics $CRT_{r^*}^{LR}$ and $CRT_{r^*}^{W}$ under the null hypothesis that the rank is r^*.

From analytical approximations of the first four moments, it can be shown that $\hat{s}_{u,i,ii}$ has a symmetric leptokurtic distribution which fairly rapidly tends to a normal distribution as the sample size increases, but $\hat{\kappa}_{u,ii,ii}$ has a very skewed distribution that hardly converges to a normal distribution (see Mardia, 1980). This implies that the finite-sample critical values to test the null hypothesis of no excess kurtosis converge extremely slowly to their asymptotic counterparts. Numerical simulations of the Jarque-Bera tests for kurtosis further suggest that the use of asymptotic critical values leads to severe size distortions, as the empirical size often substantially deviates from

---

[18] Note that $rk[S_u] = rk[S_\nu] = m_s$, $rk[K_u^e] = rk[K_\nu^e] = m_\kappa$, and $rk[\Psi_u] = rk[\Psi_\nu] = m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$ given that $S_u = \Omega^{-1} S_\nu(\Omega^{-1'} \otimes \Omega^{-1'})$ and $K_u^e = \Omega^{-1} K_\nu^e(\Omega^{-1'} \otimes \Omega^{-1'} \otimes \Omega^{-1'})$, where $\nu_t = \Omega u_t$.

[19] Note that most rank tests require non-singular asymptotic covariance matrices (see Camba-Méndez and Kapetanios, 2008).

the nominal size even for samples as large as $T = 5,000$ (see Kilian and Demiroglu, 2000; Bai and Ng, 2005).

To circumvent this problem, Kilian and Demiroglu (2000) develop a bootstrap procedure to compute finite-sample critical values of the Jarque-Bera tests for symmetry and kurtosis. Monte Carlo analyses highlight that the tests are virtually free of size distortions when the critical values are computed from the bootstrap procedure, even for samples as small as $T = 125$.

In this vein, we design a bootstrap procedure to compute the finite-sample critical values for the statistics CRT_{r*}^{LR} and CRT_{r*}^{W}. We illustrate the various steps of the procedure by focusing on the rank of S_u in order to determine m_s.

Step 1. Under the null hypothesis $rk[S_u] = r^*$ (i.e. r^* is the assumed number of asymmetric structural shocks), the vector $u_t^* = \begin{pmatrix} u_{r*,t}^{*'} & u_{n-r*,t}^{*'} \end{pmatrix}'$ is generated as follows. The elements contained in the $(r^* \times 1)$ subvector $u_{r*,t}^*$ are obtained by bootstraping those included in the vector $w_{r*,t} = C_{r*}' \hat{u}_t$ for $t = (p+1), \ldots, T$, where C_{r*} is a $(n \times r^*)$ matrix stacking the eigenvectors associated with the r^* largest eigenvalues of the quadratic form $S_u \, S_u'$ and $\hat{u}_t$ is the $(n \times 1)$ vector collecting the estimated normalized reduced-form innovations. This implies that the elements contained in $w_{r*,t}$ correspond to linear combinations of the normalized reduced-form innovations which are the most asymmetric. The elements contained in the $[(n - r^*) \times 1]$ subvector $u_{n-r*,t}^*$ are drawned from the symmetric and mesokurtic distribution $u_{n-r*,t}^* \sim N(0, I)$ for $t = (p+1), \ldots, T$.

Step 2. The bootstrap sample is generated recursively from the VAR process (8) as:

$$x_t^* = \hat{\Gamma}_0 + \sum_{\tau=1}^{p} \hat{\Gamma}_\tau x_{t-\tau}^* + \hat{\Omega} u_t^*, \tag{24}$$

for $t = (p+1), \ldots, T$. To do so, the starting values of x_t^* for $t = 1, \ldots, p$ are generated by randomly drawing a block of the actual data of length p, while $\hat{\Gamma}_0$, $\hat{\Gamma}_\tau$, and $\hat{\Omega}$ are the estimates of the reduced-form parameters obtained by applying OLS on the actual sample. Following Bose (1988), these estimates are treated as the population values of the reduced-form parameters.

Step 3. The VAR process is estimated to yield:

$$x_t^* = \hat{\Gamma}_0^* + \sum_{\tau=1}^{p} \hat{\Gamma}_\tau^* x_{t-\tau}^* + \hat{\Omega}^* \hat{u}_t^*, \tag{25}$$

22

where $\hat{\Gamma}_0^*$, $\hat{\Gamma}_\tau^*$, and $\hat{\Omega}^*$ are the estimates obtained by performing OLS on the bootstrap sample, whereas $\hat{u}_t^*$ corresponds to the normalized residuals.

Step 4. The normalized residuals $\hat{u}_t^*$ are used to compute the bootstrap analogues of the statistics (22) and (23).

Step 5. Steps 1 to 4 are repeated $2,000$ times to compute the empirical distributions of the statistics (22) and (23). Selecting the appropriate quantiles of these empirical distibutions yield the finite-sample critical values to test the null hypothesis that the rank is equal to r^* against the alternative hupothesis that the rank is larger than r^*.

Step 6. Steps 1 to 5 are repeated for $r^* = 0, 1, \ldots, n - 1$. If the null hypothesis $rk[S_u] = r^*$ is rejected for $r^* = 0, 1, \ldots, m - 1$ but is not rejected for $r^* = m$ with $m < n$, then the number of skewed structural shocks corresponds to $m_s = m$. However, if the null hypothesis $rk[S_u] = r^*$ is rejected for $r^* = 0, 1, \ldots, n - 1$, then $m_s = n$.[20]

This bootstrap procedure can also be performed to compute the finite-sample critical values for the ranks of K_u^e and Ψ_u to determine m_κ and $m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$.

To document the possible size distortions of rank tests with asymptotic and finite-sample distributions, we focus on the ranks of S_u and K_u^e to deduce m_s and m_κ. The empirical sizes of these rank tests are evaluated by simulating $10,000$ samples of size T from the bivariate system (1)–(2), where the supply shock follows a normal distribution whereas the demand shock is either generated by a normal distribution under the null hypothesis $r^* = 0$ or by non-normal distributions (i.e. a mixture of normal distributions when the shock is asymmetric and a Student's t-distribution when the shock is non-mesokurtic) under $r^* = 1$.

Table 1 presents the empirical sizes of the Wald and likelihood-ratio versions of the rank tests for symmetry with asymptotic distributions, where the limiting critical values are computed as in Robin and Smith (2000). For the Wald test, the results indicate the existence of a mild size

---

[20]The last step of the bootstrap procedure is similar to the sequential procedure proposed by Robin and Smith (2000). These authors show that, asymptotically, such a sequential procedure never selects a value of r^* that is smaller than the true rank of the matrix of interest. Admittedly, in such a sequential procedure there exists a probability, corresponding to the empirical size, to falsely reject the null hypothesis, as is common to usual testing procedures. Moreover, even for a sequential procedure providing a consistent estimate of the rank, Leeb and Pötscher (2005) show that the finite sample distribution for the subsequent inferences may not be well approximated by the pointwise asymptotic. However, this corresponds to the worst possible outcome when conducting inference, not the likely outcome (see Killian and Lutkepohl, 2017, Chapter 2).

distortion when $T = 100$, but this distortion quickly vanishes as T increases. For example, the empirical sizes of 3.92 percent under the null hypothesis $r^* = 0$ and 5.79 percent under $r^* = 1$ reported for $T = 100$ become almost equal to the nominal size of 5 percent when $T \geq 200$. For the likelihood-ratio test, however, the size distortion documented for small samples is more important than that reported for the Wald test, and such distortion is still observed for samples as large as $T = 1,000$. Table 2 shows the empirical sizes of the Wald and likelihood-ratio versions of the rank tests for kurtosis with asymptotic distributions. For both the Wald and LR tests, the size distortions are severe under the null hypotheses $r^* = 0$ and $r^* = 1$. Specifically, the empirical sizes are systematically close to zero, and, as such, they are substantially smaller than the nominal sizes even for samples as large as $T = 5,000$.

Tables 3 and 4 display the empirical sizes related to the rank tests for symmetry and kurtosis with finite-sample distributions, where the critical values are constructed from the bootstrap procedure developed above. For symmetry, both the Wald and likelihood-ratio tests are essentially free of size distortions; the empirical sizes are very close to the nominal sizes for all T. Note that, although the empirical sizes of the Wald tests with finite-sample and asymptotic distributions are similar, the empirical sizes of the likelihood-ratio test with finite-sample distributions deviate markedly from those obtained from the asymptotic counterpart. For kurtosis, the empirical sizes of the Wald and likelihood-ratio tests with finite-sample distributions are almost identical to the nominal sizes, regardless of the sample size T. Importantly, these findings are strikingly different than those reported for tests with asymptotic distributions.

Finally, we document the empirical powers of the tests with finite-sample distributions for the ranks of S_u and K_u^e. For this purpose, we simulate the bivariate system (1)–(2) for the cases in which i) the supply shock is symmetric and the demand shock is skewed when we consider the null hypothesis $r^* = 0$, and ii) the supply and demand shocks are both asymmetric when we contemplate that $r^* = 1$ — where the skewness is moderate ($s_\epsilon = -0.5253$) or pronounced ($s_\epsilon = -0.9907$). Likewise, we perform simulations for cases in which i) the supply shock is mesokurtic and the demand shock displays excess kurtosis under when we consider the null hypothesis $r^* = 0$, and ii) the supply and demand shocks are both non-mesokurtic when we contemplate that $r^* = 1$ — where

the excess kurtosis is moderate ($\kappa_\epsilon^e = 1$) or pronounced ($\kappa_\epsilon^e = 6$).

Tables 5 and 6 highlight two main features. First, as expected, the powers of the tests substantially improve as the sample size increases. For example, for the cases of moderate skewnesses and excess kurtoses, the Wald and likelihood-ratio tests for skewness (kurtosis) correctly reject, at the 5 percent level, the null hypothesis that $r^* = 0$ about 11 percent (22 percent) of the time when $T = 100$, and almost 98 percent (90 percent) of the time when $T = 1,000$. Second, the powers of the tests considerably increase as the skewnesses and excess kurtoses become more pronounced. For instance, considering a sample size of $T = 200$, the Wald and likelihood-ratio tests for skewness (kurtosis) correctly reject, at the 5 percent level, the null hypothesis that $r^* = 0$ around 26 percent (35 percent) of the time when $s_\epsilon = -0.5253$ ($\kappa_\epsilon^e = 1$), and about 96 percent (80 percent) of the time when $s_\epsilon = -0.9907$ ($\kappa_\epsilon^e = 6$).

Overall, our bootstrap procedure for rank tests always overcomes size distortions and often yields good power properties.[21] Consequently, this procedure is most useful to determine the numbers of asymmetric and/or non-mesokurtic structural shocks, in order to assess whether the order and rank conditions hold.

5. Application

We now apply the developments presented above to document the effects of fiscal policies on economic activity. The effectiveness of fiscal policies represents a classical question in macroeconomics. Also, it has received renewed interest in light of the recent Great Recession and the ongoing debate about which type of government interventions stimulate the most the economy.

We consider a trivariate SVAR process implying the relation:

$$\begin{pmatrix} \nu_{\tau,t} \\ \nu_{g,t} \\ \nu_{y,t} \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{pmatrix}, \tag{26}$$

where $\nu_{\tau,t}$, $\nu_{g,t}$, $\nu_{y,t}$ represent the reduced-form innovations capturing the unanticipated movements

---

[21]Note that when the sample size is small and/or the structural shocks exhibit negligible skewnesses and excess kurtoses, then the powers of rank tests decrease. This leads to conservative analyses: it is likely that an analyst would conclude that the entire system is under-identified (even if it is actually identified) or would under-evaluate the size of the subsystem that is identified (when the entire system is actually under-identified).

in taxes, government spending, and output, whereas $\epsilon_{1,t}$, $\epsilon_{2,t}$, and $\epsilon_{3,t}$ correspond to the structural shocks.

The relation (26) is evaluated for quarterly U.S. data from 1980-I to 2015-III.[22] Output corresponds to the logarithm of real GDP per capita, taxes are defined as the logarithm of real total government receipts net of transfer payments per capita, and government spending is the logarithm of the sum of real consumption and gross investment expenditures per capita. The series are expressed in real terms using the GDP deflator and in per capita terms using total population. Also, taxes and government spending are measured for the general government, i.e. the sum of federal (defense and non-defense), state, and local governments.[23]

As explained previously, it is crucial to verify whether the identification conditions hold before proceeding to the estimation of the structural parameters. To do so, we apply the rank tests where the finite-sample critical values are computed by the bootstrap procedure discussed in Section 4.2.[24] The results reveal that the hypothesis stipulating that the structural shocks are symmetric is not rejected (at all conventional levels) and only one shock is non-mesokurtic (i.e. $m_{ss} = m_{s\kappa} = 0$ and $m_{\kappa\kappa} = 1$), given that the likelihood-ratio and Wald versions of the tests imply that $rk[S_u] = m_s = 0$, $rk[K_u^e] = m_\kappa = 1$, and $rk[\Psi_u] = m_{ss} + m_{\kappa\kappa} + m_{s\kappa} = 1$. In this context, the number of structural parameters is $\eta = n^2 + m_\kappa = 10$, whereas the number of distinct elements in the reduced form is $\rho = \left[\frac{n(n+1)}{2}\right] + \left[\frac{n(n+1)(n+2)(n+3)}{24}\right] = 21$ and the rank associated with the reduced form is $r = r_\kappa + r_{n\kappa} + r_{\kappa_\epsilon^e} = 9$ — with $r_\kappa = n \times m_\kappa = 3$, $r_{n\kappa} = \sum_{i=0}^{n-m_k}(n-i) - m_\kappa = 5$, and $r_{\kappa_\epsilon^e} = m_\kappa = 1$. This implies that, although the order (necessary) conditions $\rho \geq \eta$ are satisfied, the rank (sufficient) condition $r = \eta$ is violated so that the entire system is not identified.

Empirically, the estimates of the parameters θ_{11}, θ_{21}, and θ_{13} of system (26) are 0.0916, 0.0001, and 0.0002.[25] The estimate of θ_{11} is statistically different from zero, whereas the hypotheses that the

---

[22] A similar starting date of the sample is selected by Perotti (2004), Favero and Giavazzi (2009), and Bouakez, Chihi, and Normandin (2014).

[23] The data are seasonally adjusted at the source and are taken from the National Income and Products Accounts (NIPA), except for total population which is obtained from the Federal Reserve Bank of Saint-Louis' FRED database.

[24] The reduced-form innovations are measured by the OLS residuals of (8). This reduced form includes a linear deterministic trend and eight lags, which correspond to the most parsimonious lag structure for which all reduced-form residuals are serially uncorrelated.

[25] The 10 parameters (θ_{ij} for $i,j = 1,2,3$ and $\kappa_{\epsilon,11,11}^e$) involved in system (26) are estimated from the 10 following unconditional moments: $\sigma_{\nu,\tau\tau}$, $\sigma_{\nu,\tau g}$, $\sigma_{\nu,\tau y}$, $\sigma_{\nu,gg}$, $\sigma_{\nu,gy}$, $\sigma_{\nu,yy}$, $\kappa_{\nu,\tau\tau,\tau\tau}^e$, $\kappa_{\nu,\tau\tau,\tau g}^e$, $\kappa_{\nu,\tau\tau,\tau y}^e$, and $\kappa_{\nu,\tau\tau,gy}^e$ — where the moments are evaluated by the sample estimates from the reduced-form residuals. The confidence intervals of the

true values of θ_{21} and θ_{31} are zero cannot be rejected (at all conventional levels). Note that the true values $\theta_{21} = \theta_{31} = 0$ mean that $\nu_{\tau,t}$ exhibits excess kurtosis, while $\nu_{g,t}$ and $\nu_{y,t}$ are mesokurtic.[26] Also, the true values imply that, at impact, the structural shock $\epsilon_{1,t}$ only affects taxes, so that this shock can be interpreted economically as a tax shock.

The results lead to the important implication that the subsytem relating all the reduced-form innovations to the tax shock is identified. Consequently, the responses of output, taxes, and government spending following a tax shock can be evaluated without imposing any restrictions on the structural parameters. Empirically, the response of output suggest that the effectiveness of the tax policy is weak. That is, the tax multiplier (i.e. the dollar change in output occuring in quarter $t+i$ resulting from a dollar cut in the exogenous component of taxes) is small; it is -0.01 at impact and it reaches a peak of 0.59 at 14 quarters (see Table 7).

In contrast, the subsytem relating the reduced-form innovations to the structural shocks $\epsilon_{2,t}$ and $\epsilon_{3,t}$ is under-identified. To achieve the idenfication of this subsystem, $(\eta - r) = 1$ restriction must be imposed. This restriction is required to assess the responses of output, taxes, and government spending following the structural shocks $\epsilon_{2,t}$ and $\epsilon_{3,t}$, where one of these shocks may correspond to the government spending shock.

To deepen the analysis of the effectiveness of the spending policy, we rely on the specification invoked in the seminal paper of Blanchard and Perotti (2002):

$$\nu_{\tau,t} = \alpha_1 \nu_{y,t} + \alpha_2 \omega_g \epsilon_{g,t} + \omega_\tau \epsilon_{\tau,t}, \tag{27}$$

$$\nu_{g,t} = \beta_1 \nu_{y,t} + \beta_2 \omega_\tau \epsilon_{\tau,t} + \omega_g \epsilon_{g,t}, \tag{28}$$

$$\nu_{y,t} = \gamma_1 \nu_{\tau,t} + \gamma_2 \nu_{g,t} + \omega_y \epsilon_{y,t}. \tag{29}$$

The structural shocks $\epsilon_{\tau,t}$ and $\epsilon_{g,t}$ represent the tax and spending shocks that reflect unexpected, exogenous, discretionary changes in taxes and government expenditures, whereas $\epsilon_{y,t}$ captures the non-fiscal shocks that affect output. Equations (27) and (28) describe the government's tax and

_____

estimated parameters are computed from 5,000 bootstrap samples.

[26] These results are confirmed by applying Jarque-Bera tests for the reduced-form innovations, where the finite-sample critical values are approximated by Kilian and Demiroglu's (2000) bootstrap procedure. Specifically, we find that the hypothesis of symmetry is not rejected for all reduced-form innovations, whereas the hypothesis of zero excess kurtosis is rejected only for the reduced-form innovation associated with taxes, $\nu_{\tau,t}$. This implies that $n_{ss} = n_{s\kappa} = 0$ and $n_{\kappa\kappa} = 1$.

spending rules. Specifically, the rule (27) highlights that taxes may vary in response to changes in output or to spending shocks. The rule (28) has an analogous interpretation for public spending. In these rules, the parameters α_1 and β_1 potentially measure the automatic and government's systematic responses of taxes and government spending to changes in output, whereas α_2 and β_2 allow for interactions between tax and spending policies. Equation (29) relates changes in output to changes in taxes and government expenditures, and to non-fiscal shocks. Finally, the terms ω_τ, ω_g, and ω_y are scaling parameters.

The specification (27)–(29) can be expressed in the form of relation (26) as:

$$
\begin{pmatrix} \nu_{\tau,t} \\ \nu_{g,t} \\ \nu_{y,t} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} (1 + \alpha_1\beta_2\gamma_2 - \beta_1\gamma_2)\omega_\tau & (\alpha_2 + \alpha_1\gamma_2 - \alpha_2\beta_1\gamma_2)\omega_g & \alpha_1\omega_y \\ (\beta_2 + \beta_1\gamma_1 - \alpha_1\beta_2\gamma_1)\omega_\tau & (1 + \alpha_2\beta_1\gamma_1 - \alpha_1\gamma_1)\omega_g & \beta_1\omega_y \\ (\gamma_1 + \beta_2\gamma_2)\omega_\tau & (\alpha_2\gamma_1 + \gamma_2)\omega_g & \omega_y \end{pmatrix} \begin{pmatrix} \epsilon_{\tau,t} \\ \epsilon_{g,t} \\ \epsilon_{y,t} \end{pmatrix}, \qquad (30)
$$

where $\Delta = (1 - \alpha_1\gamma_1 - \beta_1\gamma_2)$. Here, the element θ_{ij} of the marix (26) corresponds to the (i,j) element of the matrix in (30) divided by Δ, whereas $\epsilon_{1,t} = \epsilon_{\tau,t}$, $\epsilon_{2,t} = \epsilon_{g,t}$, and $\epsilon_{3,t} = \epsilon_{y,t}$.[27]

Blanchard and Perotti (2002) elaborate two sets of identifying restrictions. The first set fixes $\alpha_2 = 0$ such that taxes do not vary following a spending shock. It also calibrates $\alpha_1 = 2.08$ and $\beta_1 = 0$ using institutional information about tax and transfer systems, where such information allows to measure automatic adjustments of taxes and public spending rather than the government's systematic responses to fluctuations in output (see Blanchard and Perotti, 2002). In principle, this identification strategy may allow for $\theta_{21} \neq 0$ and $\theta_{31} \neq 0$. In practice, however, this apparent flexibility is illusive in the sense that the estimates of β_2 and γ_1 are close to zero to recover the true values $\theta_{21} = \theta_{31} = 0$.[28] Moreover, the misleading case $\theta_{21} \neq 0$ and $\theta_{31} \neq 0$ forces to impose three restrictions: two restrictions are required to compensate for the 'pseudo' deviations of θ_{21} and θ_{31} from zero (despite that the true values are $\theta_{21} = \theta_{31} = 0$) and, as stated above, one restriction is needed to identify the subsystem linking the reduced-form innovations to the structural shocks $\epsilon_{g,t}$ and $\epsilon_{y,t}$. Here, the three restrictions, implying that $\theta_{12} = \alpha_1\theta_{32}$, $\theta_{13} = \alpha_1\theta_{33}$, and $\theta_{23} = 0$, are placed on the subsystem relating the reduced-form innovations to the structural shocks $\epsilon_{g,t}$ and $\epsilon_{y,t}$, so that it becomes over-identified.

---

[27]Recall that the identified subsystem implies that the structural shock $\epsilon_{1,t}$ corresponds to the tax shock $\epsilon_{\tau,t}$. Also, the shocks $\epsilon_{2,t}$ and $\epsilon_{3,t}$ are ordered such that they can be interpreted as a spending shock $\epsilon_{g,t}$ and a non-fiscal shock $\epsilon_{y,t}$.

[28]These estimates are available upon request.

The second set of identifying restrictions invoked by Blanchard and Perotti (2002) imposes $\beta_2 = 0$ so that government spending is not affected by tax shocks, as well as $\alpha_1 = 2.08$ and $\beta_1 = 0$ to capture only automatic adjustments. Again, this identification strategy allows for $\theta_{31} \neq 0$. However, the estimate of γ_1 is negligible to yield the true value $\theta_{31} = 0$.[29] Moreover, two restrictions, namely $\theta_{13} = \alpha_1 \theta_{33}$ and $\theta_{23} = 0$, lead to the over-identification of the subsystem allowing to trace the responses of the variables to a spending shock.

Empirically, we place only one of the restrictions $\theta_{12} = \alpha_1 \theta_{32}$, $\theta_{13} = \alpha_1 \theta_{33}$, or $\theta_{23} = 0$ at a time, so that the subsystem linking the reduced-form innovations to the structural shocks $\epsilon_{g,t}$ and $\epsilon_{y,t}$ is just identified. This exercise reveals that the evaluation of the effectiveness of the spending policy represents a challenging task. That is, the spending multiplier (i.e. the dollar change in output occuring in quarter $t+i$ resulting from a dollar increase in the exogenous component of government spending) highly depends on the nature of the restriction; it is between 0.73 and 2.00 at impact, and it reaches a peak that ranges between 0.73 and 2.91 (see Table 7).

6. Conclusion

In this paper, we first derived the local identification conditions of SVAR processes through higher unconditional moments. These conditions are solely related to the numbers structural shocks that display skewness and/or excess kurtosis. Furthermore, these conditions establish which structural parameters are identified and which are not. For practitioners, this yields useful guidances about which structural parameters need to be restricted to achieve the identification of the entire system.

We then developed a tractable procedure to verify whether a SVAR process is identified, prior to the estimation of the structural parameters. In particular, the numbers of structural shocks exhibiting skewness and excess kurtosis correspond to the ranks of the third and fourth unconditional moment matrices of the reduced-form innovations. A bootstrap procedure is designed to improve the small-sample properties of these rank tests. The bootstrap version of the tests are virtually free of size distortions, whereas existing tests with asymptotic distributions suffer from severe size distortions even for large samples.

---

[29]This estimate is available upon request.

Conceptually, the validity of economic and statistical restrictions that are commonly placed on the entire structural system or subsystem that is identified through higher unconditional moments could be tested. However, there exists a possibility that some of the parameters are weakly identified. This calls for the development of tests (including overidentification tests), where these tests could be robust to weakly identified systems. We leave this for future research.

7. Appendix

This Appendix details the analytical partial derivatives involved in the Jacobians matrices (19), (20), and (21). First, the partial derivatives of the second unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$J_{\sigma_\nu,\theta_i} = 2D_\sigma^+(\Theta \otimes I_n)\Upsilon_{\theta_i},$$

$$J_{\sigma_\nu,s_\epsilon} = 0,$$

$$J_{\sigma_\nu,\kappa_\epsilon^e} = 0,$$

where $i = s, ns$ in (19), $i = \kappa, n\kappa$ in (20), and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (21). The vectorization of the distinct elements of the second moments yields $\sigma_\nu = D_\sigma^+ vec(\Sigma_\nu)$, where $\sigma_\nu = vech(\Sigma_\nu)$, $D_\sigma^+ = (D_\sigma' D_\sigma)^{-1} D_\sigma'$, and D_σ is the $\left(n^2 \times \frac{n(n+1)}{2}\right)$ duplication matrix such that $D_\sigma \sigma_\nu = vec(\Sigma_\nu)$. Using this vectorization, we obtain $\frac{\partial \sigma_\nu}{\partial \theta_i'} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(\Theta)'} \frac{\partial vec(\Theta)}{\partial \theta_i'}$. Equation (11) leads to $vec(\Sigma_\nu) = (\Theta \otimes \Theta)vec(I_n)$, so that $\frac{\partial vec(\Sigma_\nu)}{\partial vec(\Theta)'} = 2(\Theta \otimes I_n)$ (see Lütkepohl, 2007, p. 363). Also, $\frac{\partial vec(\Theta)}{\partial \theta_i'} = \Upsilon_{\theta_i}$ is a matrix containing the values one and zero such that only the partial derivatives with respect to the elements of the vector θ_i are selected. As an example, consider the relation (14) with $n = 2$ and $m_s = 1$ (where the asymmetric reduced-form innovation and structural shock are ordered first), then the $(n^2 \times nm_s)$ selection matrix corresponds to $\Upsilon_{\theta_s} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}'$ and $\theta_s = vec(\Theta_s)$. Moreover, $\frac{\partial \sigma_\nu}{\partial s_\epsilon'} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(S_\epsilon)'} \frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(\Sigma_\nu)}{\partial vec(S_\epsilon)'} = 0$ given that Σ_ν is not a function of the skewnesses of the structural shocks. Likewise, $\frac{\partial \sigma_\nu}{\partial \kappa_\epsilon^{e'}} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(K_\epsilon^e)'} \frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e'}}$ with $\frac{\partial vec(\Sigma_\nu)}{\partial vec(K_\epsilon^e)'} = 0$.

Next, the partial derivatives of the third unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$J_{s_\nu,\theta_i} = D_s^+\{(I_{n^2} \otimes \Theta S_\epsilon)[(I_n \otimes C_{n,n} \otimes I_n)][(I_{n^2} \otimes vec(\Theta')) + (vec(\Theta') \otimes I_{n^2})]C_{n,n}] + [(\Theta \otimes \Theta)S_\epsilon' \otimes I_n]\}\Upsilon_{\theta_i},$$

$$J_{s_\nu,s_\epsilon} = D_s^+(\Theta \otimes \Theta \otimes \Theta)\Upsilon_{s_\epsilon},$$

$$J_{s_\nu,\kappa_\epsilon^e} = 0,$$

where $i = s, ns$ in (19) and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (21). The vectorization of the distinct elements of the third moments corresponds to $s_\nu = D_s^+ vec(S_\nu)$, where $D_s^+ = (D_s' D_s)^{-1} D_s'$, and D_s is the

$\left(n^3 \times \frac{n(n+1)(n+2)}{6}\right)$ matrix such that $D_s s_\nu = vec(S_\nu)$. As an example, for a bivariate system with $n = 2$, then:

$$D_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above vectorization, we have $\frac{\partial s_\nu}{\partial \theta_i'} = D_s^+ \frac{\partial vec(S_\nu)}{\partial vec(\Theta)'} \frac{\partial vec(\Theta)}{\partial \theta_i'}$ with $\frac{\partial vec(\Theta)}{\partial \theta_i'} = \Upsilon_{\theta_i}$. Rewriting equation (12) as $vec(S_\nu) = [(\Theta \otimes \Theta) \otimes \Theta]vec(S_\epsilon)$, then $\frac{\partial vec(S_\nu)}{\partial vec(\Theta)'} = (I_{n^2} \otimes \Theta S_\epsilon)\frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} + [(\Theta \otimes \Theta)S_\epsilon' \otimes I_n]$, where $\frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_n \otimes C_{n,n} \otimes I_n)[(I_{n^2} \otimes vec(\Theta')) + (vec(\Theta') \otimes I_{n^2})]\frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$ with $\frac{\partial vec(\Theta')}{\partial vec(\Theta)'} = C_{n,n}$ (see Magnus and Neudecker, 2007, pp. 208–209), and $C_{n,m}$ is a $(nm \times nm)$ commutation matrix implying that $C_{n,m}vec(A) = vec(A')$ for the arbitrary $(n \times m)$ matrix A. Note that $\frac{\partial s_\nu}{\partial \theta_i'} = 0$ for $i = ns$ in (19) and for $i = \kappa\kappa, ns\kappa$ in (21), since S_ν is not a function of the structural parameters relating the reduced-form innovations to the symmetric structural shocks. Furthermore, $\frac{\partial s_\nu}{\partial s_\epsilon} = D_s^+ \frac{\partial vec(S_\nu)}{\partial vec(S_\epsilon)'} \frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(S_\nu)}{\partial vec(S_\epsilon)'} = (\Theta \otimes \Theta \otimes \Theta)$ and $\frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'} = \Upsilon_{s_\epsilon}$ is a $(n^3 \times m_s)$ matrix selecting the partial derivatives with respect to the non-zero elements of s_ϵ. In particular, for a system with $n = m_s = 2$, then Υ_{s_ϵ} has values one for the (1,1) and (8,2) elements, and zero elsewhere. For the system with $n = 2$ and $m_s = 1$, then Υ_{s_ϵ} has values one for the (1,1) element, and zero elsewhere. Moreover, $\frac{\partial s_\nu}{\partial \kappa_\epsilon^e} = D_s^+ \frac{\partial vec(S_\nu)}{\partial vec(K_\epsilon^e)'} \frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e'}}$, where $\frac{\partial vec(S_\nu)}{\partial vec(K_\epsilon^e)'} = 0$ given that S_ν is not a function of the excess kurtoses of the structural shocks.

Finally, the partial derivatives of the fourth unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$
\begin{aligned}
J_{\kappa_\nu^e, \theta_i} &= D_\kappa^+ \{(I_{n^2} \otimes \Theta K_\epsilon^e)(I_{n^2} \otimes C_{n,n^2} \otimes I_n)[(I_{n^4} \otimes vec(\Theta'))(I_n \otimes C_{n,n} \otimes I_n) \\
&\quad \times [(I_{n^2} \otimes vec(\Theta') + (vec(\Theta') \otimes I_{n^2})]C_{n,n} + (vec(\Theta' \otimes \Theta') \otimes I_{n^2})C_{n,n}] + [(\Theta \otimes \Theta \otimes \Theta)K_\epsilon^{e'} \otimes I_n]\}\Upsilon_{\theta_i}, \\
J_{\kappa_\nu^e, s_\epsilon} &= 0, \\
J_{\kappa_\nu^e, \kappa_\epsilon^e} &= D_\kappa^+(\Theta \otimes \Theta \otimes \Theta \otimes \Theta)\Upsilon_{\kappa_\epsilon^e},
\end{aligned}
$$

where $i = \kappa, n\kappa$ in (20) and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (21). The vectorization of the distinct ele-

ments of the fourth moments is $\kappa^e_\nu = D^+_\kappa vec(K^e_\nu)$, where $D^+_\kappa = (D'_\kappa D_\kappa)^{-1} D'_\kappa$, and D_κ is the $\left(n^4 \times \frac{n(n+1)(n+2)(n+3)}{24}\right)$ matrix such that $D_\kappa \kappa^e_\nu = vec(K^e_\nu)$. For example, when $n = 2$, then:

$$
D_\kappa = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Using the above vectorization, we have $\frac{\partial \kappa^e_\nu}{\partial \theta'_i} = D^+_\kappa \frac{\partial vec(K^e_\nu)}{\partial vec(\Theta)'} \frac{\partial vec(\Theta)}{\partial \theta'_i}$ with $\frac{\partial vec(\Theta)}{\partial \theta'_i} = \Upsilon_{\theta_i}$. Given that equation (13) implies $vec(K^e_\nu) = [(\Theta \otimes \Theta \otimes \Theta) \otimes \Theta] vec(K^e_\epsilon)$, then $\frac{\partial vec(K^e_\nu)}{\partial vec(\Theta)'} = (I_{n^2} \otimes \Theta K^e_\epsilon) \frac{\partial vec(\Theta' \otimes \Theta' \otimes \Theta')}{\partial vec(\Theta)'} +$ $[(\Theta' \otimes \Theta' \otimes \Theta') K^{e\prime}_\epsilon \otimes I_n]$, where $\frac{\partial vec(\Theta' \otimes \Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_{n^2} \otimes C_{n,n^2} \otimes I_n) \left[(I_{n^4} \otimes vec(\Theta')) \frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} + [vec(\Theta' \otimes \Theta') \otimes I_{n^2}] \right] \frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$, and, as shown above, $\frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_n \otimes C_{n,n} \otimes I_n)[(I_{n^2} \otimes vec(\Theta')) + (vec(\Theta') \otimes I_{n^2})] \frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$ and $\frac{\partial vec(\Theta')}{\partial vec(\Theta)'} = C_{n,n}$. Note that $\frac{\partial \kappa^e_\nu}{\partial \theta'_i} = 0$ for $i = n\kappa$ in (20) and for $i = ss, ns\kappa$ in (21), since K^e_ν is not a function of the structural parameters relating the reduced-form innovations to the mesokurtic structural shocks. Moreover, $\frac{\partial \kappa^e_\nu}{\partial s_\epsilon} = D^+_\kappa \frac{\partial vec(K^e_\nu)}{\partial vec(S_\epsilon)'} \frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(K^e_\nu)}{\partial vec(S_\epsilon)'} = 0$ given that K^e_ν is not a function of the skewnesses of the structural shocks. In addition, $\frac{\partial \kappa^e_\nu}{\partial \kappa^e_\epsilon} = D^+_\kappa \frac{\partial vec(K^e_\nu)}{\partial vec(K^e_\epsilon)'} \frac{\partial vec(K^e_\epsilon)}{\partial \kappa^{e\prime}_\epsilon}$, where $\frac{\partial vec(K^e_\nu)}{\partial vec(K^e_\epsilon)'} = (\Theta \otimes \Theta \otimes \Theta \otimes \Theta)$ and $\frac{\partial vec(K^e_\epsilon)}{\partial \kappa^{e\prime}_\epsilon} = \Upsilon_{\kappa^e_\epsilon}$ is a $(n^4 \times m_\kappa)$ matrix selecting the partial derivatives with respect to the non-zero elements of κ^e_ϵ. For example, when $n = m_\kappa = 2$, then $\Upsilon_{\kappa^e_\epsilon}$ has values one for the (1,1) and (16,2) elements, and zero elsewhere. For the system with $n = 2$ and $m_\kappa = 1$, then $\Upsilon_{\kappa^e_\epsilon}$ has values one for the (1,1) element, and zero elsewhere.

References

[1] Anderson, T.W. (1951), "The Asymptotic Distribution of Certain Characteristic Roots and Vectors," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, in J. Neyman (ed.), Berkeley: University of California Press, pp. 103-130.

[2] Bai, J., and S. Ng (2005), "Tests for Skewness, Kurtosis, and Normality for Time Series Data," *Journal of Business & Economic Statistics* 23, pp. 49–60.

[3] Bekaert, G., and C.R. Harvey (1997), "Emerging Equity Market Volatility," *Journal of Financial Economics* 43, pp. 29–77.

[4] Blanchard, O.J., and R. Perotti (2002), "An Empirical Characterization of the Dynamic Effects of Changes in Government Spending and Taxes on Output," *Quarterly Journal of Economics* 117, pp. 1329–1368.

[5] Blanchard, O.J., and D. Quah (1989),"The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review* 79, pp. 655–673.

[6] Blanchard, O.J., and M.W. Watson (1986), "Are Business Cycles all Alike?" *American Business Cycle: Continuity and Change*, in R.J. Gordon (ed.), Chicago: University of Chicago Press, pp. 123–180.

[7] Boothe, P., and D. Glassman (1987), "The Statistical Distribution of Exchange Rates: Empirical Evidence and Economic Implications," *Journal of International Economics* 22, pp. 297–319.

[8] Bose, A. (1988), "Edgeworth Correction by Bootstrap in Autoregressions," *The Annals of Statistics* 16, pp. 1709–1722.

[9] Bouakez, H., F. Chihi, and M. Normandin (2014), "Measuring the Effects of Fiscal Policy," *Journal of Economic Dynamics and Control* 47, pp. 123–151.

[10] Camba-Méndez, G., and G. Kapetanios (2008), "Statistical Tests and Estimators of the Rank of a Matrix and their Applications in Econometric Modelling," European Central Bank, Working Paper Series No. 850.

[11] Clark, P.K. (1973), "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," *Econometrica* 41, pp. 135–156.

[12] Comon, P. (1994), "Independent Component Analysis, A New Concept?" *Signal Processing* 36, pp. 287–314.

[13] Eriksson, J., and V. Koivunen (2004), "Identifiability, Separability and Uniqueness of Linear ICA Models," *IEEE Signal Processing Letters* 11, pp. 601–604.

[14] Favero, C., and F. Giavazzi (2009), "How Large Are the Effects of Tax Changes," NBER Working Paper No. 15303.

[15] Fuijwara, I., L.M. Körber, and D. Nagakura (2013), "Asymmetry in Government Bonds Returns," *Journal of Banking and Finance* 37, pp. 3218–3226.

[16] Gouriéroux, C., A. Monfort, and J.-P. Renne (2017), "Statistical Inference for Independent Component Analysis: Application to Structural VAR Models," *Journal of Econometrics* 196, pp. 111–126.

[17] Gouriéroux, C., A. Monfort, and J.-P. Renne (2018), "Identification and Estimation in Non-Fundamental Structural VARMA Models," manuscript.

[18] Herwartz, H. (2015), "Structural VAR Modelling with Independent Innovations — An Analysis of Macroeconomic Dynamics in the Euro Area Based on a Novel Identification Scheme," University of Göttingen, Working Paper.

[19] Kilian, L. (1998), "Confidence Intervals for Impulse Responses under Departures from Normality," *Econometric Reviews* 17, pp. 1–29.

[20] Kilian, L., and U. Demiroglu (2000), "Residual-Based Tests for Normality in Autoregressions: Asymptotic Theory and Simulation Evidence," *Journal of Business & Economic Statistics* 18, pp. 40–50.

[21] Kilian, L., and H. Lütkepohl (2017), *Structural Vector Autoregressive Analysis,* Cambridge: Cambridge University Press, 754 p.

[22] Lanne, M., H. Lütkepohl, and K. Maciejowska (2010), "Structural Vector Autoregressions with Markov Switching," *Journal of Economic Dynamics and Control* 34, pp. 121–131.

[23] Lanne, M., M. Meitz, and P. Saikkonen (2017), "Identification and Estimation of Non-Gaussian Structural Vector Autoregressions," *Journal of Econometrics* 196, pp. 288–304.

[24] Leeb, H., and B.M. Pötscher (2005), "Model Selection and Inference: Facts and Fiction," Econometric Theory 21, pp. 21–59.

[25] Lütkepohl, H. (2007), *New Introduction to Multiple Time Series Analysis,* Berlin: Springer, 764 p.

[26] Lütkepohl, H., and A. Netšunajev (2014), "Structural Vector Autoregressions with Smooth Transition in Variances: The Interaction Between U.S. Monetary Policy and the Stock Market," DIW Discussion Paper 1388.

[27] Lütkepohl, H., and T. Schlaak (2018), "Choosing Between Different Time-Varying Volatility Models for Structural Vector Autoregressive Analysis," *Oxford Bulletin of Economics and Statistics* 80, pp. 715–735.

[28] Magnus, J.R., and H. Neudecker (2007), *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Third Edition, New York: John Wiley & Sons, 450 p.

[29] Mardia, K.V. (1980), "Tests of Univariate and Multivariate Normality," *Handbook of Statistics, Volume 1: Analysis of Variance*, P. R. Krishnaiah (ed.), Amsterdam: North Holland, pp. 279–320.

[30] Moneta, A., D. Entner, P.O. Hoyer, and A. Coad (2013), "Causal Inference by Independent Component Analysis: Theory and Applications," *Oxford Bulletin of Economics and Statistics* 75, pp. 705–730.

[31] Normandin, M., and L. Phaneuf (2004), "Monetary Policy Shocks: Testing Identification Conditions Under Time-Varying Conditional Volatility," *Journal of Monetary Economics* 51, pp. 1217–1243.

[32] Perotti, R. (2004), "Estimating the Effects of Fiscal Policy in OECD Countries," Università Bocconi, Discussion Paper No. 276.

[33] Rigobon, R. (2003), "Identification through Heteroskedasticity," *Review of Economics and Statistics* 85, pp. 777–792.

[34] Robin, J.-M., and R.J. Smith (2000), "Tests of Rank," *Econometric Theory* 16, pp. 151–175.

[35] Sims, C.A. (1980), "Macroeconomic and Reality," *Econometrica* 48, pp. 1–48.

[36] Uhlig, H. (2005), "What Are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure," *Journal of Monetary Economics* 52, pp. 381–419.

Table 1. Empirical Sizes of Rank Tests with Asymptotic Distributions: Symmetry

$r^* = 0$

T	Wald 10 %	5%	1%	LR 10 %	5%	1%
100	8.72	3.92	0.53	2.68	0.63	0.01
200	9.99	4.66	0.80	5.81	1.91	0.12
500	9.93	4.69	0.81	7.97	3.36	0.41
1,000	9.73	4.63	0.70	8.65	3.94	0.52
5,000	10.03	5.22	1.09	9.90	4.97	1.02

$r^* = 1$

T	Wald 10 %	5%	1%	LR 10 %	5%	1%
100	11.83	5.79	1.52	7.86	3.22	0.51
200	10.87	5.30	1.18	8.60	3.66	0.53
500	10.89	5.20	1.06	9.74	4.42	0.63
1,000	9.97	4.82	1.03	9.45	4.36	0.86
5,000	10.61	5.59	1.02	10.05	5.47	0.99

Notes. Entries are the empirical sizes (in percentage) of the rank tests with asymptotic distributions under the null hypothesis that $rk[S_u] = r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_d = \alpha_s = 0.5$ and $\omega_d = \omega_s = 1$. Also, the distributions are $\epsilon_{s,t} \sim N(0,1)$, and i) $\epsilon_{d,t} \sim N(0,1)$ under $r^* = 0$ or ii) $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326,1)$ with probability 0.2113 under $r^* = 1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r^*}^W$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r^*}^{LR}$ associated with S_u exceed the asymptotic critical values, where the latters are computed as in Robin and Smith (2000).

Table 2. Empirical Sizes of Rank Tests with Asymptotic Distributions: Kurtosis

$r^* = 0$

	Wald			LR		
T	10 %	5%	1%	10 %	5%	1%
100	0.70	0.10	0.00	0.00	0.00	0.00
200	0.00	0.00	0.00	0.00	0.00	0.00
500	0.00	0.00	0.00	0.00	0.00	0.00
1,000	0.00	0.00	0.00	0.00	0.00	0.00
5,000	0.00	0.00	0.00	0.00	0.00	0.00

$r^* = 1$

	Wald			LR		
T	10 %	5%	1%	10 %	5%	1%
100	1.19	0.53	0.08	0.37	0.07	0.00
200	1.05	0.38	0.04	0.33	0.06	0.00
500	0.68	0.21	0.02	0.36	0.12	0.00
1,000	0.54	0.21	0.05	0.32	0.09	0.00
5,000	0.36	0.10	0.02	0.32	0.08	0.02

Notes. Entries are the empirical sizes (in percentage) of the rank tests with asymptotic distributions under the null hypothesis that $rk[K_u^e] = r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_d = \alpha_s = 0.5$ and $\omega_d = \omega_s = 1$. Also, the distributions are $\epsilon_{s,t} \sim N(0,1)$, and i) $\epsilon_{d,t} \sim N(0,1)$ under $r^* = 0$ or ii) $1.291 \times \epsilon_{d,t} \sim t(5)$ under $r^* = 1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r*}^W$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r*}^{LR}$ associated with K_u^e exceed the asymptotic critical values, where the latters are computed as in Robin and Smith (2000).

Table 3. Empirical Sizes of Rank Tests with Finite-Sample Distributions: Symmetry

		$r^* = 0$				
	Wald			LR		
T	10 %	5%	1%	10 %	5%	1%
100	9.42	4.65	0.98	9.56	4.85	1.01
200	10.17	5.25	0.98	10.19	5.20	1.00
500	10.14	5.04	1.10	10.29	4.99	1.12
1,000	9.82	4.91	0.92	9.87	4.90	0.92
5,000	10.02	5.10	1.12	9.98	5.11	1.11

		$r^* = 1$				
	Wald			LR		
T	10 %	5%	1%	10 %	5%	1%
100	11.41	6.35	1.47	11.41	6.35	1.47
200	9.11	4.86	1.42	9.11	4.86	1.42
500	9.29	4.55	1.07	9.29	4.55	1.07
1,000	8.39	4.26	1.02	8.39	4.26	1.02
5,000	9.20	4.68	0.96	9.20	4.68	0.96

Notes. Entries are the empirical sizes (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[S_u] = r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_d = \alpha_s = 0.5$ and $\omega_d = \omega_s = 1$. Also, the distributions are $\epsilon_{s,t} \sim N(0,1)$, and i) $\epsilon_{d,t} \sim N(0,1)$ under $r^* = 0$ or ii) $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326, 1)$ with probability 0.2113 under $r^* = 1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r*}^W$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r*}^{LR}$ associated with S_u exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Table 4. Empirical Sizes of Rank Tests with Finite-Sample Distributions: Kurtosis

$r^* = 0$

T	Wald			LR		
	10 %	5%	1%	10 %	5%	1%
100	10.12	5.00	0.99	10.43	4.93	1.09
200	9.74	5.14	1.23	9.75	5.19	1.21
500	9.81	4.91	1.01	9.86	4.87	1.00
1,000	9.71	4.60	1.04	9.75	4.58	1.03
5,000	9.84	4.88	1.02	9.83	4.89	1.03

$r^* = 1$

T	Wald			LR		
	10 %	5%	1%	10 %	5%	1%
100	9.90	4.98	0.93	9.90	4.98	0.93
200	10.65	5.67	1.19	10.65	5.67	1.19
500	9.88	5.08	1.15	9.88	5.08	1.15
1,000	10.10	4.95	0.95	10.10	4.95	0.95
5,000	9.71	4.76	0.95	9.71	4.76	0.95

Notes. Entries are the empirical sizes (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[K_u^e] = r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_d = \alpha_s = 0.5$ and $\omega_d = \omega_s = 1$. Also, the distributions are $\epsilon_{s,t} \sim N(0,1)$, and i) $\epsilon_{d,t} \sim N(0,1)$ under $r^* = 0$, and ii) $1.291 \times \epsilon_{d,t} \sim t(5)$ under $r^* = 1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r*}^W$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r*}^{LR}$ associated with K_u^e exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Table 5. Empirical Powers of Rank Tests with Finite-Sample Distributions: Skewness

	Skewness = −0.5231						*Skewness = −0.9907*					
			$r^* = 0$						$r^* = 0$			
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	20.71	11.44	2.42	20.88	11.46	2.53	72.05	46.66	10.43	69.95	44.82	10.53
200	41.02	26.70	8.50	40.58	26.40	8.15	99.35	96.85	74.28	99.23	96.33	67.90
500	82.98	71.28	42.66	82.82	70.93	41.24	100.0	100.0	100.0	100.0	100.0	100.0
1,000	99.11	97.66	88.94	99.10	97.64	88.51	100.0	100.0	100.0	100.0	100.0	100.0
			$r^* = 1$						$r^* = 1$			
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	16.35	8.05	1.31	16.35	8.05	1.31	88.27	78.73	41.91	89.15	78.75	41.91
200	41.12	27.24	8.06	41.12	27.24	8.06	99.70	99.20	94.65	99.70	99.20	94.65
500	86.85	78.10	53.80	86.85	78.10	53.80	100.0	100.0	100.0	100.0	100.0	100.0
1,000	99.49	98.65	94.17	99.49	98.65	94.17	100.0	100.0	100.0	100.0	100.0	100.0

Notes. Entries are the empirical powers (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[S_u] = r^*$. The empirical powers are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_d = \alpha_s = 0.5$ and $\omega_d = \omega_s = 1$. For $r^* = 0$, the distributions are: i) $\epsilon_{s,t} \sim N(0,1)$ as well $1.6808 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.5 and $1.6808 \times \epsilon_{d,t} \sim N(-1, 2.65)$ with probability 0.5 when the demand shock exhibits a skewness of -0.5231, and ii) $\epsilon_{s,t} \sim N(0,1)$ as well as $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326, 1)$ with probability 0.2113 when the demand shock exhibits a skewness of -0.9907. For $r^* = 1$, the distributions are: i) $1.6808 \times \epsilon_{s,t} \sim N(1,1)$ and $1.6808 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.5 as well as $1.6808 \times \epsilon_{s,t} \sim N(-1, 2.65)$ and $1.6808 \times \epsilon_{d,t} \sim N(-1, 2.65)$ with probability 0.5 when each shock exhibits a skewness of -0.5231, and ii) $2.1755 \times \epsilon_{s,t} \sim N(1,1)$ and $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 as well as $2.1755 \times \epsilon_{s,t} \sim N(-3.7326, 1)$ and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326, 1)$ with probability 0.2113 when each shock exhibits a skewness of -0.9907. For each parametrization, $10,000$ simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r*}^{W}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r*}^{LR}$ associated with S_u exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Table 6. Empirical Powers of Rank Tests with Finite-Sample Distributions: Kurtosis

		Excess Kurtosis = 1							*Excess Kurtosis = 6*				
			$r^* = 0$							$r^* = 0$			
		Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	32.46	22.77	10.40	32.00	22.41	10.23	64.00	54.52	37.60	63.17	53.69	36.42
200	45.88	36.18	19.45	45.39	35.64	18.74	85.58	80.00	65.25	84.96	79.23	64.38
500	73.88	65.63	46.76	73.86	65.06	45.81	99.28	98.73	96.46	99.22	98.66	96.06
1,000	93.20	89.67	78.58	93.20	89.41	78.41	100.0	100.0	100.0	100.0	100.0	100.0
			$r^* = 1$							$r^* = 1$			
		Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	18.96	12.57	3.90	18.96	12.57	3.90	53.18	44.34	23.81	53.18	44.34	23.81
200	32.72	24.31	10.27	32.72	24.31	10.27	80.78	74.44	55.78	80.78	74.44	55.78
500	99.09	98.58	95.55	99.09	98.58	95.55	99.09	98.58	95.55	99.09	98.58	95.55
1,000	100.0	99.99	99.94	100.0	99.99	99.94	100.0	99.99	99.94	100.0	99.99	99.94

Notes. Entries are the empirical powers (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[K_u^e] = r^*$. The empirical powers are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_d = \alpha_s = 0.5$ and $\omega_d = \omega_s = 1$. For $r^* = 0$, the distributions are: i) $\epsilon_{s,t} \sim N(0,1)$ and $1.118 \times \epsilon_{d,t} \sim t(10)$ when the demand shock exhibits an excess kurtosis of 1, and ii) $\epsilon_{s,t} \sim N(0,1)$ and $1.291 \times \epsilon_{d,t} \sim t(5)$ when the demand shock exhibits an excess kurtosis of 6. For $r^* = 1$, the distributions are: i) $1.118 \times \epsilon_{s,t} \sim t(10)$ and $1.118 \times \epsilon_{d,t} \sim t(10)$ when each shock exhibits an excess kurtosis of 1, and ii) $1.291 \times \epsilon_{s,t} \sim t(5)$ and $1.291 \times \epsilon_{d,t} \sim t(5)$ when each shock exhibits an excess kurtosis of 6. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r^*}^W$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r^*}^{LR}$ associated with K_u^e exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Table 7. Multipliers

Quarter	Tax	Spending		
		$\theta_{12} = \alpha_1\theta_{32}$	$\theta_{13} = \alpha_1\theta_{33}$	$\theta_{23} = 0$
1	−0.01	0.73*	1.75*	2.00*
4	0.03	0.23	2.14*	2.60*
8	0.22	−0.14	1.57	1.94
Peak	0.59	0.73*	2.51*	2.91*
	[14]	[1]	[3]	[3]

Notes. Entries correspond to the tax (spending) multipliers: the dollar change in output at a given horizon that results from a dollar decrease (increase) in the exogenous component of taxes (government spending). An asterisk indicates that the 90 percent confidence interval does not include zero, where the confidence intervals are computed from 5,000 bootstrap samples. Numbers between brackets indicate the quarters in which the maximum value of the multiplier is reached.
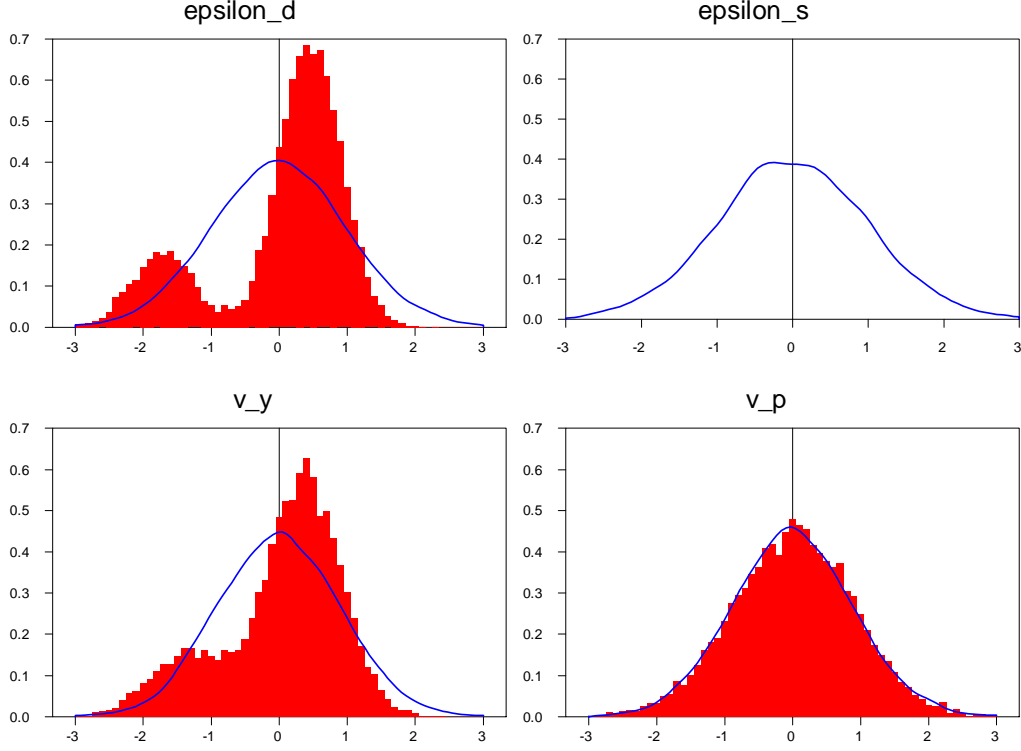
Figure 1. Densities of simulated series for the zero and negative skewness cases. Notes: The blue lines correspond to the estimates of the density functions (using a gaussian kernel and a bandwidth of $0.79Q/T^{1/5}$ where Q is the range between the 25% and 75% quantiles and T is the sample size) associated with the zero skewness case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $\epsilon_{d,t} \sim N(0,1)$, and $\epsilon_{s,t} \sim N(0,1)$. The red bars form the histograms associated with the negative skewness case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326,1)$ with probability 0.2113, whereas $\epsilon_{s,t} \sim N(0,1)$. For each parametrization, the simulated series are generated from 10,000 draws.
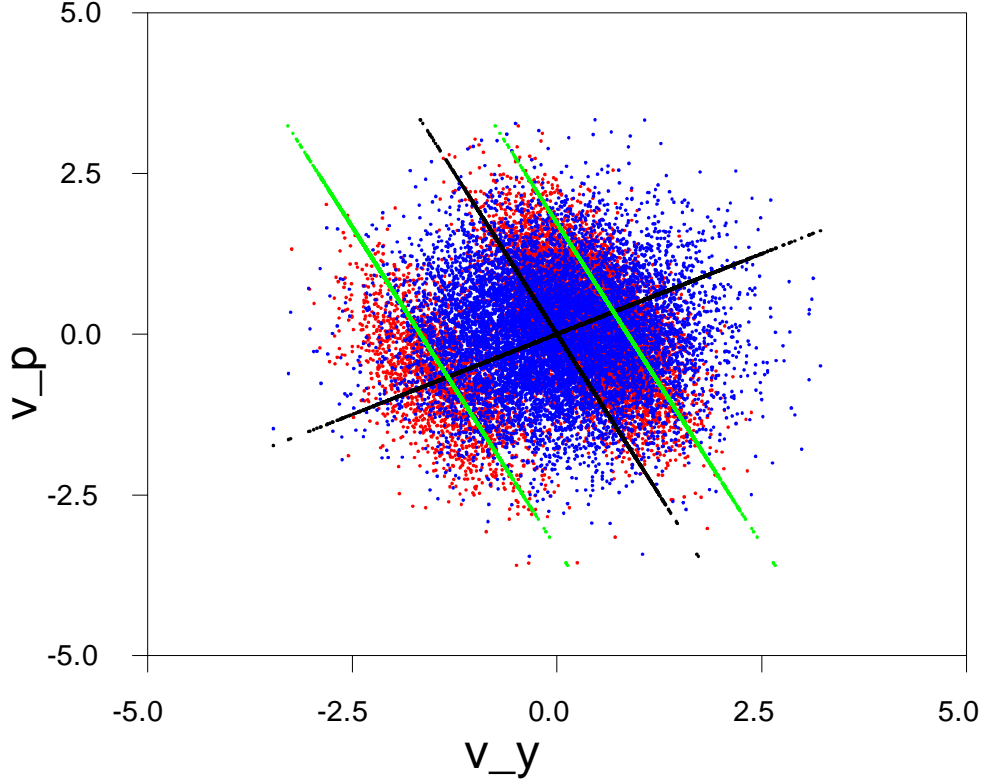
Figure 2. Scatter plots of simulated series for the zero and negative skewness cases. Notes: The blue dots correspond to the zero skewness case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $\epsilon_{d,t} \sim N(0,1)$, and $\epsilon_{s,t} \sim N(0,1)$. The red dots correspond to the negative skewness case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $2.1755 \times \epsilon_{d,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{d,t} \sim N(-3.7326, 1)$ with probability 0.2113, whereas $\epsilon_{s,t} \sim N(0,1)$. For each parametrization, the simulated series are generated from 10,000 draws. The black lines represent the downward demand curve and the upward supply curve. The green lines illustrate shifts of the demand curve.
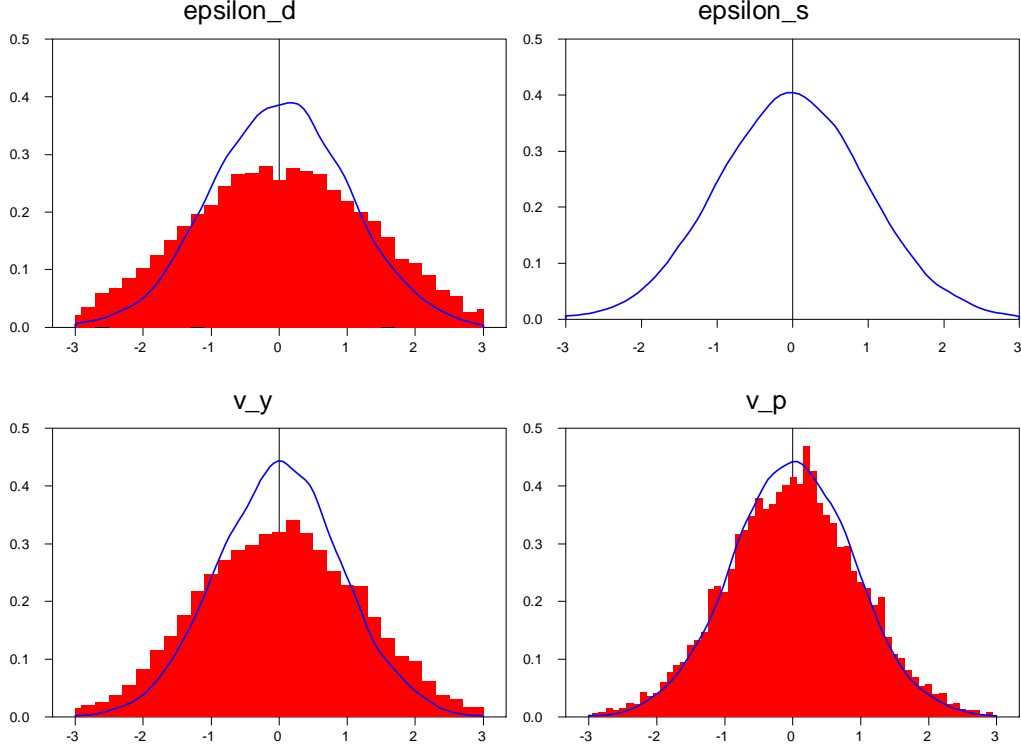
Figure 3. Densities of simulated series for the zero and positive excess kurtosis cases. Notes: The blue lines correspond to the estimates of the density functions (using a gaussian kernel and a bandwidth of $0.79Q/T^{1/5}$ where Q is the range between the 25% and 75% quantiles and T is the sample size) associated with the zero excess kurtosis case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5, \omega_d = \omega_s = 1, \epsilon_{d,t} \sim N(0,1)$, and $\epsilon_{s,t} \sim N(0,1)$. The red bars form the histograms associated with the positive excess kurtosis case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5, \omega_d = \omega_s = 1, 1.291 \times \epsilon_{d,t} \sim t(5)$, and $\epsilon_{s,t} \sim N(0,1)$. For each parametrization, the simulated series are generated from 10,000 draws.
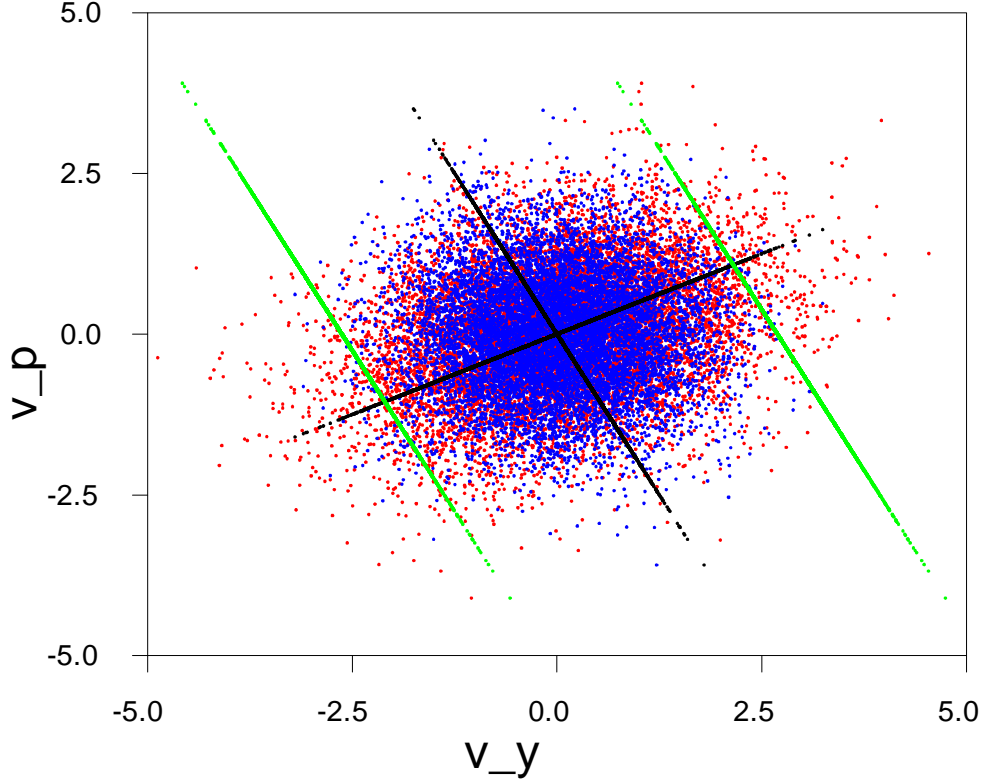
Figure 4. Scatter plots of simulated series for the zero and positive excess kurtosis cases. Notes: The blue dots correspond to the zero excess kurtosis case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $\epsilon_{d,t} \sim N(0,1)$, and $\epsilon_{s,t} \sim N(0,1)$. The red dots correspond to the positive excess kurtosis case, where the parametrization of equations (1) and (2) is $\alpha_d = \alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $1.291 \times \epsilon_{d,t} \sim t(5)$, and $\epsilon_{s,t} \sim N(0,1)$. For each parametrization, the simulated series are generated from 10,000 draws. The black lines represent the downward demand curve and the upward supply curve. The green lines illustrate shifts of the demand curve.