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Karamata Production Functions: A Unified Framework for Growth and Technical Change Beyond Uzawa

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Abstract

Uzawa (1961) showed that balanced growth is only compatible with purely labor-augmenting technological progress unless the aggregate production function is Cobb-Douglas. We show that this conclusion is not general. We identify a broader class of technologies, *regularly varying (Karamata) production functions*, that admits balanced growth with simultaneous labor- and capital-augmenting technical progress. Within this class, factor shares converge and the elasticity of substitution tends to one without requiring asymptotic Cobb-Douglas behavior. These production functions arise naturally from aggregation mechanisms and imply that purely labor-augmenting technical change is not a universal outcome.

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Keywords: balanced growth, factor-augmenting technical change, regular variation, Karamata production functions.

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1 Introduction

A central question in growth theory is whether the long-run properties of an economy impose strong restrictions on the functional form of the production technology. In particular, the existence of a balanced growth path is often viewed as a powerful constraint on admissible production functions. A classical result due to Uzawa (1961) states that balanced growth is only compatible with purely labor-augmenting technological progress, unless the production function is Cobb-Douglas. This result has profoundly shaped the literature and has led to the widespread view that Cobb–Douglas plays a special role in long-run analysis.

This paper shows that the standard interpretation is too restrictive. The key insight is that the asymptotic restrictions typically imposed in growth theory, such as the convergence of the capital share to a constant and the convergence of the elasticity of substitution to one, do not uniquely characterize the Cobb-Douglas production function. I show that this result is not universally true, as there exists an entire class of functions, known as Karamata functions or regularly varying functions, that are compatible with both labor- and capital-augmenting technological progress. Quite surprisingly, I am not the first to establish this result. Indeed, the overlooked article by Sato and Beckmann (1970) already indicated and demonstrated that the traditional result is incorrect. They constructed a class of production functions that allow for capital-augmenting technical progress while still ensuring the existence of a balanced growth path within the neoclassical framework. However, the class of production functions they obtain is actually a subset of the more general class that we propose. We thus introduce the Karamata class of production functions into the economics literature.

The Karamata class of functions, introduced by Karamata (1933), provides a general framework for studying the asymptotic behavior of functions that exhibit regular variation at large scales. Intuitively, these functions describe phenomena whose growth or decay is predictable over the long term, while still allowing for finer adjustments or slow variations. A key feature of functions in this class is their asymptotic homogeneity: at sufficiently large scales, they behave similarly to homogeneous functions, making them particularly useful for modeling scaling properties and equilibrium dynamics in complex systems. Regular variation provides a natural framework to describe asymptotic behavior. Intuitively, a regularly varying function behaves like a power function up to a slowly varying correction term. More precisely, it can be written as $f(x) = x^{\alpha} L(x)$, where $L(x)$ is a slowly varying function, that is, a function that changes more slowly than any power of x. This representation allows for systematic deviations from Cobb-Douglas behavior while preserving its key asymptotic properties. In general, such functions are not asymptotically equivalent to x^{α}, since $\frac{f(x)}{x^{\alpha}} \sim L(x)$, which need not converge to a constant. Asymptotic equivalence arises only in the special case where $L(x)$ converges to a finite non-zero constant. To the best of my knowledge, Karamata functions have never been used in economics, despite their elegant fit with the properties of neoclassical production functions. However, they are widely used in mathematical statistics, particularly in the analysis of heavy-tailed distributions (such as Pareto distributions and, more generally, all distributions within the Fréchet domain of attraction).

Jones (2005) provides deeper microfoundations for the neoclassical aggregate production function. He considers a framework in which firms have access to a finite number of complementary technologies, either Walras-Leontief or CES with low elasticity of substitution, and producing more requires the discovery of new ideas. These ideas are drawn from a Pareto distribution, as in Kortum (1997), and the resulting aggregate production function is Cobb-Douglas. The Karamata class of production functions provides the ideal framework to formalize the Houthakker-Jones approach, as we demonstrate. Our approach is also consistent with the recent contribution by Jones (2023), who, using a theorem that connects extreme value maxima to the number of draws and the shape of the upper tail of probability distributions, shows that exponential growth can emerge without being tightly linked to any specific distribution, such as the Pareto distribution. In the same spirit, we show in a very general way that if the production function belongs to the Karamata class, a balanced growth path remains feasible, even if technical progress is not purely labor-augmenting.

Several alternative approaches have been proposed to explain balanced growth, despite Uzawa's theorem. For instance, Acemoglu (2003) uses endogenous growth models, where profit-maximizing firms carry out innovations, to show that technical progress can initially be factor-augmenting along the transition path, but eventually becomes purely labor-augmenting asymptotically. He generalizes and provides microfoundations for the induced-innovation models which were criticized by Nordhaus (1973) for their lack of microfoundations. Under relatively general conditions, factor-augmenting technological progress is possible at least during the transition, and labor-augmenting technological progress emerges asymptotically along the balanced growth path. The result that asymptotic technical change is purely labor-augmenting relies critically on the structure of the final-good aggregator. In Acemoglu (2003), this aggregator is of the CES type, which imposes a strong restriction on the elasticity of substitution between capital- and labor-intensive inputs. In particular, when the elasticity of substitution is strictly below one, the equilibrium dynamics lead to a balanced growth path in which factor shares converge and technical change becomes asymptotically labor-augmenting. In the present framework, we replace the CES aggregator with a regularly varying production function. While such functions preserve the key asymptotic properties required for balanced growth, most notably, the convergence of factor shares, they allow for a much richer class of substitution patterns. As a result, the asymptotic structure of technical change is no longer restricted to be purely labor-augmenting. In particular, we show that, both labor- and capital-augmenting technological improvements may arise simultaneously along a balanced growth path. This shows that the asymptotic direction of technical change is not a universal property of the model, but depends sensitively on the functional form of the final-good technology.

More recently, Grossman *et al.* (2017) extended Uzawa's theorem by introducing human capital accumulation, where human capital is endogenously determined and capital is more complementary with human capital than with raw labor. They obtained a class of production functions for which the neoclassical growth model remains compatible with capital-augmenting technological progress, while also allowing for endogenous human capital growth, ensuring the existence of a balanced growth path. Léon-Ledesma and Satchi (2018) use the shape of the technological frontier and adjustment costs in firms' technology choices to distinguish between short-run and long-run elasticities of substitution. When their technological frontier becomes log-linear, the long-run production function reduces to a Cobb-Douglas form, even though the short-run elasticity of substitution remains below one. Our approach is fully compatible with theirs, since our production function also exhibits an elasticity of substitution different from one in the short run, while converging to one in the long run. However, such a production function is not necessarily Cobb-Douglas.

The remainder of the paper is organized as follows. Section 2 introduces the concept of regularly varying functions. In Section 2.1, we present the main definitions and fundamental properties of regular variation. In Section 2.2, we discuss the structure of the production function $Y(t) = F(B(t)K(t), A(t)L(t))$, which incorporates both capital- and labor-augmenting technical progress. We then establish a first key result, which is well known: defining the ratio of effective capital to effective labor as $x(t) = \frac{B(t)K(t)}{A(t)L(t)}$, the intensive production function can be written as $y(t) = A(t)f(x(t))$. This representation allows us to derive the asymptotic properties of the capital share, which converges to a constant, and of the elasticity of substitution, which converges to one. While such production functions satisfy Inada conditions at infinity, the converse does not generally hold. A particularly interesting result is the relationship between the elasticity of substitution, the capital share, and the elasticity of the capital share itself, a form of "super-elasticity" also discussed, in a different context, in Jones (2023).

Section 3 provides examples and further insights. In Section 3.1, we present production functions commonly used in the literature that satisfy the properties of Karamata production functions but are asymptotically equivalent to Cobb-Douglas. These examples help explain why the literature often associates asymptotically unit elasticity of substitution with Cobb–Douglas behavior. In Section 3.2, we construct examples of production functions that are not asymptotically equivalent to Cobb-Douglas. In particular, we show how to derive a "super-log" production function using fractional derivatives in the sense of Caputo (1967), providing an asymptotically meaningful alternative to the translog specification. We also show that infinite controlled oscillations of the capital share and the elasticity of substitution are possible within the Karamata framework. In Section 3.3, we demonstrate that the class of production functions introduced by Sato et Beckmann (1970) can be interpreted as a special case of our framework.

Section 4 studies the implications for growth theory. In Section 4.1, we generalize the results of Jones (2005) by showing that if ideas are drawn from a heavy-tailed distribution (of which the Pareto distribution is a special case), then the aggregate production function belongs to the Karamata class, providing microfoundations to it. In Section 4.2, we first show that balanced growth is asymptotically feasible when the production function is regularly varying. We then reverse the question to show that asymptotic production function compatible with balanced growth does not necessary exhibits labor-augmenting technological progress. After noting that the classical approach initiated by Schlicht (2006) is not compatible with standard continuous-time growth models, we adopt an explicitly asymptotic perspective. In models where equilibrium dynamics are governed by autonomous differential equations, the steady state is never reached in finite time: convergence occurs only as $t \to \infty$. This distinction is not merely technical. It plays a key role in the characterization of long-run dynamics. Within this framework, we extend the argument of Acemoglu (2009) by allowing for convergence rates slower than $1/t$. While the standard analysis focuses on sufficiently fast convergence, we show that slower convergence can generate non-trivial asymptotic dynamics and allow for both labor- and capital-augmenting technical change. Moreover, when the production function is regularly varying, the elasticity of substitution depends only on the capital share and its elasticity. This places us precisely in case (XIV) of Sato and Beckmann (1968), where "*Inventions are neutral in the sense that the elasticity of substitution remains unchanged as long as the income shares of factors are constant (Factor-Augmenting Technical Progress).*", corresponding to factor-augmenting technical progress. Finally, Section 4.3 revisits the endogenous growth framework of Acemoglu (2003) within this more general setting and shows that purely labor-augmenting technical change is not a universal outcome, but depends on the functional form of the production technology.

2 The Karamata production function

2.1 Preliminaries and Definitions

Before introducing the Karamata production function itself, it is helpful to recall a few fundamental notions from the theory of regular variation. These concepts provide the analytical framework that will later allow us to characterize the asymptotic behaviour of neoclassical production functions and, in particular, to connect their shape to the behaviour of the elasticity of substitution, the capital share and the equilibrium growth path. We begin with the classical definition of a regularly varying function.

Definition 1 (Regularly Varying Function, First Definition, Bingham *et al.*, 1987).

> A positive measurable function f defined on some neighborhood $[x_0, \infty)$ of infinity is called regularly varying at infinity with index α if, for each $\lambda > 0$ and some $\alpha \in \mathbb{R}$,
>
> $$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha} \quad \forall \lambda > 0.$$
>
> The real number α is called the index of regular variation.

The central object in this definition is the behavior of the ratio $f(\lambda x)/f(x)$ as $x \to \infty$. When this ratio converges to a pure power of λ, the asymptotic structure of f becomes remarkably tractable. In economic applications, functions of this type naturally arise in growth models.

However, working directly with Definition 1 is often inconvenient. The power function x^{α} captures the dominant asymptotic trend, but the residual deviations from exact power behaviour play an essential role, particularly when linking a production function to its underlying economic primitives. To isolate these deviations, it is useful to introduce a finer concept describing functions that vary more slowly than any power.

Definition 2 (Slowly Varying Function, Bingham *et al.*, 1987, p.6).

> Let L be a positive measurable function, defined on some neighborhood $[x_0, \infty)$ of infinity, and satisfying:
>
> $$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1 \quad \forall \lambda > 0$$
>
> Then L is said to be slowly varying (in Karamata's sense).

Let us note that the neighbourhood $[x_0, \infty)$ in both definitions is of little importance and we may suppose L defined on $[0, \infty)$ without lost of generality.[2] Slowly varying functions constitute the 'index-zero' case of regular variation. They evolve at a pace that is negligible compared to any power of x, a property that will allow us to refine the structure of regularly varying production functions. The restriction to a neighborhood of infinity is technically standard but economically innocuous, since our analysis will focus on large-scale behaviour.

Definition 2 also prepares the ground for an alternative characterization of regular variation. Instead of verifying property of definition 1 for all $\lambda > 0$, one may express a regularly varying function as the product of a power function and a slowly varying component. This decomposi-

[2] The notation L is customarily used for such functions because of the first letter of the French word "*lentement*" which means "*slowly*". These functions were introduced and studied by Karamata (1933) in a pioneering paper, written in French, with continuity in place of measurability.

tion is crucial for our purposes because the slowly varying term will later encode economically meaningful information about factor shares and elasticities.

Definition 3 (Regularly Varying Function, Second Definition), Bingham *et al.*, 1987.

> A positive measurable function f defined on some neighborhood $[x_0, \infty)$ of infinity is called regularly varying at infinity with index α if it can be represented in the form:
>
> $f(x) = x^{\alpha} L(x)$
>
> where $L(x)$ is a slowly varying function.

The representation above shows that understanding regular variation ultimately reduces to understanding the structure of slowly varying functions. Fortunately, these functions admit a powerful integral representation, originally due to Karamata (1933). This result not only clarifies the asymptotic behaviour of L but also provides a direct link between its elasticity and a small auxiliary function $\varepsilon(x)$, which tends to zero at infinity. The following Representation Theorem will provide deeper insight into the structure of slowly varying production functions and explicitly link them to the capital share:[3]

Representation Theorem (Bingham *et al.*, 1987).

> The function L is slowly varying if and only if it can be written in the form:
>
> $$L(x) = c(x)e^{\int_{x_0}^{x} \frac{\varepsilon(u)}{u} du}$$
>
> for $x \geq x_0 > 0$, where $c(x)$ is measurable and satisfies $\lim_{x\to\infty} c(x) = c > 0$, and $\varepsilon(x)$ is a measurable function such that $\lim_{x\to\infty} \varepsilon(x) = 0$.
> Moreover, when $c(x) = c > 0$ is a constant function, L is called normalized slowly varying.

This representation has two important consequences for our analysis. First, it makes explicit the way in which a slowly varying function deviates from a constant: all such deviations are encoded in the vanishing function $\varepsilon(x)$. Second, normalized slowly varying functions were introduced by Kohlbecker (1958). As claimed by Bingham *et al.* (1987, p.15), and as can be easily shown, when L is normalized, its elasticity is given almost everywhere by:

$$\varepsilon(x) = \frac{xL'(x)}{L(x)}.$$

Conversely, given a function L with $\varepsilon(x) := \frac{xL'(x)}{L(x)}$ continuous and of order $o(1)$ at infinity, integrat-

[3] The second definition of a regularly varying function makes it clear that a slowly varying function L is regularly varying with index $\alpha = 0$. Thus the set of slowly varying functions forms a subset of the set of regularly varying functions. However, as discussed below, we will work with a set of production functions with indexes $\alpha \in (0, 1)$.

ing this expression yields:

$$L(x) = ce^{\int_{x_0}^{x} \frac{\varepsilon(u)}{u} du},$$

thereby showing that L is a normalized slowly varying function. This observation will be central when we later relate the asymptotic elasticity of the production function to the capital share.

2.2 The neoclassical Karamata production function

Let $F : (0, \infty)^2 \to (0, \infty)$ be a constant returns to scale production function, defined as:

$$Y(t) = F(B(t)K(t), A(t)L(t))$$

where $B(t)$ and $A(t)$ are technology parameters scaling capital $K(t)$ and labor $L(t)$, respectively. These terms represent capital-augmenting and labour-augmenting technological progress. These technology terms evolve according to:

$$A(t) = A(t_0)e^{g_A(t-t_0)} \quad \text{and} \quad B(t) = B(t_0)e^{g_B(t-t_0)},$$

where $A(t_0), B(t_0) > 0$ are given initial values, and $g_A > 0$ and $g_B > 0$ denote, respectively, labor-augmenting and capital-augmenting technological progress. When $g_B = 0$, technological change is purely labor-augmenting and purely Harrod-neutral, and the production function reduces to the familiar form $Y(t) = F(K(t), A(t)L(t))$. Conversely, if $g_B \neq 0$, the technological progress is referred to as factor-augmenting technological change.

We assume that F exhibits strictly positive marginal productivities ($F_K > 0$ and $F_L > 0$) and strictly decreasing marginal productivities ($F_{KK} < 0$ and $F_{LL} < 0$).[4] Under constant returns to scale, the cross-derivatives can be shown to be strictly positive ($F_{KL} = F_{LK} > 0$) [5]

We now turn to the intensive form of the technology. Following standard practice, we define $y(t) := \frac{Y(t)}{L(t)}$ as the per-worker output, $k(t) := \frac{K(t)}{L(t)}$ as the per-worker physical capital, and $x(t) := \frac{B(t)}{A(t)}k(t)$ as the effective per-worker physical capital. By constant returns to scale, there exists a function $f : (0, \infty) \to (0, \infty)$ such that the per-worker output is written as $y(t) = A(t)F(x(t), 1) := A(t)f(x(t))$. The marginal productivity of capital is expressed as $F_K = B(t)f'(x)$, the marginal productivity of labour is given by $F_L = A(t)\left(f(x(t)) - x(t)f'(x)\right)$, and the output-capital ratio (i.e., the average productivity of capital) is written as $\frac{Y(t)}{K(t)} = B(t)\frac{f(x(t))}{x(t)}$. The function f inherits the following properties from the function F: $f'(x(t)) > 0$ (strictly increasing) and $f''(x(t)) < 0$ (strictly concave).

[4] To simplify the text, we will sometimes use the following abbreviations:

$$F_j := \frac{\partial F}{\partial j}(B(t)K(t), A(t)L(t)) \quad \text{and} \quad F_{jj} := \frac{\partial^2 F}{\partial j^2}(B(t)K(t), A(t)L(t)),$$

for $j \in \{K(t), L(t)\}$.

[5] These results are the well-known Wicksell's Laws (see Appendix A).

We now introduce four auxiliary elasticities that will play a central role in the asymptotic characterization of the production function.

1. $\alpha(x(t)) := \dfrac{f'(x(t))x(t)}{f(x(t))}$ is the elasticity of output with respect to capital. Under competitive factor markets, this expression equals the capital income share.

2. $\theta(x(t)) := \dfrac{\alpha(x(t))}{1-\alpha(x(t))}$ is the ratio of factors shares.

3. $\beta(x(t)) := \dfrac{f''(x(t))x(t)}{f'(x(t))}$ is the elasticity of marginal productivity of capital with respect to capital.

4. Finally, $\sigma(x(t)) := \dfrac{1-\alpha(x(t))}{-\beta(x(t))}$ is the elasticity of substitution between the factors of production, capital and labor.[6]

Definition (Neoclassical Karamata Production Function).

> Let $f : (0, \infty) \to (0, \infty)$ be a positive measurable production function, defined on some neighborhood $[x_0, \infty)$ of infinity, and satisfying $f'(x) > 0$ (strictly increasing), $f''(x) < 0$ (strictly concave). We say that f is a *Neoclassical Karamata Production Function* if it is regularly varying at infinity with index $\alpha \in (0, 1)$.

Regular variation provides a complete asymptotic characterization of the behaviour of the technology and its induced elasticities. If f is regularly varying with index $\alpha \in (0, 1)$, then its derivatives inherit the same asymptotic structure: f' is regularly varying with index $\alpha - 1$, and f'' with index $\alpha - 2$. Consequently, the elasticities $\alpha(x)$, $\beta(x)$, and $\sigma(x)$ all become slowly varying functions whose limits are determined entirely by the index of regular variation. In particular, the elasticity of substitution converges to 1.

[6]The elasticity of substitution can be written as follows:

$$\sigma(x) = \frac{F_L F_K}{F F_{KL}} = -\frac{L F_L F_K}{K F F_{KK}} = \frac{A(t)\left(f(x(t)) - x(t)f'(x(t))\right) B(t) f'(x(t))}{-A(t) L f(x(t)) \frac{B(t)}{L} x(t) f''(x(t))}$$

$$= -\frac{\left(f(x(t)) - x(t)f'(x(t))\right) f'(x(t))}{f(x(t)) x(t) f''(x(t))}.$$

The first equality is the definition of the elasticity of substitution under constant returns to scale. The second equality uses Wicksell's Laws. The third equality employs the quantities defined in the text. The last equality demonstrates that the elasticity of substitution can also be expressed solely as a function of x. Simplifying the fraction by $f(x)f'(x)$ yields the result.

Proposition 1.

> Let f be regularly varying at infinity with index $\alpha \in (0,1)$. Then f', f'' are regularly varying at infinity with indices $\alpha - 1$ and $\alpha - 2$, respectively. Moreover, the elasticities $\alpha(x)$, $\beta(x)$, $\sigma(x)$ are slowly varying, and:
>
> $\lim_{x\to\infty} \alpha(x) = \alpha,\ \lim_{x\to\infty} \beta(x) = \alpha - 1$ and $\lim_{x\to\infty} \sigma(x) = 1,$
>
> Thus, the capital share converges to a non-degenerate limit and the elasticity of substitution tends asymptotically to 1.

Proof. See Appendix A.

■

Finally, regular variation also delivers a simple and useful sufficient condition for the neoclassical Inada limits at infinity. If the production function grows asymptotically like a power function with positive index, then output necessarily diverges and the marginal product of capital vanishes along large capital paths. Importantly, this result is purely asymptotic: regular variation provides no information about the behaviour of the technology near the origin, and the corresponding Inada limits at zero do not follow from this assumption.

Proposition 2 (Inada sufficient Conditions at infinity).

> Let f be a regularly varying function with index $\alpha > 0$. Then:
>
> $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} f'(x) = 0$

Proof. See Appendix A.

■

Finally, we conclude with a useful property of the elasticity of substitution that highlights its link with the capital share. Since $\alpha(x)$ is slowly varying (Proposition 1), we may apply the Representation Theorem to write

$$\alpha(x) = \frac{xf'(x)}{f(x)} = Ae^{\int_{x_0}^{x} \varepsilon_\alpha(u)\frac{du}{u}}$$

where $\varepsilon_\alpha(x)$ can be interpreted as the elasticity of $\alpha(\cdot)$, i.e. the "elasticity of the elasticity" of the Neoclassical Karamata Production Function (referred to as a "superelasticity" by Jones (2023) in a related context). Differentiating $\alpha(x)$ with respect to x yields

$$\frac{xf''(x)}{f'(x)} - \frac{xf'(x)}{f(x)} + 1 = \varepsilon_\alpha(x)$$

which implies that the (negative of the) elasticity of the marginal product of capital can be written as $-\beta(x) = 1 - \alpha(x) - \varepsilon_\alpha(x)$. The elasticity of substitution then becomes

$$\sigma(x) = \frac{1 - \alpha(x)}{1 - \alpha(x) - \varepsilon_\alpha(x)} = \frac{1}{1 - \frac{\varepsilon_\alpha(x)}{1-\alpha(x)}}$$

We have therefore established the following proposition.

Proposition 3 (Expression of the elasticity of substitution).

Let f be a Neoclassical Karamata Production Function. Then the elasticity of substitution satisfies

$$\frac{\sigma(x) - 1}{\sigma(x)} = \frac{\varepsilon_\alpha(x)}{1 - \alpha(x)}$$

where $\alpha(x)$ is the elasticity of f with respect to x (i.e. the capital share under competitive factor markets), and $\varepsilon_\alpha(x)$ is the elasticity of $\alpha(\cdot)$ with respect to x.

This proposition shows that, for a neoclassical Karamata production function, the elasticity of substitution is not an independent object. Instead, it is entirely determined by the capital share $\alpha(x)$ and its elasticity $\varepsilon_\alpha(x)$. In particular, $\sigma(x)$ depends on x only through the behavior of $\alpha(x)$ and its local variation. This property imposes a strong structural restriction on admissible technologies, ruling out specifications in which the elasticity of substitution evolves independently of factor shares. This restriction contrasts with the standard non-identification result of Diamond, McFadden and Rodriguez (1978), which relies on the absence of such structural constraints. In other words, once the path of the capital share is given, the elasticity of substitution is pinned down. There is no additional degree of freedom for $\sigma(x)$.

3 Some examples and counterexamples

3.1 Production functions asymptotically equivalent to Cobb-Douglas

Off course, the Cobb-Douglas production function $Y(t) = \beta(B(t)K(t))^\alpha(A(t)L(t))^{1-\alpha}$ is a regularly varying function, as in intensive form $y = Af(x)$, with $f(x) = \beta x^\alpha$, we have:

$$\lim_{x\to\infty} \frac{\beta(\lambda x)^\alpha}{\beta x^\alpha} = \lambda^\alpha,$$

where the parameter α is the index of regular variation. Using the Representation Theorem, one can also see why the Cobb-Douglas production function naturally emerges as a candidate for an asymptotic production function. Indeed, if the slowly varying component satisfies $L(x) = \beta$, then $f(x) = x^\alpha L(x)$ and $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$. Thus, the Cobb-Douglas corresponds to the special case where the slowly varying component is constant. While this specification is consistent with a steady state, it remains a very particular case within the class of regularly varying functions.

Let us now consider CES technologies: $Y(t) = [\beta(A(t)L(t))^\rho + (1-\beta)(B(t)K(t))^\rho]^{\frac{1}{\rho}}$, with $\rho < 1$, $0 < \beta < 1$. When $\rho < 0$ (i.e., $\sigma = \frac{1}{1-\rho} < 1$), we have $x^\rho \to 0$ as $x \to \infty$, so that $f(x) \to \beta^{1/\rho}$. Hence: $\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = 1$ and the function is slowly varying (with index 0). In this case, the capital share degenerates to 0. The limiting case $\rho \to -\infty$ yields the Leontief technology $f(x) = \min\{1, x\}$, for which the same result holds. For $\rho > 0$ (i.e. $\sigma = \frac{1}{1-\rho} > 1$), we instead have $\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda$, so that the function is regularly varying with index 1. In this case, the production function becomes asymptotically linear (AK-type), and therefore falls outside the class of neoclassical production functions considered in this paper, as the capital share converges to 1. Therefore, CES production functions do not belong to the class of neoclassical Karamata production functions considered in this paper, as they fail to generate a non-degenerate asymptotic capital share in $(0, 1)$.

Another production function proposed in the literature, which belongs to the Karamata class, is a hybrid between the Cobb-Douglas and CES production functions. This type of production function originates from the unpublished work of Bruno (1962) and has been used in the empirical studies of Liu and Hildebrand (1965) and Nerlove (1967). It was also derived by Sato and Beckmann (1970) as a special case within their class of production functions, which is compatible with the existence of balanced growth despite the presence of factor-augmenting technology. We consider the following function:

$$Y = \left[\beta(A(t)L(t))^\rho + (1-\beta)\left[(B(t)K(t))^\alpha (A(t)L(t))^{1-\alpha}\right]^\rho\right]^{\frac{1}{\rho}}$$

with $0 < \alpha < 1$ and $\rho > 0$. In intensive form: $y = Af(x)$, with $f(x) = [\beta + (1-\beta)x^{\alpha\rho}]^{\frac{1}{\rho}}$, where $\alpha(x) = \frac{\alpha}{1+\frac{\beta}{1-\beta}x^{-\alpha\rho}}$, $\beta(x) = \alpha(x) - 1 - \rho(\alpha(x) - \alpha)$, $\frac{\sigma(x)-1}{\sigma(x)} = \frac{\varepsilon_\alpha(x)}{1-\alpha(x)}$ with $\varepsilon_\alpha(x) = \rho(\alpha(x) - \alpha)$.[7]

When $\beta = 0$, this function reduces to the Cobb-Douglas production function, when $\alpha = 1$ this function reduces to the CES production function, and when $\rho = 1$, this function reduces to the CEDD (Constant Elasticity of Derived Demand) production function proposed by Sato (1970).[8] In this case, we have:

$$Y(t) = \beta(A(t)L(t)) + (1-\beta)(B(t)K(t))^\alpha (A(t)L(t))^{1-\alpha}$$

with $0 < \alpha < 1$. In intensive form: $y = Af(x)$, with $f(x) = \beta + (1-\beta)x^\alpha$, where $\alpha(x) = \frac{\alpha}{1+\frac{\beta}{1-\beta}x^{-\alpha}}$,

[7]This type of production function also supports the approach proposed by Jones (2003, 2005). Indeed, we can propose combining the CES and Cobb-Douglas production functions in the following manner (with an another choice of parameters), by factoring out the Cobb-Douglas function from the CES function:

$$f(x) = \left[\beta\left(\frac{x_0}{x}\right)^\rho + 1 - \beta\right]^{1/\rho} x^\alpha$$

where $\rho < 0$. Here, x represents the long-term quantities of factors used, while x_0 epresents the short-term quantities of factors. This approach is fully compatible with the Karamata production function that we propose in this paper, which clearly distinguishes short-term production functions (in the context of transitional dynamics) from their asymptotic behavior.

[8]Bruno (1968) also presented a special case of this function, calling it the Constant-marginal-shares production function, as it generates constant shares of production factors.

$\beta(x) = \alpha - 1$, $\frac{\sigma(x)-1}{\sigma(x)} = \frac{\varepsilon_\alpha(x)}{1-\alpha(x)}$ with $\varepsilon_\alpha(x) = \alpha(x) - \alpha$. The CEDD, denoted E_k, is the elasticity of derived demand for capital per unit of labor (i.e., for a given number of labor units) and is defined as: $E_k = -\frac{F_K}{KF_{KK}} = -\frac{1}{\beta(x)} = 1-\alpha$ and the elasticity of substitution becomes $\sigma(x) = (1-\alpha(x))E_k(x)$.

It is worth noting that this type of CEDD function can take other forms, which may not be regularly varying. For instance, one can symmetrically define the elasticity of derived demand for labor per unit of capital as $E_l := \frac{F_L}{LF_{LL}}$. Following a similar reasoning, the resulting production function takes the form:

$$Y(t) = (B(t)K(t))^\alpha (A(t)L(t))^{1-\alpha} + \beta(B(t)K(t))$$

This function was used by Kurz (1968), and later by Jones and Manuelli (1990) (with $A(t)$ and $B(t)$ constant) to generate transitional dynamics in the AK endogenous growth model. However, the intensive form of this function, $f(x) = x^\alpha + \beta x$ is regularly varying but with an index of $\alpha = 1$ which means it does not belong to our class of production functions (which, moreover, implies Inada conditions are not satisfied by this type of production function). Additionally, as pointed out by Sato (1970), if both types of elasticities are constant and such that $\sigma = (1-\alpha)E_k = \alpha E_l$ is a constant, the function belongs to the CES production function class. These functions, however, are not regularly varying with a paramer $\alpha \in (0,1)$. Therefore, the CEDD-type functions that belong to our class of Karamata-type production functions are those where E_k is constant but not E_l. It should also be noted that $E_k \neq 1$, otherwise, the production function is of the "Bernoulli" type (as named by A. Marshall), and expressed as (see Sato, 1970):

$$Y(t) = (A(t)L(t)) \ln\left(\frac{A(t)K(t)}{B(t)L(t)}\right) + \beta(A(t)L(t))$$

where $\beta > 0$. In intensive form: $y = Af(x)$, with $f(x) = \ln x + \beta$. This Bernoulli-type production function is only slowly varying, i.e., regularly varying with index $\alpha = 0$. Indeed: $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} f'(x) = 0$. This provides a counterexample of Proposition 2, showing that satisfying the Inada condition at infinity, is not sufficient to ensure that the production function is regularly varying with a non-degenerate index $\alpha \in (0,1)$. In particular, it illustrates that the asymptotic behavior of marginal products alone does not characterize the class of Karamata production functions considered in this paper.

The Revankar (1971)'s *Variable Elasticity of Substitution* (VAR) combines a linear term inside a Cobb-Douglas envelope:

$$Y = (A(t)L(t))^{1-\alpha} \left[B(t)K(t) + (\rho-1)A(t)L(t)\right]^\alpha$$

with $0 < \alpha < 1$ and $\rho > 1$. When $\rho = 1$, the production function reduces to the Cobb-Douglas case, while for $\alpha = 1$ it becomes linear. In intensive form: $y = Af(x)$, with $f(x) = [x + (\rho-1)]^\alpha$, where the associated quantities are: $\alpha(x) = \frac{\alpha}{1+\frac{\rho-1}{x}}$, $\beta(x) = \frac{\alpha-1}{1+\frac{\rho-1}{x}}$, $\sigma(x) = 1 + \frac{\rho-1}{1-\alpha}\frac{1}{x}$, $\varepsilon_\alpha(x) = -\frac{1}{\alpha x}(\alpha(x) - \alpha)$.

These examples share a common structure: they all admit a representation $f(x) = x^\alpha L(x)$

with $L(x) \to c > 0$. As a result, they are asymptotically equivalent to a Cobb–Douglas production function. This observation explains why the literature often concludes that if $\lim_{x\to\infty} \alpha(x) = \alpha$ or $\lim_{x\to\infty} \sigma(x) = 1$, then the production function behaves asymptotically like Cobb-Douglas. However, this conclusion only holds under the additional restriction that the slowly varying component converges to a constant. In general, regularly varying functions do not satisfy this property. Indeed,

$$\lim_{x\to\infty} \frac{f(x)}{x^{\alpha}} = L(x),$$

and $L(x)$ need not converge. Therefore, Cobb-Douglas asymptotics correspond to a very specific subclass of regularly varying production functions. In the next subsections, we show that it is possible to construct production functions for which the capital share converges to a non-degenerate constant and the elasticity of substitution converges to one, while $L(x)$ does not converge to a constant, so that the function is not asymptotically equivalent to Cobb-Douglas.

3.2 The Karamata production function is not asymptotically Cobb-Douglas

A first simple example. A common interpretation in the literature is that if the capital share converges to a constant, or equivalently if the elasticity of substitution converges to one, then the production function must be asymptotically Cobb-Douglas. This conclusion is, however, incorrect. Indeed, consider the following production function:

$$F(BK, AL) = (B(t)K(t))^{\alpha}(A(t)L(t))^{1-\alpha} \ln\left(\frac{B(t)K(t)}{A(t)L(t)}\right)$$

In intensive form, this becomes $f(x) = x^{\alpha} \ln x$, with $x > \max\{1, e^{\frac{2\alpha-1}{\alpha(1-\alpha)}}\}$ so as to guarantee that the production function is strictly increasing and concave. Moreover, $\alpha(x) = \alpha + \frac{1}{\ln x}$, $\beta(x) = \alpha - 1 + \frac{\alpha}{\alpha \ln x + 1}$ and $\frac{\sigma(x)-1}{\sigma(x)} = \frac{\varepsilon_{\alpha}(x)}{1-\alpha(x)}$ with $\varepsilon_{\alpha}(x) = -\frac{1}{\ln x(\alpha \ln x+1)}$. However, it is not asymptotically Cobb-Douglas. Indeed, $\frac{f(x)}{x^{\alpha}} = \ln x \to \infty$ even though $\alpha(x) \to \alpha$ and $\sigma(x) \to 1$. This provides a simple counterexample showing that the convergence of the capital share to a non-degenerate constant, and even the convergence of the elasticity of substitution to one, do not imply that the production function is asymptotically equivalent to Cobb-Douglas. This example shows that the convergence of the capital share and of the elasticity of substitution is not sufficient to characterize the asymptotic form of the production function. In general, regularly varying functions admit a non-trivial slowly varying component, which prevents asymptotic equivalence to a pure power function.

From the Translog to the Super-Log Production Function The Translog (Transcendental Logarithmic) production function, $f(x) = x^{\alpha} e^{\left(\frac{\beta}{2}(\ln x)^2\right)}$ is widely used for its ability to provide a flexible second-order approximation in logarithmic space. However, its use in long-run analysis (as

$x \to \infty$) raises several fundamental issues. First, a production function consistent with balanced growth must belong to the class of *regularly varying functions*. For the Translog specification, the elasticity of output with respect to the input is given by $\alpha(x) = \alpha + \beta \ln x$, and as $x \to \infty$, $|\alpha(x)| \to \infty$. Consequently, the Translog does not admit a finite index of regular variation and exhibits implausible asymptotic behavior. In particular, it cannot be reconciled with a stable long-run distribution of factor shares. Secondly, standard neoclassical conditions require $f'(x) > 0$ and $f''(x) < 0$. In the Translog model, monotonicity fails whenever $\alpha(x)$ becomes negative (e.g., if $\beta < 0$ and $\ln x$ is sufficiently large). Moreover, global concavity is generally not ensured, as the second derivative depends quadratically on $\ln x$, leading to regions of non-convexity. This undermines the well-posedness of the producer's optimization problem. Finally, by construction, the Translog imposes a quadratic structure in $\ln x$, corresponding to an integer-order expansion ($\gamma = 2$). This implies that curvature evolves linearly in $\ln x$, which may be too restrictive in settings where adjustment or saturation effects occur more gradually. These limitations suggest that the Translog should be interpreted as a local approximation, valid in a neighborhood of a reference point, but not as a global representation of technology. A natural way to address this issue is to replace the integer-order expansion with a fractional one, allowing for intermediate curvature. To this end, we propose a fractional approach based on Caputo (1967) derivatives, for $0 < \gamma < 1$.[9]

In this case, the function In this case, the obtained function $L(x) := e^{(\beta(\ln x)^\gamma)}$ is slowly varying. Consequently, the production function $f(x) = x^\alpha L(x)$, belongs to the class of regularly varying functions with index α. This property sharply contrasts with the Translog case and ensures full compatibility with Karamata's theory. In particular, the capital share of the Super-Log specification is given by $\alpha(x) = \alpha + \beta\gamma(\ln x)^{\gamma-1}$ which converges to α as $x \to \infty$. Moreover, the elasticity of substitution satisfies $\frac{\sigma(x)-1}{\sigma(x)} = \frac{\varepsilon_\alpha(x)}{1-\alpha(x)}$, with $\varepsilon_\alpha(x) := \frac{1-\gamma}{\ln x}\frac{\alpha(x)-\alpha}{\alpha(x)}$. It is important to note that, although the factor share converges to the constant α and the elasticity of substitution converges to one, the Super-Log function is not asymptotically Cobb-Douglas in levels. Indeed, $f(x)/x^\alpha = A\exp(\beta(\ln x)^\gamma)$ does not converge to a finite nonzero constant in general. Instead, the function remains regularly varying with index α, with a nontrivial slowly varying component.

The Super-Logarithmic production function therefore provides a flexible alternative to the Translog specification that preserves local adaptability while ensuring global consistency. By replacing the integer-order expansion with a fractional one, it reconciles: (i) local flexibility in fitting the data, (ii) bounded and convergent elasticities, and (iii) asymptotic compatibility with balanced growth through regular variation.[10]

[9] See Appendix E. This construction shows that the Super-Log specification arises naturally as a fractional Taylor expansion of the logarithm of the production function along the growth path, rather than as an ad hoc functional form.

[10] A common criticism of flexible functional forms is that Taylor expansions are only valid locally. The fractional approach proposed here mitigates this limitation. For $\gamma < 1$, we have $(\ln x)^\gamma = o(\ln x)$ as $x \to \infty$, so the curvature term remains asymptotically negligible relative to the linear term. In this sense, the Super-Log specification can

Generalized production function and Lambert function. Zellner and Revankar (1969) proposed a generalized production function to allow for flexible estimation of returns to scale. We consider here a particular adaptation to the case of constant returns to scale, defined implicitly by $y + \ln y = \beta x^\alpha$ with $\beta > 0$ and $\alpha \in (0, 1)$. This specification can be rewritten as $ye^y = e^{\beta x^\alpha}$ so that the solution is given by the Lambert function: $y = W\left(e^{\beta x^\alpha}\right)$. The Lambert W function admits the well-known asymptotic expansion $W(z) = \ln z - \ln \ln z + o(1)$, when $z \to \infty$. Applying this expansion with $z = e^{\beta x^\alpha}$ yields

$$y(x) = \beta x^\alpha - \ln(\beta x^\alpha) + o(1)$$

which shows that the function is regularly varying at infinity with index α. However, this production function is not asymptotically Cobb-Douglas in levels. Indeed, the correction term $-\ln(\beta x^\alpha)$ implies that the convergence toward the asymptotic power law is governed by a nontrivial slowly varying component. In particular, the ratio $y(x)/(\beta x^\alpha)$ converges to one at a logarithmic rate, rather than being constant. Therefore, this specification provides another example of a production function that is asymptotically regularly varying, while allowing for nontrivial deviations from Cobb-Douglas behavior along the transition path. It illustrates how implicit functional forms can generate Karamata-type technologies in a natural way.

Infinite controlled oscillations. Moreover, in Karamata's theory, infinite oscillations is possible. Indeed, the definition of a slowly varying function remains valid even when $L(x)$ exhibits infinite oscillations, including cases where:

$$\lim_{x\to\infty} \inf L(x) = 0 \qquad \lim_{x\to\infty} \sup L(x) = \infty$$

Such behavior arises in functions which exhibit oscillations between 0 and ∞ while still satisfying the slow variation condition. The key requirement is the asymptotic stability under scaling, meaning that the relative growth between $L(tx)$ and $L(x)$ tends to 1, regardless of the absolute amplitude of oscillations. In contrast, when dealing with functions that exhibit more complex oscillations, we define generalized regular variation as:

$$0 < \lim_{x\to\infty} \inf L(x) \leq \lim_{x\to\infty} \sup L(x) < \infty$$

The strict inequalities here are crucial. They ensure that the function's growth is controlled: the ratio $\frac{f(tx)}{f(x)}$ does not collapse to zero or diverge to infinity, preserving the regularity of the asymptotic behavior. Regular variation requires a more rigid structure, maintaining bounded oscillations to reflect predictable scaling behavior. Thus, while slowly varying functions can exhibit extreme oscillations, regularly varying functions demand controlled asymptotic behavior to ensure consistent growth across scales.

be interpreted as a controlled asymptotic perturbation, rather than a purely local approximation. By contrast, the Translog case ($\gamma = 2$) generates a curvature term that dominates asymptotically, leading to diverging elasticities and unstable behavior.

This property implies that the existence of a regular growth path is compatible not only with the presence of factor-augmenting technical progress but also with oscillations in both the factor shares and the elasticity of substitution. To illustrate this possibility, consider the capital share given by the following function:

$$\alpha(x) = \frac{1}{1 + e^{\frac{\cos(x)}{2}}}$$

which is a function bounded between 0 and 1, as the capital share fluctuates approximately between 0.38 and 0.62. Using the ratio of the capital share to the labor share, we obtain: $\theta(x) = e^{\frac{-\cos(x)}{2}}$ which we can (log-)integrate to obtain:

$$\frac{\sigma(x) - 1}{\sigma(x)} = \frac{\sin(x)}{2}$$

Thus, the elasticity of substitution varies within the range $\left[\frac{2}{3}, 2\right]$, which is consistent with empirically estimated values in the literature. The corresponding production function is then given by the Representation Theorem: $f(x) = x^{\alpha} L(x)$ where $\varepsilon(x) = \frac{1}{1+e^{\frac{\cos(x)}{2}}} - \alpha$.[11]

3.3 The Sato and Beckmann (1970)'s production function family: the elasticity of substitution revisited

We now turn to a class of production functions introduced by Sato and Beckmann (1970), whose original purpose was to show that balanced growth is compatible with capital-augmenting technical progress (i.e., $g_B > 0$), without requiring the production function to be Cobb-Douglas. Although their formulation is very general and not particularly convenient for direct applications, it provides an important theoretical insight: the existence of balanced growth does not uniquely characterize the Cobb-Douglas technology. It is worth noting that their class can be interpreted as a particular case of the Karamata-type neoclassical production functions considered in this paper. In this sense, their result can be reinterpreted within a broader framework based on regular variation. To make this connection more explicit, we adopt an alternative approach based on the elasticity of substitution. In particular, we rely on the following necessary and sufficient condition, due to Sato and Beckmann (1970):

[11] However, one important point to note is that the derivative of f is always strictly positive, but its second derivative can change sign because of fluctuations, meaning that it is not always concave along the balanced growth path.

Proposition 4 (Asymptotic Elasticity of Substitution).

A necessary and sufficient condition for the capital share to converge to a non-degenerate value, i.e., $\lim_{x\to\infty}\alpha(x)=\alpha\in(0,1)$, is that the following relation holds:
$$\lim_{x\to\infty}\int_{x_0}^{x}\frac{\sigma(u)-1}{\sigma(u)}\frac{du}{u}=C,$$
where $C\in\mathbb{R}$ is a finite real constant.

Proof. See Appendix B.

■

As Sato and Beckmann (1970) rightly emphasized, this condition is too general to fully clarify the role of the elasticity of substitution within the class of production functions we are interested in. Ideally, one would like to characterize the set of elasticity functions $\sigma(\cdot)$ that satisfy this necessary and sufficient condition. This "ideal set" can be approximated via the following sufficient condition (Sato and Beckmann, 1970):

Proposition 5 (Sufficient Condition).

If there exists a constant N such that
$$\lim_{x\to\infty}\left|\frac{\sigma(x)-1}{\sigma(x)}\right|(\ln x)^k=N,$$
with $0\leq N<\infty$ and $k>1$, then $\lim_{x\to\infty}\alpha(x)=\alpha\in(0,1)$.

Proof. See Appendix B.

■

Using this sufficient condition, we can construct an entire family of production functions that satisfy it. Sato and Beckmann (1970) claimed that the most general form of this class of production functions that they could obtain is (their equation (20), p.396, written here in our notation):
$$f(x)=\begin{cases}Ce^{\displaystyle\int_{x_0}^{x}\frac{1}{1+B^{-1}e^{\frac{N}{k-1}(\ln u)^{1-k}}}\frac{du}{u}}, & \text{if } \sigma(x)=\frac{1}{1+\frac{N}{(\ln x)^k}}:=\bar{\sigma}(x)>1,\\ Ce^{\displaystyle\int_{x_0}^{x}\frac{1}{1+B^{-1}e^{-\frac{N}{k-1}(\ln u)^{1-k}}}\frac{du}{u}}, & \text{if } \sigma(x)=\frac{1}{1-\frac{N}{(\ln x)^k}}:=\underline{\sigma}(x)<1\end{cases}$$
with $B>0$ given by
$$B=\begin{cases}\theta(x_0)e^{\frac{(\ln x_0)^{1-k}}{k-1}},, & \text{if } \sigma(x)=\bar{\sigma}(x)\\ \theta(x_0)e^{-\frac{(\ln x_0)^{1-k}}{k-1}}, & \text{if } \sigma(x)=\underline{\sigma}(x)\end{cases}$$

However, this family is in fact a particular case of the broader class of Karamata production functions. Indeed, in their construction, the auxiliary function $\theta(x)$ converges to a finite positive constant, $\lim_{x\to\infty}\theta(x) = B \in \mathbb{R}_+^\star$, which implies:

$$\lim_{x\to\infty}\alpha(x) = \lim_{x\to\infty}\frac{\theta(x)}{1+\theta(x)} = \frac{B}{1+B} := \alpha \in (0,1)$$

Moreover, by the Representation Theorem for slowly varying functions, $\alpha(x)$ is slowly varying and we may write

$$\varepsilon(x) = \alpha(x) - \alpha = \frac{\theta(x)}{1+\theta(x)} - \frac{B}{1+B}$$

Thus, the capital share converges to a non-degenerate constant. Moreover, the deviations $\varepsilon(x) := \alpha(x) - \alpha$ vanish asymptotically and are driven by the behavior of $\theta(x)$, so that the corresponding production function admits a representation of the form $f(x) = x^\alpha L(x)$, with L slowly varying. Therefore, the Sato-Beckmann family is contained in the Karamata class.

At the same time, their construction provides a useful characterization of the admissible paths of the elasticity of substitution. In particular, it shows that $\sigma(x)$ must converge to 1 at a rate governed by inverse powers of $\ln x$, and that both upper and lower bounds can be constructed around this limit. However, visualizing these restrictions directly is delicate, since they only apply asymptotically as $x \to \infty$. To overcome this difficulty, we use the following duality result.

Proposition 6 (Limit of σ at zero).

Let $\sigma : (0,\infty) \to (0,\infty)$ be a slowly varying function at infinity such that $\lim_{x\to\infty}\sigma(x) = 1$. Then $\lim_{x\to 0^+}\sigma(1/x) = 1$.

Proof. See Appendix B.

■

This result allows us to represent the admissible region for the elasticity of substitution in terms of the inverse variable $1/x = \frac{AL}{BK}$ as it converges to zero. The figure below illustrates these admissible bounds. In particular, it becomes immediately clear that a constant-elasticity-of-substitution (CES) specification does not belong to this class, as it fails to satisfy the required asymptotic convergence toward unity.

This result allows us to represent the admissible region for the elasticity of substitution in terms of the inverse variable $z := \frac{1}{x} = \frac{AL}{BK}$ as it converges to zero. The figure 1 illustrates these admissible bounds. The shaded area represents the set of admissible trajectories for $\sigma(z)$ that are consistent with the asymptotic restrictions derived above. In particular, any admissible path must remain within these bounds and converge to unity as $z \to 0$. The upper and lower envelopes illustrate how fast this convergence can occur, and show that the approach to $\sigma = 1$ is tightly constrained by logarithmic rates. Several examples are plotted within this region. They highlight that

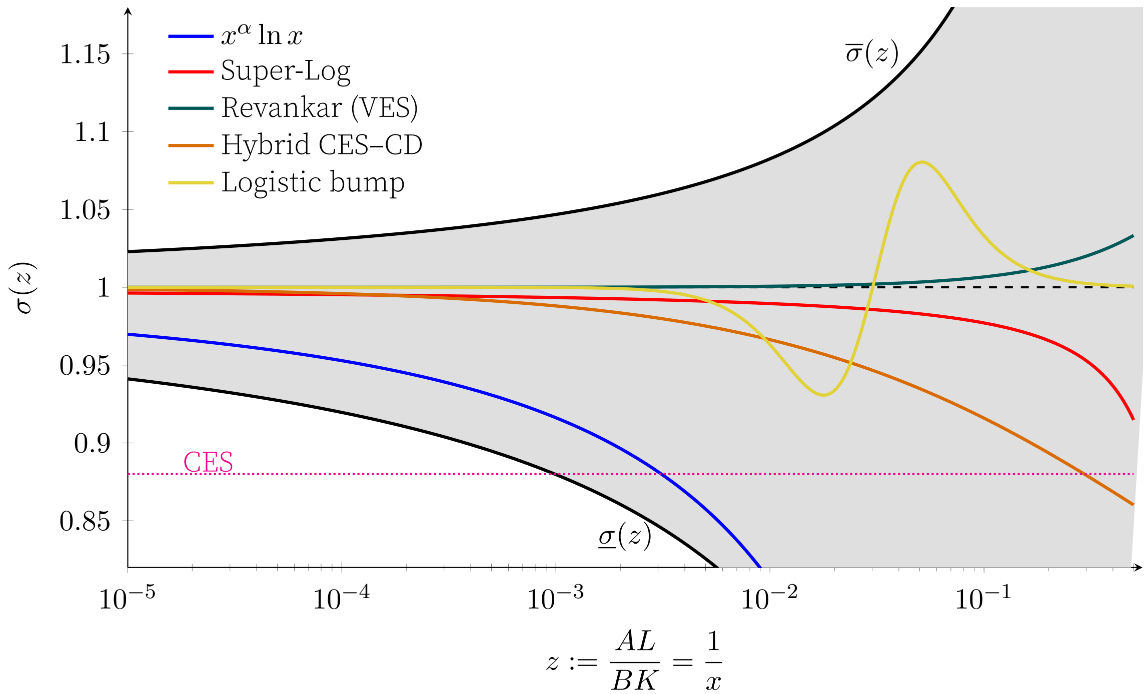

Figure 1: Asymptotic admissible region for the elasticity of substitution and illustrative examples.

a wide variety of behaviors are compatible with these asymptotic restrictions: some elasticities converge from below, others from above, and at different speeds. In contrast, a constant-elasticity-of-substitution (CES) specification does not belong to this class, as it fails to satisfy the required convergence toward unity. Moreover, the figure suggests that admissible trajectories need not be monotonic. The elasticity of substitution may exhibit non-trivial transitional dynamics before converging asymptotically to one.

To illustrate that the elasticity of substitution may exhibit non-monotonic behavior, consider a production function of the form $f(x) = x^\alpha L(x)$, $0 < \alpha < 1$, where the slowly varying component is defined as $L(x) := \exp(\delta S(\ln x))$, $S(u) := \frac{1}{1+e^{-\kappa(u-m)}}$, with $\kappa > 0$, $m \in \mathbb{R}$, and $\delta \in \mathbb{R}$. In this case, the capital share is given by $\alpha(x) = \alpha + \delta\kappa\, S(\ln x)\big(1 - S(\ln x)\big)$, which corresponds to a localized deviation from the constant level α. Since $S(1-S)$ is bell-shaped, the capital share exhibits a smooth "bump" around $\ln x = m$. Its elasticity is $\varepsilon_\alpha(x) = \frac{\delta\kappa^2 S(\ln x)\big(1-S(\ln x)\big)\big(1-2S(\ln x)\big)}{\alpha(x)}$, which changes sign around the inflection point of the logistic function. Using the identity $\frac{\sigma(x)-1}{\sigma(x)}\frac{\varepsilon_\alpha(x)}{1-\alpha(x)}$, it follows that the elasticity of substitution $\sigma(x)$ may be above one for some values of x, below one for others, and still converge to one as $x \to \infty$. Indeed, since $S(\ln x) \to 1$ as $x \to \infty$. This example shows that convergence of the elasticity of substitution toward one need not be monotonic. Even simple specifications can generate temporary deviations from the unit-elastic benchmark, while remaining fully consistent with balanced growth.

4 Balanced growth

4.1 The Houthakker-Jones (2005) extreme-value approach

At this point, one may naturally ask: does our class of production functions admit a microeconomic foundation? More specifically, generalizing the work of Jones (2005) shows that if firms draw local technologies from a Pareto distribution, either of the Leontief type or of the CES type with elasticity of substitution less than 1, then the resulting aggregate production function becomes asymptotically Cobb–Douglas. Our aim here is to extend this insight. The key observation is that the Karamata class of functions is the natural analytic tool used in the study of heavy-tailed distributions and extreme-value phenomena. The Pareto distribution is only the simplest example. Thus, by linking firm-level heterogeneity to regularly varying tails, we are able to provide microeconomic foundations for the whole class of regularly varying production functions, not only the Cobb–Douglas case emphasized by Jones.

We proceed by considering the upper envelope

$$F(x;N) = \max_{i=1,\ldots,N} a_i f\left(\frac{b_i}{a_i}k\right)$$

generated by local technologies of the form $y_i = a_i f(x_i)$, with $x_i := \frac{b_i}{a_i}k$. Jones (2005) obtains closed-form results under a Walras–Leontief specification (elasticity of substitution approaching 0) and numerical results under CES technologies with elasticity less than 1.[12] A central observation for our generalization is that both of these technologies are slowly varying functions. Our first generalization is to assume that $f(x_i)$ is an arbitrary slowly varying function. The CES and Leontief cases considered by Jones are then special cases. Note that f may also be regularly varying, i.e., $f(x_i) \sim x_i^\alpha L(x_i)$, where $L(x_i)$ is slowly varying, but this is not required for the following argument. The second generalization is to assume that a_i and b_i are drawn from arbitrary heavy-tailed distributions, not necessarily Pareto. Heavy-tailed distributions are those whose survival functions decay more slowly than any exponential, including Pareto, Cauchy, Fréchet-type Weibull mixtures, and many others. To formalize this, we rely on the classical Fisher–Tippett extreme-value theorem.

[12]There is a minor inconsistency in Jones (2005), who assumes the Inada conditions although neither the Leontief nor the CES functions satisfy them globally. This does not affect the validity of his asymptotic results.

Fisher–Tippett extreme-value theorem, Bingham *et al.* (1987).

Under suitable regularity conditions on the distribution function F, there exist $\alpha \in \mathbb{R}$ and normalizing sequences $(a_n)_{n\geq 1} > 0$, $(b_n)_{n\geq 1} \in \mathbb{R}$ such that, for all $y \in \mathbb{R}$ with $1 + \alpha y > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\max\{Y_1, \dots, Y_n\} - b_n}{a_n} \leq y\right] = \begin{cases} e^{-(1+\alpha y)^{-\frac{1}{\alpha}}} & \text{if } \alpha \neq 0 \\ e^{-e^{-y}} & \text{if } \alpha = 0 \end{cases} := G_\alpha(y)$$

where function G_α is the generalized extreme-value (GEV) distribution, and α is the extreme-value index.

From Fisher–Tippett extreme-value theorem, when $\alpha > 0$, the limit is of Fréchet type and corresponds to regularly varying tails. A nonnegative random variable X is said to be regularly varying with index $\alpha \geq 0$ if its survival function satisfies $\bar{G}(x) := 1 - G(x) \sim x^{-\alpha} L(x)$ with L slowly varying. Such distributions belong to the domain of attraction of the Fréchet law (see Bingham *et al.*, 1987). The parameter α directly controls the tail behavior of the survival function, and thus that of extreme values.

We now assume that the variables a_i and b_i are drawn from heavy-tailed distributions

$$\mathbb{P}(a_i > a) \sim L_a(a) a^{-\beta_1}, \quad \mathbb{P}(b_i > b) \sim L_b(b) b^{-\beta_2},$$

with $\beta_1, \beta_2 > 0$ and L_a, L_b slowly varying. As in Jones (2005), we assume that at each date, only the most productive technique (the upper envelope) is used. The aggregate is $F(x; N) = \max_{i=1,\dots,N} a_i f(x_i)$. Because f is slowly varying, for large x_i we have the approximation $f(x_i) \sim L(x_i)$ and therefore $y_i = a_i f(x_i) \sim a_i L(x_i)$. Since a_i and b_i are heavy-tailed, the dominant contribution to the maximum comes from the largest realizations of both. Extreme-value theory then yields, for suitable normalization constants c_N, $F(x; N) \sim c_N x^\alpha L(x)$ with $\alpha = \frac{\beta_1}{\beta_1 + \beta_2}$. Thus, even if f is only slowly varying, the upper envelope becomes regularly varying with index α, entirely determined by the tail parameters of the distributions of a_i and b_i. For any fixed $\lambda > 0$,

$$\frac{F(\lambda x; N)}{F(x; N)} \to \lambda^\alpha, \quad \forall \lambda > 0.$$

thanks to the slowly varying property of L. Hence $F(x; N)$ is regularly varying with index α. The asymptotic behaviour of the aggregate production function is governed by the tail behaviour of firm-level heterogeneity, not by the precise form of f.

This justifies our use of the Karamata class for macroeconomic production: the global production function lies in the regularly varying class under extremely general and economically plausible assumptions, far beyond the specific Pareto–Leontief case considered by Jones (2005).

4.2 Neoclassical Economics and Balanced Growth

Consider a neoclassical economy starting at time t_0, where time $t \in [t_0, \infty)$ is assumed to be continuous. A single good, referred to as the output, is produced using physical capital $K(t)$ and labor $L(t)$. In each period, the labor market clears, and the entire available workforce, which grows at an exogenous rate $n \geq 0$, is employed, such that $L(t) = L(t_0)e^{n(t-t_0)}$, with $L(t_0) > 0$ given. Physical capital accumulates as follows: $\dot{K}(t) = I(t) - \delta K(t)$, where $I(t)$ denotes gross investment and $0 \leq \delta < 1$ represents the depreciation rate of physical capital. There is a given initial level of capital $K(t_0) > 0$. Output is allocated either to consumption $C(t)$ or to investment $I(t)$. The resource constraint is written as $Y(t) = C(t) + I(t)$.

Given this standard neoclassical framework, we are ready to state the asymptotic version of Uzawa's Theorem (Uzawa, 1961), adapted from Schlicht (2006), Jones and Scrimgeour (2008), and Acemoglu (2009, Chapter 2).

Proposition 7 (Steady-State Growth).

Consider an economy starting at date t_0, where output, physical capital, consumption and gross investment grow at asymptotically constant rates, given respectively by:

$$\lim_{t\to+\infty} \frac{\dot{Y}(t)}{Y(t)} = g_Y > 0, \quad \lim_{t\to+\infty} \frac{\dot{K}(t)}{K(t)} = g_K > 0, \quad \lim_{t\to+\infty} \frac{\dot{C}(t)}{C(t)} = g_C > 0$$

and $$\lim_{t\to+\infty} \frac{\dot{I}(t)}{I(t)} = g_I > 0$$

under the following conditions: (i) $I(t) > 0$, or $0 < C(t) < Y(t)$, for all $t \in [t_0, \infty)$ and (ii) $\lim_{t\to\infty} \frac{I(t)}{Y(t)} = s \in (0,1)$, or $\lim_{t\to\infty} \frac{C(t)}{Y(t)} = 1 - s \in (0,1)$. Then, all growth rates are identical: $g_Y = g_K = g_C = g_I$.

Proof See Appendix C.

■

As often noted in the literature, this theorem can be stated and proven without any prior reference to the production function or the presence of equilibrium conditions. We now combine the production function we have constructed with the insights from the Balanced Growth Theorem to show that balanced growth is possible in the presence of factor-augmenting technological progress. The key contribution of the proposition lies in its final statement: it characterizes the class of production functions compatible with balanced growth.

Proposition 8.

In a neoclassical economy, where asymptotically $g_Y = g_K = g_C = g_I$, and characterized by the intensive production function $y(t) = A(t)f(x(t))$, where $x(t) := \frac{B(t)}{A(t)}k(t)$ is the ratio of effective capital to effective labor, we have:

$$(i) \lim_{t\to\infty} x(t) = \infty$$

$$(ii) \lim_{t\to\infty} \frac{\dot{x}(t)}{x(t)} = \frac{1}{1-\alpha} g_B > 0$$

$$(iii) g_Y := \lim_{t\to\infty} \frac{\dot{Y}(t)}{Y(t)} = n + g_A + \frac{\alpha}{1-\alpha} g_B$$

$$(iv) \lim_{t\to\infty} \alpha(x) = \alpha = \frac{g_Y - n - g_A}{g_Y - n + g_B - g_A} \in (0,1)$$

Moreover, f is regularly varying at infinity with index $\alpha \in (0,1)$, or a *neoclassical Karamata production function*.

Proof. See Appendix C.

■

It is immediately evident that in the absence of capital-augmenting technological progress (i.e., $g_B = 0$), output (and capital, consumption, and investment) grows at the rate $n+g_A$, $\lim_{t\to\infty} \frac{\dot{x}(t)}{x(t)} = 0$, and thus effective capital per worker, defined in our notation as $x(t) = \frac{A(t)}{B(0)} \frac{K(t)}{L(t)}$, converges to a steady-state constant. This corresponds to the classical result under a production technology with purely labor-augmenting technological progress.

To complete the last part of the proof, we must ensure the existence of a production function that simultaneously generates balanced growth, constant factor shares, a constant output-capital ratio, and a constant marginal productivity of capital (equal to the rate of return on capital in equilibrium in a competitive economy). From the result that the capital share must be asymptotically constant, we are inclined to show that the asymptotic production function compatible with this result is the Cobb-Douglas production function. This is because, with a constant capital share, it suffices to solve the following differential equation:

$$\frac{f'(x)}{f(x)} = \frac{\alpha}{x}$$

This leads to $\ln f(x) = \alpha \ln x + \ln \beta$, where β is an integration constant. Naturally, this implies $f(x) = \beta x^\alpha$ and thus, the production function must be Cobb-Douglas. This is how the literature often concludes.[13]

Another approach is based on the elasticity of substitution. For example, Grossman *et al.* (2017) show that a steady-state equilibrium is achieved if the following condition is satisfied: $[1 - \sigma(x)]g_B = 0$. They conclude that this holds if $g_B = 0$ or if $\sigma(x) = 1$, which corresponds to the Cobb-Douglas production function. However, we refine this result by noting that $\lim_{x\to\infty}[1 - \sigma(x)]g_B = 0$. Assuming $g_B > 0$, this relation is satisfied if the production function admits an asymptotically unitary elasticity of substitution. However, production functions where the capital share asymptotically converges to a constant or the asymtptotic elasticity of substitution is equal to 1 are not limited to the Cobb-Douglas form. Instead, they belong to the broader class of functions we defined. In the proposed proof of Proposition 8, we adopted the classic argument of a "$\delta - \varepsilon$" limit approach.

Using the elegant and intuitive property of normalized slowly varying functions, we can now provide a second more constructive proof of our Proposition and establishes that the neoclassical production function compatible with a steady state, where the capital share is constant and non-degenerate, belongs to the Karamata class. Indeed, from $\frac{f'(x)}{f(x)} = \frac{\alpha(x)}{x}$, we have along the entire path:

$$\ln f(x) = \ln(x_0) + \int_{x_0}^{x} \frac{\alpha(u)}{u} du,$$

Taking the exponential of both sides, we obtain:

$$f(x) = f(x_0)e^{\int_{x_0}^{x} \frac{\alpha(u)}{u} du}.$$

Now, since by assumption $\lim_{x\to\infty} \alpha(x) = \alpha$, we can write $\alpha(u) = \alpha + \varepsilon(u)$, where $\lim_{u\to\infty} \varepsilon(u) = 0$. Thus, the integral in the exponent can be rewritten as follows:

$$\int_{x_0}^{x} \frac{\alpha(u)}{u} du = \alpha \ln\left(\frac{x}{x_0}\right) + \int_{x_0}^{x} \frac{\varepsilon(u)}{u} du.$$

Let us define $L(x)$, a slowly varying function, as:

$$L(x) = x_0^{-\alpha} f(x_0)e^{\int_{x_0}^{x} \frac{\varepsilon(u)}{u} du}.$$

Thus, we can express $f(x)$ as: $f(x) = x^{\alpha}L(x)$, which provides the representation of the function f. This confirms that f is indeed a regularly varying function.

Finally, we require that, along the balanced growth path, both the average and marginal pro-

[13] Barro and Sala-i-Martin (2003) consider the output-capital ratio $\frac{Y(t)}{K(t)} = B(t)\frac{f(x(t))}{x(t)}$, which must be constant in the steady state. If we differentiate with respect to time, we obtain $0 = g_B + (\alpha(x(t)) - 1)(g_B + g_K(t) - n - g_A)$. If on a balanced growth path $g_K(t) \to g_Y$, the differential equation from Barro and Sala-i-Martin (2003) is written, in our notation, as: $\frac{f'(x(t))}{f(x(t))} = \frac{1}{x(t)}\frac{g_Y - n - g_A}{g_Y - n + g_B - g_A} = \frac{\alpha}{x(t)}$ which is equivalent to the direct approach using the capital share.

ductivity of capital remain constant. Indeed, as we have already shown, since the production function can be written in intensive form as $y(t) = A(t)f(x(t))$, the average productivity of capital is given by: $\frac{Y(t)}{K(t)} = B(t)\frac{f(x(t))}{x(t)}$. For its growth rate to be zero, the following condition must hold:

$$g_B + (\alpha(x(t)) - 1)\frac{\dot{x}(t)}{x(t)} = 0 \iff \frac{\dot{x}(t)}{x(t)} = \frac{g_B}{1 - \alpha(x(t))}$$

Similarly, the marginal productivity of capital is given by $F_K = B(t)f'(x(t))$, and its growth rate is zero if the following condition is satisfied:

$$g_B + \beta(x(t))\frac{\dot{x}(t)}{x(t)} = 0 \iff \frac{\dot{x}(t)}{x(t)} = -\frac{g_B}{\beta(x(t))}$$

Now, since we have shown that asymptotically $\lim_{x(t)\to\infty} \alpha(x(t)) = \alpha$ (Proposition 8), $\lim_{x(t)\to\infty} \beta(x(t)) = \alpha - 1$ (Proposition 1) and $\lim_{x(t)\to\infty} \frac{\dot{x}(t)}{x(t)} = \frac{g_B}{1-\alpha}$ (Proposition 8), these two growth rates are asymptotically zero. Thus, once again, since the implied production function belongs to the Karamata class, it is possible to reconcile the existence of a balanced growth path with the presence of factor-augmenting technical progress, and a constant average and marginal productivity of capital.

Note that up to this point, we have worked with a production function where technical progress is factor-augmenting. However, we could have considered the case of an investment-specific technological change, such that $Y(t) = C(t) + I(t)/B(t)$, where g_B represents an embodied technical change that allows for less forgone consumption. If the production function is written as $Y(t) = F(K(t), A(t)L(t))$, then, by defining $x(t) := \frac{K(t)}{A(t)L(t)}$, such that $y(t) = A(t)f(x(t))$, if $g_B > 0$, we would have $g_K = n + g_B + \frac{1}{1-\alpha}g_B$ and $g_Y = n + g_B + \frac{\alpha}{1-\alpha}g_B$, with $\alpha = \frac{g_Y - n - g_A}{g_Y - n - g_A + g_B}$. Note then that we have $g_Y = g_K - g_B$ (see also Grossmann *et al.* (2017)). In this case, the average productivity of capital is no longer asymptotically constant since $g_Y - g_K = -g_B$. For the capital (and labor) share to remain constant, it then becomes necessary, and even sufficient, that the marginal productivity of capital also grows asymptotically at the rate $-g_B$. This is precisely the axiomatic framework proposed by the unpublished and overlooked paper of Phelps (1965) to show the conditions under which factor-augmenting technical progress is possible. With $K(t)$ growing at the rate $g_Y + g_B$, $x(t) \to \infty$ and our framework can be used with investment-specific technological change.[14]

The previous results establish that, along a balanced growth path, the intensive production function is regularly varying. We now turn to the converse question. Suppose that the production function is given in the general form $Y(t) = \widetilde{F}(K(t), L(t), t)$, and that the economy converges to

[14] According to the Phelps' axiomatic (1965), *if there exists a function $B(t)$ that depends only on time, such that the average productivity of capital $Y(t)/K(t)$ increases proportionally to $B(t)$, and the marginal productivity of capital also increases proportionally to $B(t)$, then there exists a balanced growth path along which the shares of the factors of production remain constant*.

a balanced growth path such that $g_Y(t) \to g$ and $g_K(t) \to g$. What restrictions does this impose on the asymptotic structure of technical change? A common approach in the literature, based on Schlicht (2006), Jones and Scrimgeour (2008), and Acemoglu (2009, Chapter 2), is to assume that the economy reaches its steady state at some finite date $T < \infty$, and to characterize the production function from that point onward. While convenient, this formulation is not fully consistent with the continuous-time dynamics typically underlying growth models (as Solow or the neoclassical growth model). Indeed, in autonomous dynamical systems with locally Lipschitz dynamics, solutions are unique. As a consequence, a non-stationary trajectory cannot reach a steady state in finite time: if it did, uniqueness would imply that the trajectory coincides with the stationary solution at all dates.[15] Therefore, convergence to the steady state can only occur asymptotically as $t \to \infty$. In line with this observation, and following the asymptotic perspective of Acemoglu (2009, exercise 2.14), we formulate the analysis directly in the limit as $t \to \infty$, rather than at a finite terminal date. This leads to a representation of technical change that is inherently asymptotic. Thus, let's begin by rewriting the production function for any $t \geq T$ as:

$$Y(t) = \widetilde{F}\left(e^{\int_T^t (g_Y(s) - g_K(s))ds} K(t), e^{\int_T^t (g_Y(s) - n)ds} L(t), T\right)$$

Define: $A(t) := \exp\left(\int_T^t (g_Y(s) - n)ds\right)$ and $B(t) := \exp\left(\int_T^t (g_Y(s) - g_K(s))ds\right)$, then: $Y(t) = F_T(B(t)K(t), A(t)L(t))$. Hence, the asymptotic structure of technical change is governed by the behavior of the integrals $\int_T^t (g_Y(s) - g)ds$ and $\int_T^t (g_K(s) - g)ds$. To characterize these integrals, we adopt a more general asymptotic framework. In Acemoglu (2009), the analysis implicitly focuses on the case in which convergence to the balanced growth path is sufficiently fast, in the sense that the deviations $g_Y(t) - g$ and $g_K(t) - g$ decay faster than $1/t$. In that case, the associated integrals converge, and technical change becomes asymptotically labor-augmenting. However, this leaves open the behavior of the economy when convergence is slower. In particular, when the deviations from the balanced growth path decay at a rate comparable to or slower than $1/t$, the integrals need not converge, and the resulting asymptotic structure of technical change may differ substantially. To capture both cases within a unified framework, we assume that the deviations from the balanced growth rate are regularly varying:

$$g_Y(t) - g \sim t^{\beta_Y} L_Y(t), \quad g_K(t) - g \sim t^{\beta_K} L_K(t)$$

where $L_Y(t)$ and $L_K(t)$ are slowly varying functions. We now state a standard result.

[15] Consider the autonomous system $\dot{x}(t) = F(x(t))$, where F is locally Lipschitz, ensuring existence and uniqueness of solutions. Let $x^\star$ be a steady state, i.e. $F(x^\star) = 0$. Suppose that there exists a finite time $T < \infty$ such that $x(T) = x^\star$. Then the constant function $x(t) \equiv x^\star$ is a solution to the differential equation. By uniqueness, any solution passing through $x^\star$ at time T must coincide with this constant solution. Therefore, $x(t) = x^\star$ for all t. It follows that a non-stationary trajectory cannot reach the steady state in finite time.

Karamata's Theorem (Bingham *et al.*, 1987).

Let L be a slowly varying function and $\alpha > -1$. Then:
$$\int_{x_0}^{x} t^{\alpha} L(t)dt \sim \frac{x^{\alpha+1}}{\alpha+1} L(x)$$
as $x \to \infty$.

We can now distinguish several regimes of convergence. In the *fast convergence* case ($\beta_Y, \beta_K < -1$), both integrals converge to finite constants. Therefore:
$$\int_{T}^{t} (g_Y(s) - g_K(s))ds \to \text{constant}$$
and $B(t)$ converges to a constant. The asymptotic representation reduces to: $Y(t) \sim F(K(t), A(t)L(t))$ which corresponds to purely labor-augmenting technical change, as in Acemoglu (2009). In the *critical convergence* case ($\beta_Y = \beta_K = -1$), we have: $\int_{T}^{t} (g_Y(s) - g)ds \sim L_Y(t) \ln t$ and $\int_{T}^{t} (g_K(s) - g)ds \sim L_K(t) \ln t$ and therefore:
$$\int_{T}^{t} (g_Y(s) - g_K(s))ds \sim (L_Y(t) - L_K(t)) \ln t$$
so that $B(t)$ grows polynomially. In the *slow convergence* case ($-1 < \beta_Y, \beta_K < 0$), we have, by Karamata's theorem: $\int_{T}^{t} (g_Y(s) - g)ds \sim \frac{t^{\beta_Y+1}}{\beta_Y+1} L_Y(t)$ and $\int_{T}^{t} (g_K(s) - g)ds \sim \frac{t^{\beta_K+1}}{\beta_K+1} L_K(t)$. Hence:
$$\int_{T}^{t} (g_Y(s) - g_K(s))ds \sim \frac{t^{\beta_j+1}}{\beta_j+1} L_j(t)$$
where $\beta_j = \min\{\beta_Y, \beta_K\}$. The integral diverges sub-exponentially and $B(t)$ becomes non-trivial. Finally, in the *persistent deviations* case ($\beta_Y = \beta_K = 0$), if $g_Y(t) - g \to L_Y$ and $g_K(t) - g \to L_K$, then:
$$\int_{T}^{t} (g_Y(s) - g_K(s))ds \sim (L_Y - L_K)(t - T)$$
so that $B(t)$ grows exponentially. We can now summarize these results.

Proposition 9.

> Let $Y(t) = \widetilde{F}(K(t), L(t), t)$ be a production function with constant returns to scale. Assume that $g_Y(t) \to g$ and $g_K(t) \to g$ along a balanced growth path. Define: $A(t) := \exp\left(\int_T^t (g_Y(s) - n)ds\right)$ and $B(t) := \exp\left(\int_T^t (g_Y(s) - g_K(s))ds\right)$. Then the production function admits the asymptotic representation: $Y(t) \sim F(B(t)K(t), A(t)L(t))$.
> Moreover, if $g_Y(t) - g$ and $g_K(t) - g$ converge faster than $1/t$, then $B(t)$ converges to a constant and the asymptotic technology is purely labor-augmenting; if the convergence is slower than $1/t$, then $B(t)$ is non-trivial and the asymptotic technology is factor-augmenting.

The previous analysis allows for a wide range of asymptotic behaviors for the factor-augmenting components $A(t)$ and $B(t)$, depending on the speed of convergence of $g_Y(t)$ and $g_K(t)$ toward their common limit. In particular, when convergence is slower than $1/t$, the induced technical change need not follow an exponential path. This feature is consistent with recent empirical evidence. Using a normalized CES production function with factor-augmenting technical change, Klump *et al.* (2007) estimate a flexible specification in which the growth rates of $A(t)$ and $B(t)$ are not restricted to be constant. They find that labor-augmenting technical progress follows an approximately exponential pattern, whereas capital-augmenting technical progress exhibits a markedly different behavior, being either hyperbolic or logarithmic. These results provide empirical support for the view that factor-augmenting technical change may display non-exponential dynamics, thereby justifying the more general asymptotic framework developed above.

However, if the asymptotic production function is regularly varying at infinity with index $\alpha \in (0, 1)$, then our Proposition 3 implies that the elasticity of substitution depends only on the capital share and its elasticity. This places us precisely in case (XIV) of Sato and Beckmann (1968): "*Inventions are neutral in the sense that the elasticity of substitution remains unchanged as long as the income shares of factors are constant (Factor-Augmenting Technical Progress).*" Since, on a balanced growth path, the income shares are asymptotically constant and the elasticity of substitution inherits the same property, the asymptotic technology behaves as factor-augmenting technical change. We can therefore combine our Proposition 3 (i.e., $\sigma(k) = G(\alpha(k))$, so that the elasticity of substitution depends on k only throughout the capital share) with the proof of Sato and Beckmann (1968), p.63, to write de the following Proposition:

Proposition 10.

In a neoclassical economy, where asymptotically $g_Y = g_K = g_C = g_I$, if $f(k(t), t)$ is regularly varying at infinity with index $\alpha \in (0, 1)$, i.e., a *neoclassical Karamata production function*. Then inventions are asymptotically neutral in the sense of Sato and Beckmann (1968): the elasticity of substitution remains unchanged as long as the factor income shares are constant. Hence, the implied technical change along the balanced growth path is factor-augmenting.

4.3 Directed technological change (Acemoglu, 2003)

The approach proposed by Jones (2005), or in the Uzawa's approach, to justify why long-run growth should exhibit purely labor-augmenting technical change does not rely on an equilibrium outcome derived from firms' innovation incentives: the Cobb–Douglas production function arises purely from aggregation. We have generalized this result by showing that the asymptotic production function obtained from aggregation is not necessarily Cobb–Douglas, but a *neoclassical Karamata production function*.

Acemoglu (1998) introduced a general framework for *directed technological change*, which he applied in Acemoglu (2003) to investigate why technological progress may be predominantly labor-augmenting in the long run. He shows[16] that even when both labor- and capital-augmenting technological improvements are possible, asymptotic growth is characterized by purely labor-augmenting technical change, whereas along the transition path capital-augmenting improvements and non-constant factor shares may arise. The key mechanism is that, if the final-good aggregator is CES with elasticity of substitution strictly below one, a balanced growth path exists and necessarily displays purely labor-augmenting progress.

In the present framework, we replace the CES final-good aggregator by a regularly varying function. Proposition 1 shows that such technologies retain the essential asymptotic structure of the CES case, while allowing for a much richer class of substitution patterns. In particular, a regularly varying aggregator is consistent with *both* labor- and capital-augmenting technical progress along a balanced growth path.

We now briefly recall the structure of Acemoglu's (2003) model, focusing on the equilibrium system and on the characterization of balanced growth. The final-good sector uses labor-intensive and capital-intensive intermediate goods to produce a homogeneous good according to a constant-returns-to-scale technology $Y(t) = F(Y_L(t), Y_K(t))$ where F is regularly varying at infinity. Denot-

[16] More precisely, the proof of Lemma A1 in Acemoglu (2003), pp.28–29, is not mathematically correct, as it implicitly uses the identity $\frac{\int f(x)dx}{\int g(x)dx} = \int \frac{f(x)}{g(x)} dx$ which does not hold in general.

ing $F_L := \frac{\partial F}{\partial Y_L}(Y_L, Y_K)$ and $F_K := \frac{\partial F}{\partial Y_K}(Y_L, Y_K)$ and defining the capital share in the final-good sector as $\lambda_K(t) := \frac{F_K Y_K(t)}{Y(t)}$. Proposition 1 implies that

$$\lim_{t\to\infty} \lambda_K(t) = \lambda \in (0,1)$$

Equilibrium factor prices satisfy $P_K(t) = F_K(Y_L(t), Y_K(t))$ and $P_L(t) = F_L(Y_L(t), Y_K(t))$, so the prices are regularly varying with index $\lambda - 1$, and their ratio

$$p := \frac{P_K(t)}{P_L(t)} = \frac{F_K(Y_L(t), Y_K(t))}{F_L(Y_L(t), Y_K(t))}$$

is slowly varying. Labor-intensive and capital-intensive goods are produced competitively from CES production function, with elasticity $\nu = 1/(1-\beta)$:

$$Y_L(t) = \left[\int_0^{n(t)} y_l(i)^\beta\right]^{1/\beta} \text{ and } Y_K(t) = \left[\int_0^{m(t)} y_k(i)^\beta\right]^{1/\beta}$$

where $y(i)$ denote the intermediate goods. The equilibrium inverse demand curves are:

$$\frac{p_l(i)}{p_L(t)} = \left(\frac{y_l(i)}{Y_L(t)}\right)^{-\frac{1}{\nu}} \text{ and } \frac{p_k(i)}{p_K(t)} = \left(\frac{y_k(i)}{Y_K(t)}\right)^{-\frac{1}{\nu}}$$

where prices are given by: $p_l(i) = \frac{\nu}{\nu-1}w = \frac{w}{\beta}$ and $p_k(i) = \frac{\nu}{\nu-1}r = \frac{r}{\beta}$. We have: $y_l(i) = L/n(t)$ and $y_k(i) = K(t)/m(t)$. We also have: $Y_L(t) = n^{\frac{1-\beta}{\beta}}L = N(t)L$ and $Y_K(t) = m^{\frac{1-\beta}{\beta}}L = M(t)K(t)$ and: $\beta N(t)P_L(t)$ and $\beta M(t)P_K(t)$, so that

$$y(t) = N(t)F(1, x(t)) = N(t)f(x(t))$$

and the relative price is given by:

$$p := \frac{P_K(t)}{P_L(t)} = \frac{F_K(Y_L(t), Y_K(t))}{F_L(Y_L(t), Y_K(t))} = \frac{f'(x(t))}{f(x(t)) - x(t)f'(x(t))}$$

The equilibrium factor prices are: $r(t) = \beta M(t)p_K(t) = \beta M(t)f'(x(t))$ and $w(t) = \beta N(t)p_L(t) = \beta N(t)\left[f(x(t)) - x(t)f'(x(t))\right]$, with their ratio

$$\sigma_K(t) = \frac{r(t)K(t)}{w(t)L(t)} = p(t)x(t) = \frac{x(t)f'(x(t))}{f(x(t)) - x(t)f'(x(t))} = \frac{\lambda(x(t))}{1-\lambda(x(t))}$$

We now derive the equilibrium system of the model. Let $S_K(t)$ and $S - S_K(t)$ denote, respectively, the number of scientists engaged in discovering new capital-intensive and labor-intensive intermediate goods, with the market-clearing condition $S_K(t) + S_L(t) = S$, where S is the total (constant) number of scientists. The invention technologies are given by:

$$\begin{cases} \dfrac{\dot m(t)}{m(t)} = b_K\phi(S_K(t))S_K(t) - \delta \\ \dfrac{\dot n(t)}{n(t)} = b_L\phi(S - S_K(t))(S - S_K(t)) - \delta \end{cases}$$

where $b_L, B_K, \delta > 0$. The function $\phi(s)$ is strictly decreasing ($\phi'(s) < 0$), the product $\phi(s)s$ is strictly increasing ($\phi'(s)s + \phi(s) > 0$), and $\phi(0) < \infty$. We also assume $n(0) > 0$ and $m(0) > 0$. Moreover, there exist thresholds $S_L^\star$ and $S_K^\star$ satisfying $S_L^\star + S_K^\star < S$ such that $b_L\phi(S_L^\star)S_L^\star = \delta$ and

$b_K\phi(S_K^\star)S_K^\star = \delta$.

We define the relative intensity variable $x(t) := \dfrac{M(t)K(t)}{N(t)L(t)}$ where $M(t) := m(t)^{\frac{1-\beta}{\beta}}$ and $N(t) := n(t)^{\frac{1-\beta}{\beta}}$. Defining in addition the capital-consumption ratio $c_K(t) := \dfrac{C(t)}{K(t)}$, we obtain

$$\frac{\dot{x}(t)}{x(t)} = \left(M(t)\frac{f(x(t))}{x(t)} - c_K(t)\right) + \frac{1-\beta}{\beta}\left[b_K\phi(S_K(t))S_K(t) - b_L\phi(S - S_K(t))(S - S_K(t))\right]$$

From the Euler equation (since the economy features a representative household with CRRA preferences), we have

$$\frac{\dot{c_K}(t)}{c_K(t)} = \left(\frac{M(t)\beta f'(x(t))}{\theta} - \frac{\rho}{\theta}\right) - \left(M(t)\frac{f(x(t))}{x(t)} - c_K(t)\right)$$

Equilibrium factor prices are given by $r(t) = \beta M(t)f'(x(t))$ et $w(t) = \beta N(t)\left[f(x(t)) - x(t)f'(x(t))\right]$ which allows us to rewrite the growth rate of the value functions as:

$$\begin{cases} \dfrac{\dot{V}_K(t)}{V_K(t)} = -\dfrac{\beta n(t)^{\frac{1-\beta}{\beta}} L\left[f(x(t)) - x(t)f'(x(t))\right]}{m(t)V_K(t)}\sigma_K(x(t)) + (\beta m(t)^{\frac{1-\beta}{\beta}} f'(x(t)) + \delta) \\ \dfrac{\dot{V}_L(t)}{V_L(t)} = -\dfrac{\beta n(t)^{\frac{1-\beta}{\beta}} L\left[f(x(t)) - x(t)f'(x(t))\right]}{n(t)V_L(t)} + (\beta m(t)^{\frac{1-\beta}{\beta}} f'(x(t)) + \delta) \end{cases}$$

recalling that labor is constant, $L(t) = L$, and defining $\sigma_K(x(t)) := \dfrac{x(t)f'(x(t))}{f(x(t)) - x(t)f'(x(t))}$. We therefore obtain a system of six differential equations in seven variables: $V_K(t)$, $V_L(t)$, $m(t)$, $n(t)$, $x(t)$, $c_K(t)$ et $S_K(t)$. To close the system, we include the static allocation condition for researchers

$$\omega_S(t) = \max\left\{b_K\phi(S_K(t))m(t)V_K(t), b_L\phi(S - S_K(t))n(t)V_L(t)\right\} \text{ avec } S_K(t) + S_L(t) = S$$

We define a balanced growth path (BGP) as a trajectory along which output, consumption, and capital grow at finite constant rates. These rates may coincide, as in Acemoglu (2003), or capital may grow at a different rate from output and consumption, in accordance with the axiomatic approach of Phelps, without changing the core result. Based on the equilibrium system derived above, we can state the following proposition.

Proposition 11.

> Along any balanced growth path of the economy described above, both labor- and capital-augmenting technological improvements arise simultaneously.

Proof. See Appendix D.

■

The result that asymptotic technical change is purely labor-augmenting relies critically on the structure of the final-good aggregator. In Acemoglu (2003), this aggregator is of the CES type, which imposes a strong restriction on the elasticity of substitution between capital- and labor-intensive inputs. In particular, when the elasticity of substitution is strictly below one, the equilibrium dynamics lead to a balanced growth path in which factor shares converge and technical change becomes asymptotically labor-augmenting. In the present framework, we replace the CES aggregator with a regularly varying production function. While such functions preserve the key asymptotic properties required for balanced growth, most notably, the convergence of factor shares, they allow for a much richer class of substitution patterns. As a result, the asymptotic structure of technical change is no longer restricted to be purely labor-augmenting. In particular, as shown in Proposition 11, both labor- and capital-augmenting technological improvements may arise simultaneously along a balanced growth path. This shows that the asymptotic direction of technical change is not a universal property of the model, but depends sensitively on the functional form of the final-good technology.

A natural direction for future research would be to characterize, without relying on restrictive assumptions on the functional form of the production function, the conditions under which technical change becomes asymptotically labor-augmenting or factor-augmenting. In this respect, regularly varying production functions provide a flexible and tractable framework to study the asymptotic behavior of growth models beyond the standard CES specification.

5 Conclusion

Do two functions with the same asymptotic properties necessarily coincide? The literature has often suggested an affirmative answer in the context of production theory, implicitly identifying asymptotic Cobb–Douglas behavior with the convergence of the capital share and of the elasticity of substitution. This paper shows that such a conclusion is too restrictive. A production function whose capital share converges to a constant and whose elasticity of substitution converges to one need not be asymptotically Cobb–Douglas. More generally, it belongs to the class of regularly varying functions, which we refer to as *neoclassical Karamata production functions*. This observation has important implications for growth theory. First, it implies that balanced growth paths are compatible with a much broader class of technologies than is usually assumed. The long-run behavior of production is not uniquely pinned down by the convergence of key elasticities, but depends on finer asymptotic features captured by slowly varying components. Second, it shows that the direction of technical change is not a universal property of growth models, but depends on the structure of the production technology. When the final-good aggregator is restricted to a CES specification, strong conclusions emerge regarding the nature of technical progress. By contrast, allowing for regularly varying technologies relaxes these restrictions and admits a richer set of asymptotic behaviors. Importantly, regularly varying production functions are not introduced here as an ad hoc generalization. Rather, they arise naturally in two distinct ways. On the one hand, they characterize the asymptotic structure of neoclassical production functions along balanced growth paths. On the other hand, they emerge from aggregation mechanisms based on heterogeneity and extreme values, in the spirit of Houthakker (1955) and Jones (2005). Taken together, these results suggest that the Karamata class provides a unified and robust foundation for the analysis of long-run production and technical change, encompassing both neoclassical dynamics and aggregation-based approaches. From this perspective, Cobb–Douglas appears as a special case rather than a general benchmark, and regular variation offers a natural and tractable framework to distinguish structural features of growth from artifacts of specific functional-form assumptions.

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Appendix

Appendix A.

Proposition (Wicksell's Laws).

> Given that the production function $F : (0,\infty)^2 \to (0,\infty)$ is homogeneous of degree 1, and that marginal productivities are strictly decreasing, the cross-partial derivatives are positive. Furthermore, the following relationships hold:
> $F_{LK} = -\frac{L}{K}F_{LL}, \quad F_{KL} = -\frac{K}{L}F_{KK} \quad \text{and} \quad F_{LL} = \left(\frac{K}{L}\right)^2 F_{KK}$

Proof. Since the production function F exhibits constant returns to scale and is therefore homogeneous of degree 1, its partial derivatives, which define the marginal productivities, are homogeneous of degree 0. By applying Euler's theorem to the marginal productivities, we obtain the desired results:

$$0 = LF_{LL} + KF_{LK} \quad \Rightarrow \quad F_{LK} = -\frac{L}{K}F_{LL} > 0$$

$$0 = LF_{KL} + KF_{KK} \quad \Rightarrow \quad F_{KL} = -\frac{K}{L}F_{KK} > 0$$

Since $F_{KL} = F_{LK}$, it follows directly that: $F_{LL} = \left(\frac{K}{L}\right)^2 F_{KK}$.

■

Proof (Proposition 1). Using the representation of the production function $f(x) = x^\alpha L(x)$, where $L(x)$ is slowly varying, we compute the first derivative: $f'(x) = \alpha x^{\alpha-1}L(x) + x^\alpha L'(x)$. The ratio of the first derivative to $x^{\alpha-1}L(x)$ is:

$$\alpha(x) = \frac{f'(x)}{x^{\alpha-1}L(x)} = \alpha + \frac{xL'(x)}{L(x)} = \alpha + \varepsilon(x)$$

Since $L(x)$ is slowly varying, $\frac{xL'(x)}{L(x)} = \varepsilon(x) \to 0$ as $x \to \infty$. Thus, $\alpha(x) \to \alpha$, and asymptotically, the marginal productivity behaves as: $f'(x) \sim x^{\alpha-1}L(x)$. This implies that $f'(x)$ is regularly varying with index $\alpha - 1$. Therefore, the capital share is slowly varying, since for $\lambda > 0$, we have:

$$\lim_{x\to\infty} \frac{\alpha(\lambda x)}{\alpha(x)} = \lim_{x\to\infty} \frac{\lambda x f'(\lambda x)}{f(\lambda x)} \cdot \frac{f(x)}{xf'(x)} = 1$$

as $\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha$ and $\lim_{x\to\infty} \frac{f'(\lambda x)}{f'(x)} = \lambda^{\alpha-1}$. Since $f'(x)$ is regularly varying with index $\alpha - 1$, we can write $f'(x) = x^{\alpha-1}L_2(x)$ with $L_2 := \alpha L(x)$. Consider the second derivative: $f''(x) = (\alpha - 1)x^{\alpha-2}L_2(x) + x^{\alpha-1}L_2'(x) = \alpha(\alpha-1)x^{\alpha-2}L(x) + \alpha x^{\alpha-1}L'(x)$, the ratio of the second derivative to

$\alpha x^{\alpha-2}L(x)$ is:

$$\beta(x) = \frac{f''(x)}{\alpha x^{\alpha-2}L(x)} = (\alpha-1) + \frac{xL'(x)}{L(x)} = (\alpha-1)+\varepsilon(x)$$

Since $L(x)$ is slowly varying, $\frac{xL'(x)}{L(x)} = \varepsilon(x) \to 0$ as $x \to \infty$. Thus, $\beta(x) \to \alpha - 1$, and asymptotically, the second derivative behaves as: $f''(x) \sim \alpha x^{\alpha-2}L(x)$. This implies that $f''(x)$ is regularly varying with index $\alpha - 2$. Finally, $\sigma(x) = \frac{1-\alpha(x)}{-\beta(x)}$ is a slowly varying function, since the ratio of two slowly varying functions is also a slowly varying function (Bingham *et al.*, 1987).

■

Proof (Proposition 2, Inada sufficient Conditions at infinity). Since $f(x)$ is regularly varying with index $\alpha > 0$, it satisfies: $f(x) \sim x^\alpha L(x)$, where $L(x)$ is a slowly varying function. The term x^α grows unbounded as $x \to \infty$ because $\alpha > 0$. The function $L(x)$ is positive and slowly varying, meaning it does not grow faster than any power of $\ln(x)$. Therefore, $L(x)$ remains positive and grows sufficiently slowly compared to x^α. Thus: $f(x) \to \infty$ as $x \to \infty$.

From the derivative of $f(x) = x^\alpha L(x)$, we have: $f'(x) = [\alpha + \varepsilon(x)]\, x^{\alpha-1}L(x)$. Since $\alpha - 1 < 0$, $x^{\alpha-1} \to 0$ as $x \to \infty$. $L(x)$ is slowly varying and remains positive and $\varepsilon(x) \to 0$ as $x \to \infty$. Combining these results: $f'(x) \to 0$ as $x \to \infty$.

■

Appendix B

Proof (Proposition 4). Let $\theta(x)$ denote the elasticity of output per worker $y(t)$ with respect to the ratio of effective capital to effective labor $x(t)$, which is equal to the ratio of the capital and labor shares:

$$\theta(x) = \frac{\alpha(x)}{1-\alpha(x)}.$$

This measure has proven very useful in empirical studies as well as in the proof of Uzawa's theorem (see Uzawa (1961) or Jones and Scrimgeour (2004) for example). We can show that:

$$\frac{\theta'(x)}{\theta(x)} = \frac{1}{x}\frac{\sigma(x)-1}{\sigma(x)}.$$

This linear differential equation can be solved from $x_0 > 0$ to x as follows:

$$\theta(x) = \theta(x_0)e^{\displaystyle\int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)}\frac{du}{u}}.$$

Furthermore, note that $\theta(x)$ is a slowly varying function since $\lim_{x\to\infty} \frac{\sigma(u)-1}{\sigma(u)} = 0$, according to the necessary condition. We now establish the following equivalence:

$$\lim_{x\to\infty} \alpha(x) = \alpha \in (0,1) \Leftrightarrow \lim_{x\to\infty} \theta(x) = \theta := \frac{\alpha}{1-\alpha} \in \mathbb{R}_+^\star.$$

Next, since $\theta > 0$, we have:

$$\lim_{x\to\infty} \ln(\theta(x)) = \ln\theta \in \mathbb{R},$$

where we substitute $\theta(x)$ as computed above:

$$\lim_{x\to\infty} \ln\theta(x_0) + \lim_{x\to\infty} \int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)}\frac{du}{u} = \ln\theta \in \mathbb{R}.$$

This directly leads to the desired result:

$$\lim_{x\to\infty} \int_{x_0}^{x} \frac{\sigma(u)-1}{\sigma(u)}\frac{du}{u} = C,$$

where $C := \ln\frac{\theta}{\theta_0} \in \mathbb{R}$ (a finite real constant).

■

Proof (Proposition 5). Let us perform the following change of variables: $u = e^t$, so that $t = \ln u$. This implies $\eta(t) := \frac{\sigma(e^t)-1}{\sigma(e^t)}$ and $du = e^t dt$. Updating the bounds, when $u = x_0$, $t = \ln x_0 := t_0$, and when $u = x$, $t = \ln x$. We have:

$$\lim_{t\to\infty} \int_{t_0}^{t} \eta(u)du = C.$$

This equation converges if there exists a number $0 \leq N < +\infty$ such that:

$$\lim_{t\to\infty} \frac{|\eta(t)|}{t^\beta} = N,$$

with $\beta < -1$.

Indeed, the condition on the function $\eta(t)$ implies that for sufficiently large t, $|\eta(t)| \sim Nt^\beta$, where N is a finite constant. This means that $\eta(t)$ decays as a power of t as $t \to \infty$. Consequently, the integral becomes asymptotically equivalent to:

$$\int_{t_0}^{\infty} |\eta(u)|du \sim \int_{t_0}^{\infty} Nu^\beta du = \lim_{t\to\infty} \frac{t^{\beta+1}}{\beta+1} - \frac{(\ln x_0)^{\beta+1}}{\beta+1}.$$

For this integral to converge, we must have $t^{\beta+1} \to 0$ as $t \to \infty$, which holds if $\beta < -1$.

Thus, we have:

$$\lim_{t\to\infty} \frac{|\eta(t)|}{t^\beta} = N.$$

Substituting $t = \ln x$, $k = -\beta$, and $\eta(t) = \frac{\sigma(x)-1}{\sigma(x)}$, we recover the stated condition.

■

Proof (Proposition 6). Recall the standard duality in regular variation theory: a positive measurable function is regularly varying at infinity with index α if and only if its reciprocal transform $x \mapsto f(1/x)$ is regularly varying at zero with index $-\alpha$ (see Bingham *et al.*, 1987). Now, define

$g : (0, \infty) \to (0, \infty)$ by $g(x) := \sigma(1/x)$. For any $\lambda > 0$ and $x > 0$,

$$\frac{g(\lambda x)}{g(x)} = \frac{\sigma\left(\frac{1}{\lambda x}\right)}{\sigma\left(\frac{1}{x}\right)}$$

Setting $y := 1/x$, we have $x \to 0^+$ if and only if $y \to \infty$, and

$$\frac{g(\lambda x)}{g(x)} = \frac{\sigma\left(\frac{1}{\lambda} y\right)}{\sigma(y)}$$

Since σ is slowly varying at infinity, the ratio on the right-hand side converges to 1 as $y \to \infty$ for every $\lambda > 0$. Hence g is slowly varying at zero.

For the limit, let (x_n) be any sequence with $x_n \to 0^+$. Then $y_n := 1/x_n \to \infty$ and $g(x_n) = \sigma(1/x_n) = \sigma(y_n) \to 1$, because $\sigma(x) \to 1$ as $x \to \infty$. Therefore $\lim_{x \to 0^+} \sigma(1/x) = 1$.

■

Appendix C

Proof (Proposition 7). The condition $\lim_{t\to\infty} \frac{I(t)}{Y(t)} = s \in (0, 1)$ represents the idea that, even asymptotically, the economy does not invest (and therefore does not consume) the entirety of the income generated by output $Y(t)$. This condition naturally implies that $g_I = g_C = g_Y$. Without this condition, it is possible to construct counterexamples that invalidate the balanced growth result (see Acemoglu (2005), Exercise 2.14 for a counterexample, and the proof below).

Let $\chi(t) := \frac{K(t)}{Y(t)}$ denote the capital-output ratio. Its growth rate is written as:

$$\frac{\dot{\chi}(t)}{\chi(t)} = \frac{\dot{K}(t)}{K(t)} - g_Y = \frac{I(t)}{K(t)} - \delta - g_Y = \frac{I(t)}{Y(t)}\frac{Y(t)}{K(t)} - \delta - g_Y$$

We have:

$$\lim_{t\to\infty} \frac{\dot{\chi}(t)}{\chi(t)} = g_K - g_Y = \lim_{t\to\infty} \frac{I(t)}{Y(t)} \lim_{t\to\infty} \frac{1}{\chi(t)} - (\delta + g_Y) = s \lim_{t\to\infty} \frac{1}{\chi(t)} - (\delta + g_Y)$$

Thus, finally:

$$\lim_{t\to\infty} \chi(t) = \lim_{t\to\infty} \frac{K(t_0)}{Y(t_0)} e^{(g_K - g_Y)t} = \frac{s}{\delta + g_K}$$

which is satisfied if $g_K = g_Y$. Therefore, we have shown that $g_Y = g_K = g_C = g_I$.

■

Proof (Proposition 8). The growth rate of $x(t)$ is given by:

$$\frac{\dot{x}(t)}{x(t)} = g_B - g_A - n - \frac{\dot{K}(t)}{K(t)} = g_B - g_A - (n + \delta) - \frac{I(t)}{K(t)}.$$

Asymptotically, this growth rate becomes:

$$\lim_{t\to\infty} \frac{\dot{x}(t)}{x(t)} = g_B - g_A - (n + \delta) + \lim_{t\to\infty} \frac{I(t)}{K(t)}.$$

Using the Balanced Growth Theorem, we know:

$$\lim_{t\to\infty} \frac{I(t)}{K(t)} = \lim_{t\to\infty} \frac{I(t)}{Y(t)}\frac{Y(t)}{K(t)} = s \cdot \frac{\delta + g_Y}{s} = \delta + g_Y.$$

Thus, we obtain:

$$\lim_{t\to\infty} \frac{\dot{x}(t)}{x(t)} = g_B - g_A - (n+\delta) + \delta + g_Y = g_Y - n - g_A + g_B.$$

From the production function $Y(t) = F(B(t)K(t), A(t)L(t))$, we derive:

$$\begin{aligned}\frac{\dot{Y}(t)}{Y(t)} &= \alpha(x(t))\left(g_B + \frac{\dot{K}(t)}{K(t)}\right) + (1-\alpha(x(t)))(g_A+n)\\ &= n + g_A + \alpha(x(t))\left(\frac{\dot{K}(t)}{K(t)} - n - g_A + g_B\right).\end{aligned}$$

Taking the limit on both sides of the equation, we have:

$$\lim_{t\to\infty} \frac{\dot{Y}(t)}{Y(t)} = n + g_A + \lim_{t\to\infty} \alpha(x(t))\left(\lim_{t\to\infty} \frac{\dot{K}(t)}{K(t)} - n - g_A + g_B\right).$$

Recalling that $g_Y = g_K$, we establish point (iv):

$$\lim_{t\to\infty} \alpha(x(t)) = \alpha = \frac{g_Y - n - g_A}{g_Y - n - g_A + g_B}.$$

This allows us to derive the asymptotic growth rate of output (iii):

$$g_Y = n + g_A + \frac{\alpha}{1-\alpha} g_B.$$

From this, we deduce the growth rate of $x(t)$ (ii):

$$\lim_{t\to\infty} \frac{\dot{x}(t)}{x(t)} = \frac{1}{1-\alpha} g_B > 0.$$

Thus, $x(t) = \frac{B(t_0)K(t_0)}{A(t_0)L(t_0)} e^{\frac{1}{1-\alpha} g_B t}$, and therefore (i):

$$\lim_{t\to\infty} x(t) = \infty.$$

Let $\phi(x) := \ln f(x)$, so that $\phi'(x) = \frac{f'(x)}{f(x)}$. The capital share in income can then be expressed as:

$$\alpha(x) = \frac{f'(x)x}{f(x)} = x\phi'(x).$$

Thus:

$$\phi'(x) = \frac{\alpha(x)}{x}.$$

By integrating both sides between a constant $x_0 > 0$ and x, we obtain:

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} \frac{\alpha(u)}{u} du.$$

Given that $\lim_{x\to\infty} \alpha(x) = \alpha$, for any $\varepsilon > 0$, there exists x_1 such that for all $x \geq x_1$, $|\alpha(x) - \alpha| < \varepsilon$.

Thus, for $x \geq x_1$, the capital share satisfies: $\alpha - \varepsilon < \alpha(x) < \alpha + \varepsilon$.

For $x \geq x_1$, we can bound $\phi(x)$ as follows:

$$(\alpha - \varepsilon) \int_{x_1}^{x} \frac{du}{u} \leq \phi(x) - \phi(x_1) \leq (\alpha + \varepsilon) \int_{x_1}^{x} \frac{du}{u}.$$

After evaluating the integrals, we get:

$$(\alpha - \varepsilon) \ln \left(\frac{x}{x_1} \right) \leq \phi(x) - \phi(x_1) \leq (\alpha + \varepsilon) \ln \left(\frac{x}{x_1} \right).$$

Taking the exponential of both sides yields:

$$\left(\frac{x}{x_1} \right)^{\alpha - \varepsilon} e^{\phi(x_1)} \leq f(x) \leq \left(\frac{x}{x_1} \right)^{\alpha + \varepsilon} e^{\phi(x_1)}.$$

Now, consider the ratio $\frac{f(\lambda x)}{f(x)}$. Using the above bounds, we can write:

$$\left(\frac{\lambda x}{x} \right)^{\alpha - \varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq \left(\frac{\lambda x}{x} \right)^{\alpha + \varepsilon}.$$

Simplifying gives:

$$t^{\alpha - \varepsilon} \leq \frac{f(\lambda x)}{f(x)} \leq t^{\alpha + \varepsilon}.$$

Finally, taking the limit as $x \to \infty$ (with λ fixed) and noting that $\varepsilon > 0$ can be made arbitrarily small, we obtain:

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha},$$

which proves that f is a regularly varying function.

■

Appendix D

Proof (Proposition 11). Along a balanced growth path, the equilibrium interest rate is given by $r(t) = \beta M(t) f'(x(t))$ and must converge to a constant in steady state. Therefore,

$$\lim_{t \to \infty} \frac{\dot{r}(t)}{r(t)} = \lim_{t \to \infty} \left(\frac{\dot{M}(t)}{M(t)} + \frac{f''(x(t))x(t)}{f'(x(t))} \frac{\dot{x}(t)}{x(t)} \right) = \lim_{t \to \infty} \frac{\dot{M}(t)}{M(t)} - (1 - \lambda) \lim_{t \to \infty} \frac{\dot{x}(t)}{x(t)} = 0$$

Hence,

$$\lim_{t \to \infty} \frac{\dot{x}(t)}{x(t)} = (1 - \lambda)^{-1} \frac{1 - \beta}{\beta} \lim_{t \to \infty} \frac{\dot{m}(t)}{m(t)}$$

This expression is constant whenever $\frac{\dot{m}(t)}{m(t)}$ converges to a constant, which requires that $S_K(t) \to S_K \in [0, S]$. Consequently, $\frac{\dot{n}(t)}{n(t)}$ also converges to a constant, and thus:

$$\begin{cases} \lim_{t \to \infty} \dfrac{\dot{m}(t)}{m(t)} = b_K \phi(S_K) S_K - \delta = g_m \\ \lim_{t \to \infty} \dfrac{\dot{n}(t)}{n(t)} = b_L \phi(S - S_K)(S - S_K) - \delta = g_n \end{cases}$$

The growth rate of $x(t)$ then satisfies:

$$\frac{\dot{x}(t)}{x(t)} = \left(M(t)\frac{f(x(t))}{x(t)} - c_K(t)\right) + \frac{1-\beta}{\beta}\left[g_m - g_n\right] = (1-\lambda)^{-1}\frac{1-\beta}{\beta}g_m$$

which implies

$$\left(M(t)\frac{f(x(t))}{x(t)} - c_K(t)\right) = \frac{1-\beta}{\beta}\left[g_n - g_m\right] + (1-\lambda)^{-1}\frac{1-\beta}{\beta}g_m = \frac{1-\beta}{\beta}\left(g_n + \frac{\lambda}{1-\lambda}g_m\right)$$

Similarly, the Euler equation (using CRRA preferences) gives:

$$\frac{\dot{c_K}(t)}{c_K(t)} = \left(\frac{r-\rho}{\theta}\right) - \left(M(t)\frac{f(x(t))}{x(t)} - c_K(t)\right) = \left(\frac{r-\rho}{\theta}\right) - \frac{1-\beta}{\beta}\left(g_n + \frac{\lambda}{1-\lambda}g_m\right)$$

Output per capita is $y(t) = N(t)f(x(t))$ with growth rate

$$\frac{\dot{y}(t)}{y(t)} = \frac{1-\beta}{\beta}g_n + \frac{\lambda}{1-\lambda}\frac{1-\beta}{\beta}g_m = \frac{1-\beta}{\beta}\left(g_n + \frac{\lambda}{1-\lambda}g_m\right)$$

Now consider their growth rates when $t \to \infty$:

$$\begin{cases} \dfrac{\dot{V}_K(t)}{V_K(t)} = -\dfrac{\lambda}{1-\lambda}\dfrac{w(t)}{m(t)V_K(t)} + (r+\delta) \\ \dfrac{\dot{V}_L(t)}{V_L(t)} = -\dfrac{w(t)}{n(t)V_L(t)} + (r+\delta) \end{cases}$$

Equivalently,

$$\begin{cases} V_K(t) = \displaystyle\int_t^\infty e^{-(r+\delta)(s-t)}\frac{w(s)}{m(s)}ds = \frac{\lambda}{1-\lambda}\frac{w(t)}{m(t)}\int_t^\infty e^{-\int_t^s (r+\delta-g_w(s)+g_m)}ds \\ V_L(t) = \displaystyle\int_t^\infty e^{-(r+\delta)(s-t)}\frac{w(s)}{n(s)}ds = \frac{w(t)}{n(t)}\int_t^\infty e^{-\int_t^s (r+\delta-g_w(s)+g_n)}ds \end{cases}$$

To compute these integrals, note that the real wage, $w(t) = \beta N(t)\left[f(x(t)) - x(t)f'(x(t))\right]$, must grow at a constant rate:

$$\frac{\dot{w}(t)}{w(t)} = \frac{1-\beta}{\beta}g_n(t) - \beta(x(t))\frac{\lambda(t)}{1-\lambda(t)}\frac{\dot{x}(t)}{x(t)}$$

It indeed converges to the constant (using Proposition 1)

$$\frac{\dot{w}(t)}{w(t)} = \frac{1-\beta}{\beta}\left[g_n + \frac{\lambda}{1-\lambda}g_m\right]$$

Thus the integrals yield:

$$\begin{cases} m(t)V_K(t) = \dfrac{\lambda}{1-\lambda}\dfrac{w(t)}{r+\delta-g_w+g_m} \\ n(t)V_L(t) = \dfrac{w(t)}{r+\delta-g_w+g_n} \end{cases}$$

Hence, as $t \to \infty$

$$\Delta(t) = \frac{m(t)V_K(t)}{n(t)V_L(t)} = \frac{\lambda}{1-\lambda}\frac{r+\delta-g_w+g_n}{r+\delta-g_w+g_m} = \Delta$$

Finally, the asymptotic wage of scientists is:

$\omega_S(t) = \max\{b_K\phi(S_K)\Delta, b_L\phi(S - S_K)\}\, n(t)V_L(t)$ avec $S_K \in [0, 1]$

Therefore, the equilibrium allocation of research effort is asymptotically given by

$$\frac{\gamma}{1-\gamma} = \Delta\frac{b_L\phi(S - S_K)}{b_K\phi(S_K)}$$

and using the Acemoglu (2003) definition of balanced growth path, we have: $\frac{\dot{c_K}(t)}{c_K(t)} = 0$, and the unique asymptotic equilibrium is

$$\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} = \lim_{t\to\infty}\frac{\dot{Y}(t)}{Y(t)} = \lim_{t\to\infty}\frac{\dot{C}(t)}{C(t)} = \frac{1-\beta}{\beta}\left(g_n + \frac{\lambda}{1-\lambda}g_m\right)$$

■

Appendix E

Starting from the identity $\frac{f'(x)}{f(x)} = \frac{\alpha}{x}$, we have shown that the production function can be written as $f(x) = x^\alpha L(x)$, where $L(x)$ is slowly varying. Integrating along the path yields:

$$\ln f(x) = \ln f(x_0) + \alpha(\ln x - \ln x_0) + \int_{x_0}^{x}\frac{(\alpha(u) - \alpha)}{u}du$$

Let $z = \ln\frac{x}{x_0}$, so that $x = x_0e^z$. Define

$$\phi(z) := \int_{x_0}^{x_0e^z}\frac{(\alpha(u) - \alpha)}{u}du = \int_0^z [\alpha(x_0e^\nu) - \alpha]\, d\nu$$

Then $\ln f(x) = \ln f(x_0) + \alpha z + \phi(z)$.

Theorem (Generalized Taylor formula, Odibat and Shawagfeh (2007, Theorem 3)).

> Let $0 < \gamma \leq 1$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a function such that the sequential Caputo derivatives $({}^CD_a^\gamma)^k f$ exist and are continuous on $[a, b]$ for $k = 0, 1, \ldots, N+1$. Then, for every $x \in [a, b]$, one has
>
> $$f(x) = \sum_{k=0}^{N}\frac{({}^CD_a^\gamma)^k f(a)}{\Gamma(k\gamma + 1)}(x-a)^{k\gamma} + R_N^\gamma(x),$$
>
> where the remainder term is given by
>
> $$R_N^\gamma(x) = \frac{({}^CD_a^\gamma)^{N+1} f(\xi)}{\Gamma((N+1)\gamma + 1)}(x-a)^{(N+1)\gamma}, \qquad a \leq \xi \leq x$$
>
> where $\Gamma(\cdot)$ denotes the Euler Gamma function.

The notation ${}^CD_a^\gamma$ refers to the Caputo derivative of order γ with lower bound a. This operator is non-local and depends on the entire past trajectory of the function over $[a, x]$. The notation $({}^C D_a^\gamma)^k f$ denotes the k-fold sequential Caputo derivative, and should not be confused with the Caputo derivative of order $k\gamma$, since in general $({}^C D_a^\gamma)^k f \neq {}^C D_a^{k\gamma} f$.

We now apply the fractional Taylor expansion to ϕ around $z = 0$. For $N = 1$, we obtain:

$$\phi(z) = \phi(0) + \frac{({}^C D_0^\gamma)\phi(0)}{\Gamma(\gamma+1)} z^\gamma + R_1^\gamma(z)$$

Since $\phi(0) = 0$, this yields:

$$\ln f(x) = \ln f(x_0) + \alpha \ln \frac{x}{x_0} + \beta \left(\ln \frac{x}{x_0}\right)^\gamma + R_1^\gamma \left(\ln \frac{x}{x_0}\right)$$

avec $\beta := \frac{({}^C D_0^\gamma)\phi(0)}{\Gamma(\gamma+1)}$. Setting $\ln A := \ln f(x_0) - \alpha \ln x_0$, we obtain the Super-Logarithmic specification: $\ln f(x) = \ln A + \alpha \ln x + \beta (\ln x)^\gamma$, or, equivalently, $f(x) = A x^\alpha e^{(\beta(\ln x)^\gamma)}$.